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An Efficient Method for Solving Coupled Time Fractional Nonlinear Evolution Equations with Conformable Fractional Derivatives

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Abstract

In this article, an efficient reliable method, which is the residual power series method (RPSM), is used in order to investigate the approximate solutions of conformable time fractional nonlinear evolution equations with conformable derivatives under initial conditions. In particular, two types of equations are considered, which are time coupled diffusion-reaction equations (CD-REs) and MKdv equations coupled with conformable fractional time derivative of order α . The attitude of RPSM and the influence of different values of α are shown graphically.

Keywords: Fractional differential equations (FDEs); Residual power series method (RPSM); Nonlinear evolution equations.

طريقة كفوءة لحل معادلات التطور غير الخطية الزمنية ذات الرتبة الكسرية التوافقية

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الخلاصة

في هذه المقالة ، استخدمت طريقة فعالة تسمى طريقة سلسلة الطاقة المتبقية (RPSM) من أجل إيجاد الحلول التقريبية لمعادلات التطور غير الخطية الجزئية ذات رتبة الزمن الكسرية المطابقة في ظل الشروط الابتدائية. بصورة خاصة ، تم أخذ نوعين من المسائل في الفيزياء مع مشتق زمني كسري مطابق من الرتبة α . عرض سلوك طريقة (RPSM) وتأثيرها وقيم α مختلفة وقد تم استعراض ذلك بيانياً.

1. Introduction

Recently, the subject of fractional calculus has been gaining a considerable attention from various authors due to its important role in many applications. Fractional derivatives can be considered as a generalization of integer derivatives that have been widely used in characterizing the biological and physical phenomena[1-4]. There are many definitions for the fractional order derivatives, such as those reported by Riemann-Liouville, Caputo, and Grunwald-Letnikov, etc.[5- 7]. Khalil *et al.*[8, 9] were the first who proposed a new fractional derivative, viz. the conformable fractional derivative (CFD) to take control of the remarkable problem that had occurred in the Riemann-Liouville and Caputo fractional derivatives, which is the inheritance of the nonlocal properties from the integral. Consequently, various numerical or semi numerical methods, for example the Adomain decomposition method [10], generalized Mittag-Leffler function method and Sumudu transform method [11],

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homotopy perturbation method [12], and variational iteration method [13], were discovered to provide the approximate solutions of FDEs. The RPSM was used successfully to produce a series of solutions for tumor models[14]. RPSM was used to investigate a numerical solution for the fractional Burger equation[15]. The exact analytical solution of the time-fractional Schrodinger equation was found in another work [16]. The main aim of this paper is to employ RPSM for two models of nonlinear FDEs of special interest physically, in terms of the convergent fractional power series. The rest of this article is arranged as follows: In section 2 some preliminaries are given. In section 3 we describe the RPSM. The models of the proposed study are described in sections 4 and 5. Numerical simulations are drawn in section 6. Finally, the conclusions are presented in section 7.

2. Preliminaries

Definition 2.1[8]: Given a function $y : [0, \infty) \rightarrow \mathbb{R}$, then the conformable fractional derivative of order α of y is defined by

$$CD_t^\alpha (y)(t) = \lim_{\varepsilon \rightarrow 0} \frac{y(t + \varepsilon t^{1-\alpha}) - y(t)}{\varepsilon}, \quad \forall t > 0, \alpha \in (0, 1] \tag{1}$$

And the conformable integral of order α is defined by

$$CI_t^\alpha (y)(t) = \int_\alpha^t y(x) x^{\alpha-1} dx, \quad \alpha \geq 0 \tag{2}$$

Theorem 2.2 [8]: Let $\alpha \in (0, 1]$ and f, g be α -differentiable at a point $t > 0$. then

1. $CD_t^\alpha (af + bg) = a(CD_t^\alpha f) + b(CD_t^\alpha g), \forall a, b \in \mathbb{R}$
2. $CD_t^\alpha (t^p) = pt^{p-\alpha}, \forall p \in \mathbb{R},$
3. $CD_t^\alpha (\lambda) = 0$, for all constant function $f(t) = \lambda$
4. $CD_t^\alpha (fg) = f(CD_t^\alpha g) + g(CD_t^\alpha f)$
5. $CD_t^\alpha (f / g) = (g - f) / g^2$
6. If, in addition, f is differentiable, then $(CD_t^\alpha f)(t) = t^{1-\alpha} \frac{df}{dt}(t)$.

3. An Overview of CRPSM

We consider the following fundamental concept of RPSM operator:

$$CD^\alpha u(x, t) = N(u) + R(u). \tag{4}$$

Where $N(u)$ and $R(u)$ are nonlinear and linear terms, respectively, with an initial condition (IC):

$$u(x, 0) = f(x). \tag{5}$$

The RPSM suggests the solution for Eqs.(4) as a fractional power series (FPS) about the (IC) $t=0$ as:

$$u(x, t) = \sum_{n=0}^{\infty} f_n(x) \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)}, \quad 0 < \alpha \leq 1, -\infty < x < \infty, 0 \leq t < R. \tag{6}$$

Truncating the infinite series (6) after k^{th} terms implies:

$$u_k(x, t) = \sum_{n=0}^k f_n(x) \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)}. \tag{7}$$

For the convergence of the FBS, refer for instance to a previous work [17].

Using Equation (5), then Eq. (7) may be expressed as:

$$u_k(x, t) = f(x) + \sum_{n=1}^k f_n(x) \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)}, \quad k = 1, 2, 3, \dots \tag{8}$$

The residual function (RF) for Eq.(4) is defined by:

$$Res_u(x, t) = CD_t^\alpha u(x, t) - N(u) - R(u). \tag{9}$$

Hence, the k^{th} (RF) $Res_{u,k}$ is

$$Res_{u,k}(x, t) = CD_t^\alpha u_k(x, t) - N(u_k) - R(u_k). \tag{10}$$

As in an earlier work [18], $\text{Res}(x, t) = 0$ and $\lim_{k \rightarrow \infty} \text{Res}_k(x, t) = \text{Res}(x, t)$. Therefore, $D_t^{n\alpha} \text{Res}(x, t) = 0$, which comes from theorem 2.2, property 3, and since $D_t^{n\alpha}$ of RF coincides at $t = 0$, for all $n=0,1,\dots,k$, then,

$$C D_t^{n\alpha} \text{Res}(x, 0) = C D_t^{n\alpha} \text{Res}_k(x, 0) = 0, \forall n = 0,1,\dots,k \tag{11}$$

However, finding $f_1(x), f_2(x), f_3(x), \dots$ needs to solve the algebraic equations:

$$C D_t^{(k-1)\alpha} \text{Res}_{u,k}(x, 0) = 0, k = 1,2,3,\dots \tag{12}$$

4. RPSM for solving CD-REs

The performing of the RPSM for finding the solution of the CD-REs in terms of the FPS is represented in this section.

First consider that:

$$C D_t^\alpha u = u(1-u-v) + u_{xx}, \tag{13}$$

$$C D_t^\alpha v = v_{xx} - uv, \tag{14}$$

With
$$u(x, 0) = \frac{e^{kx}}{[1 + e^{0.5kx}]^2}, \tag{15}$$

$$v(x, 0) = \frac{1}{[1 + e^{0.5kx}]}. \tag{16}$$

and the exact solution when $\alpha=1$ is given by a previous work [19] as:

$$u(z) = \frac{e^{kz}}{[1 + e^{0.5kz}]^2}, \tag{17}$$

$$v(z) = \frac{1}{[1 + e^{0.5kz}]}. \tag{18}$$

where, $z = x + ct$, k is constant, and $C D_t^\alpha u, C D_t^\alpha v$ symbolize the conformable fractional order derivative w.r.t. t for the functions u and v , respectively. The solution of problems (13) - (16) in a FPS expansion about the (IC) $t = 0$ is given as follows:

$$u(x, t) = \sum_{n=0}^k f_n(x) \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)}, 0 < \alpha \leq 1, \tag{19}$$

$$v(x, t) = \sum_{n=0}^k g_n(x) \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)}, 0 < \alpha \leq 1. \tag{20}$$

The k^{th} truncated series of $u(x, t)$ and $v(x, t)$ is defined by:

$$u_k(x, t) = \sum_{n=0}^\infty f_n(x) \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)}, \tag{21}$$

$$v_k(x, t) = \sum_{n=0}^\infty g_n(x) \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)}. \tag{22}$$

It is obvious that $f_0(x)$ and $g_0(x)$ can be obtained directly from the initial conditions given by Eqs. (15) and (16), hence:

$$f_0(x) = \frac{e^{kx}}{[1 + e^{0.5kx}]^2}, \tag{23}$$

$$g_0(x) = \frac{1}{[1 + e^{0.5kx}]}. \tag{24}$$

Furthermore, we rewrite Eqs. (21) and (22) as :

$$u_k(x,t) = \frac{e^{kx}}{[1+e^{0.5kx}]^2} + \sum_{n=1}^k f_n(x) \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \tag{25}$$

$$v_k(x,t) = \frac{1}{[1+e^{0.5kx}]} + \sum_{n=1}^k g_n(x) \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}. \tag{26}$$

Let us define the RF of Eqs. (19) and (20) as follows:

$$Res_u(x,t) = CD_t^\alpha u - u(1-u-v) - u_{xx}, \tag{27}$$

$$Res_v(x,t) = CD_t^\alpha v - v_{xx} + uv. \tag{28}$$

Then the k^{th} residual function becomes:

$$Res_{u,k}(x,t) = CD_t^\alpha u_k - u_k(1-u_k-v_k) - (u_k)_{xx}, \tag{29}$$

$$Res_{v,k}(x,t) = CD_t^\alpha v_k - (v_k)_{xx} + u_k v_k. \tag{30}$$

It is clear that $Res(x,t) = 0$ and $\lim_{k \rightarrow \infty} Res_k(x,t) = Res(x,t)$. Therefore

$CD_t^{n\alpha} Res_u(x,t) = CD_t^{n\alpha} Res_v(x,t) = 0, \forall x,t$, see theorem (2.2, property 3). Hence

$$CD_t^{n\alpha} Res(x,0) = CD_t^{n\alpha} Res_k(x,0) = 0, \forall k = 0,1,\dots,n.$$

The coefficients of $f_n(x)$ and $g_n(x), n = 1,2,\dots,k$, respectively, can be computed by solving the following system:

$$\begin{aligned} CD_t^{(k-1)\alpha} Res_{u,k}(x,0) &= 0, \quad k = 1,2,\dots \\ CD_t^{(k-1)\alpha} Res_{v,k}(x,0) &= 0, \quad k = 1,2,\dots \end{aligned} \tag{31}$$

By putting the obtained values of Eq. (31) in Eqs. (19) and (20) , we get the desired approximate solution of (CD-REs).

The numerical behaviour of the approximate solution of $u(x,t)$ and $v(x,t)$ of problems (13) – (16), obtained by RPSM) with different values of fractional time derivative of order α , are shown graphically in Figures- (1) - (6).

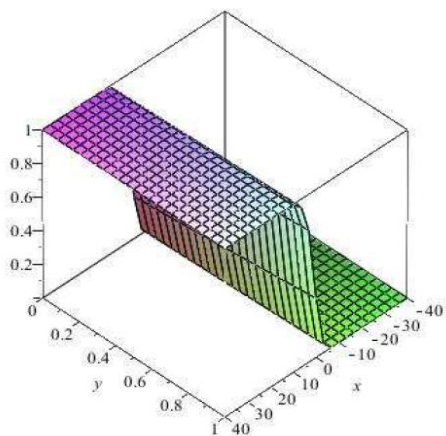


Figure 1-Result of u of Eq.(13) when $\alpha=1$

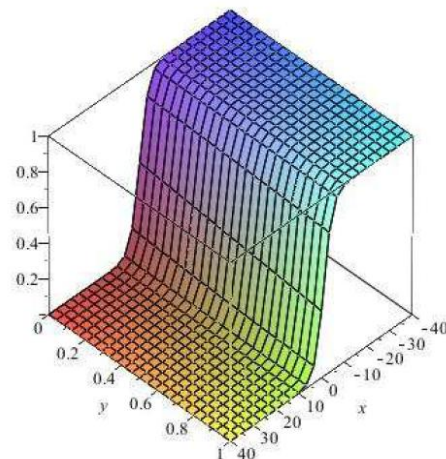


Figure 2-Result of v of Eq.(14) when $\alpha=1$

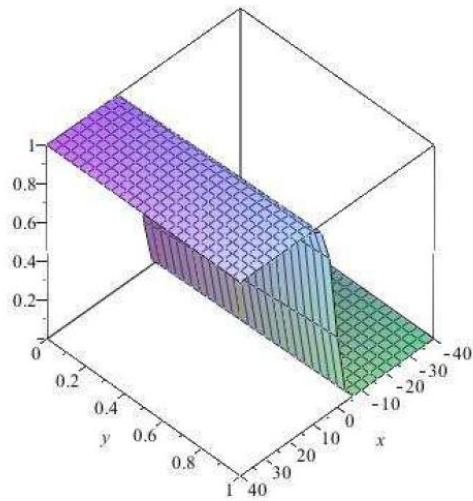


Figure 3-Closed form of u of Eq.(13) when $\alpha=1$

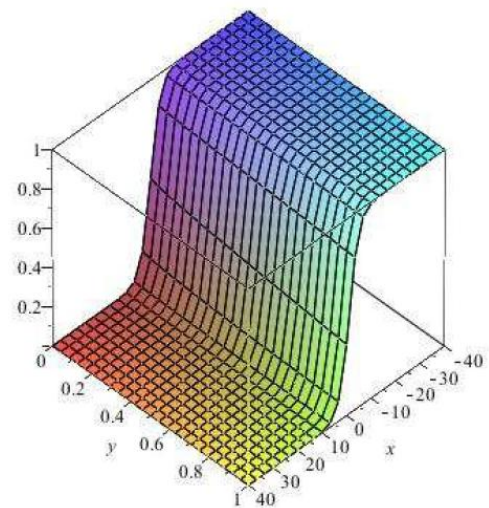


Figure 4-Closed form of v of Eq.(14) when $\alpha=1$

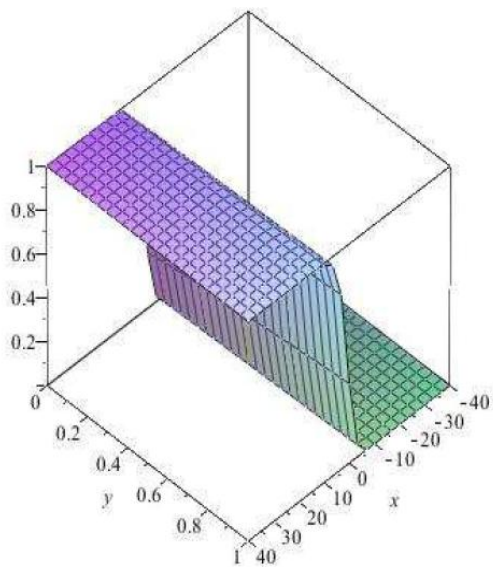


Figure - 5: Result of u of Eq.(13) when $\alpha=3/4$

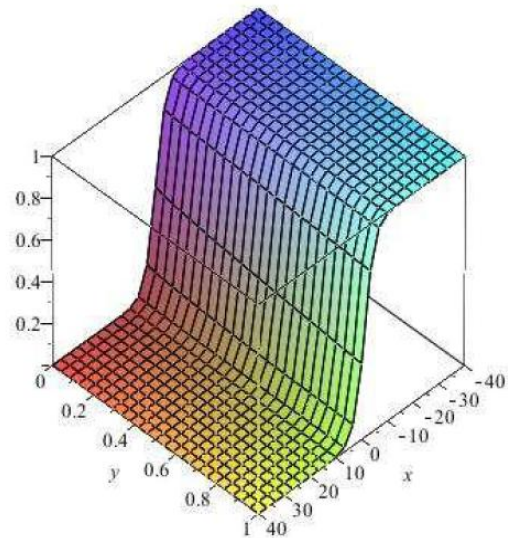


Figure - 6: Result of v of Eq.(14) when $\alpha=3/4$

The below Figures- (7 and 8) represent the absolute error between the approximate solution and the closed form for different values of x and $y=0.1$.

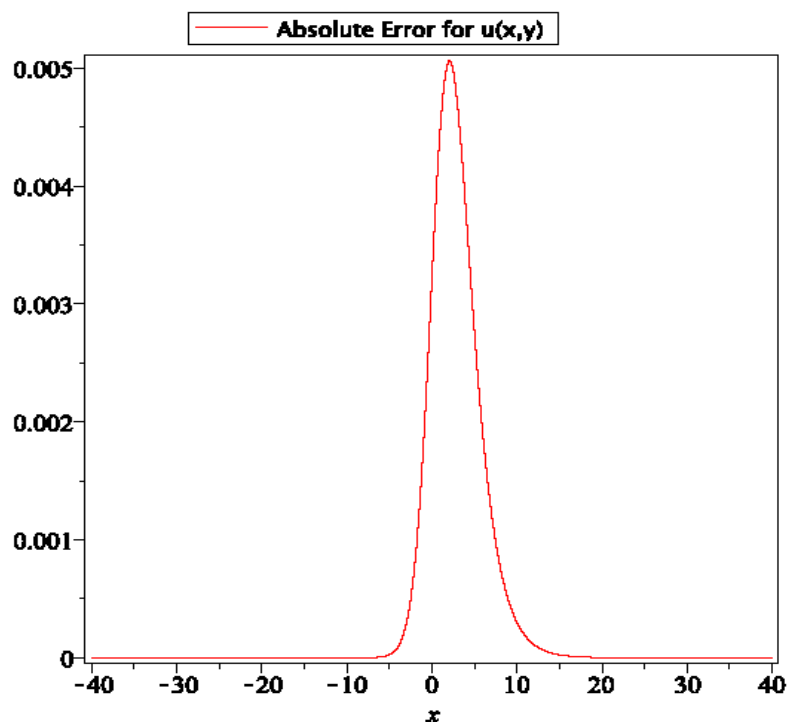


Figure 7-The absolute error between the approximate solution of $u(x,t)$ and the closed form for problems (13)-(16).

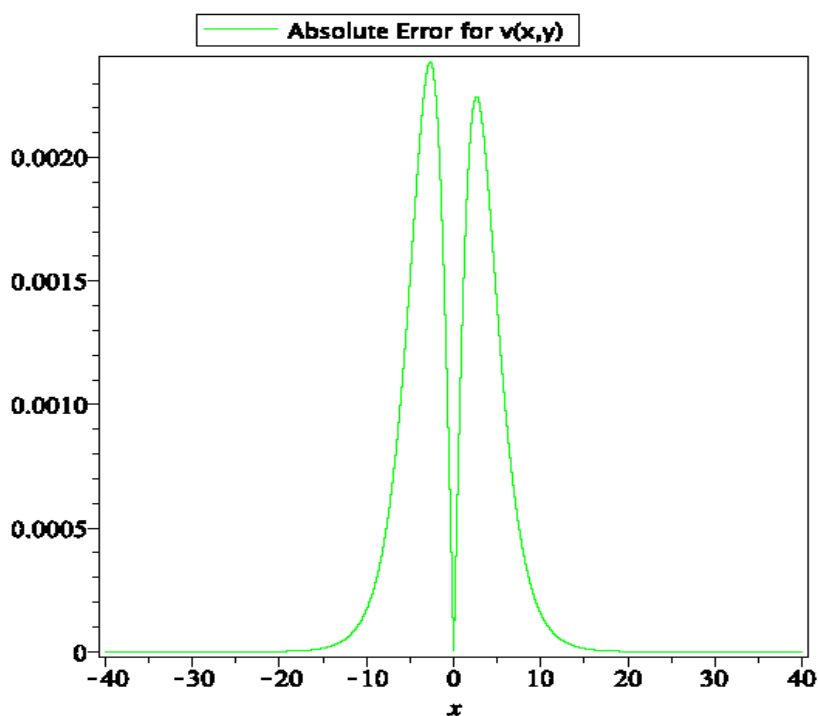


Figure 8-The absolute error between the approximate solution of $v(x,t)$ and the closed form for problems (13)-(16).

5. RPSM for solving coupled MKdv equations with a conformable fractional order derivative (CMKdv)

In this segment, the RPSM will also be considered in order to get the solution of the CD-R as FBS, as follows:

$$CD_t^\alpha u = \frac{1}{2}u_{xxx} - 3u^2u_x + \frac{3}{2}v_{xx} + 3(uv)_x - 3\lambda u_x, \tag{32}$$

$$CD_t^\alpha v = -v_{xxx} - 3vv_x - 3u_xv_x + 3u^2v_x + 3\lambda v_x. \tag{33}$$

With $u(x, 0) = \frac{b_1}{2k} + k \tanh[kx]$, (34)

$$v(x, 0) = \frac{\lambda}{2} \left(1 + \frac{b_1}{2}\right) + b_1 \tanh[kx]. \tag{35}$$

The exact solution when $\alpha=1$ is given by a previous work [20], as:

$$u(x, t) = \frac{b_1}{2k} + k \tanh[k\xi], \tag{36}$$

$$v(x, t) = \frac{\lambda}{2} \left(1 + \frac{b_1}{2}\right) + b_1 \tanh[k\xi]. \tag{37}$$

with $\xi = x + \frac{1}{4}[-4k^2 - 6\lambda + \frac{6k\lambda}{b_1} + \frac{3b_1^2}{k^2}]t$, where $k, b_1 \neq 0, \lambda$ is an arbitrary constant, $CD_t^\alpha u$,

$CD_t^\alpha v$ symbolize the conformable fractional order derivative w.r.t. t for the functions u and v , respectively. Then the solutions of problems (32) - (35) in a fractional power series expansion about the initial point $t = 0$ are given by Eqs. (19) and (20). Also $u_k(x, t)$ and $v_k(x, t)$ are defined by Eqs. (21) and (22), respectively. Clearly, for the case $n = 0$, we have from Eqs. (34) and (35) that:

$$f_0(x) = \frac{b_1}{2k} + k \tanh[kx], \tag{38}$$

$$g_0(x) = \frac{\lambda}{2} \left(1 + \frac{b_1}{2}\right) + b_1 \tanh[kx]. \tag{39}$$

Now, from Eqs. (38) and (39), we have:

$$u_k(x, t) = \frac{b_1}{2k} + k \tanh[kx] + \sum_{n=1}^k f_n(x) \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)}, \tag{40}$$

$$v_k(x, t) = \frac{\lambda}{2} \left(1 + \frac{b_1}{2}\right) + b_1 \tanh[kx] + \sum_{n=1}^k g_n(x) \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)}. \tag{41}$$

We define the RF of Eqs. (32) and (33) as follows:

$$Res_u(x, t) = CD_t^\alpha u - \frac{1}{2}u_{xxx} + 3u^2u_x - \frac{3}{2}v_{xx} - 3(uv)_x + 3\lambda u_x, \tag{42}$$

$$Res_v(x, t) = CD_t^\alpha v + v_{xxx} + 3vv_x + 3u_xv_x - 3u^2v_x - 3\lambda v_x. \tag{43}$$

Then the k^{th} (RF) becomes:

$$Res_{u,k}(x, t) = CD_t^\alpha u - \frac{1}{2}u_{xxx} + 3u^2u_x - \frac{3}{2}v_{xx} - 3(uv)_x + 3\lambda u_x, \tag{44}$$

$$Res_{v,k}(x, t) = CD_t^\alpha v + v_{xxx} + 3vv_x + 3u_xv_x - 3u^2v_x - 3\lambda v_x. \tag{45}$$

The coefficient of the approximate solutions u and v of problems (32)-(35), which is represented by Eqs. (21) and (22), where $f_n(x)$ and $g_n(x), n = 1, 2, \dots, k$, may be obtained by solving the following algebraic system:

$$CD_t^{(k-1)\alpha} Res_{u,k}(x, 0) = 0, \quad k = 1, 2, \dots$$

$$CD_t^{(k-1)\alpha} Res_{v,k}(x, 0) = 0, \quad k = 1, 2, \dots$$

By putting the obtained values in Eqs. (19) and (20), we get the desired approximate solution of CMKdv.

Figures.(9) - (14) present the dynamic and attitude of the (RPSM) solutions $u(x,t)$ and $v(x,t)$ of problem (32) – (35) under the influence of the replacing in the values of fractional order α .

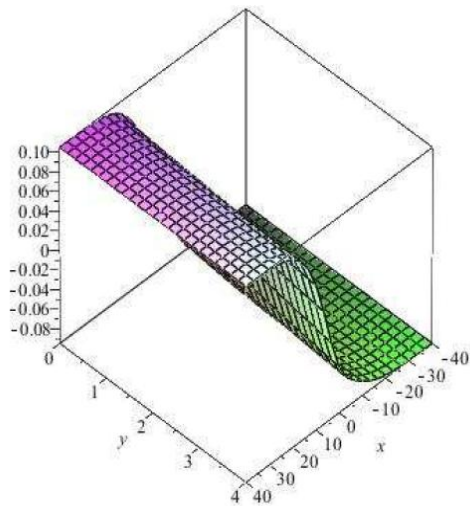


Figure - 9: Result of u of Eq.(32) when $\alpha=1$

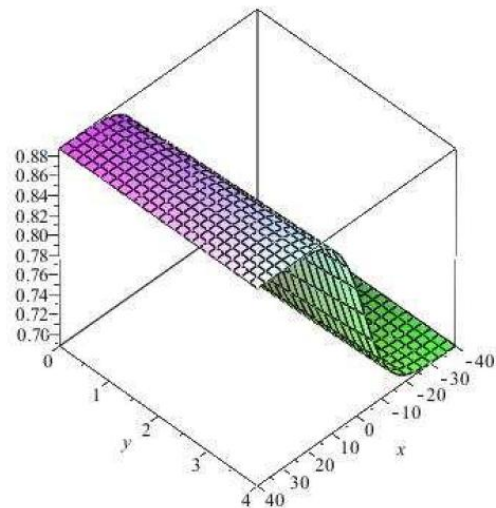


Figure - 10: Result of v of Eq.(33) when $\alpha=1$

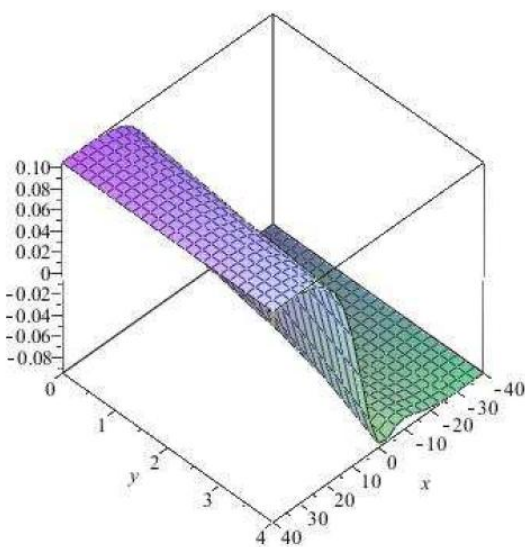


Figure 11-Closed form of u of Eq.(32) when $\alpha=1$

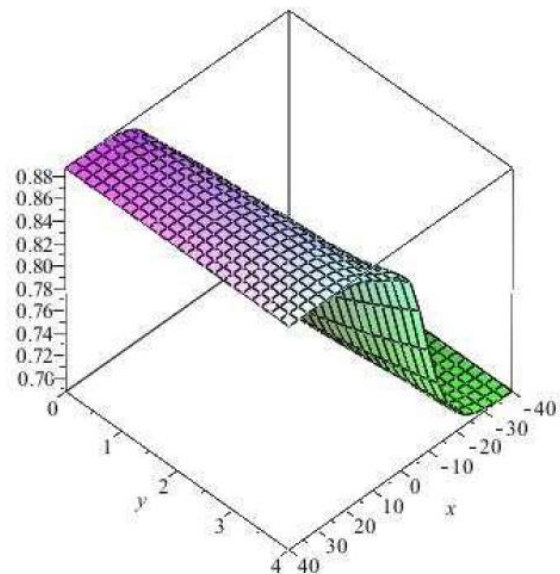


Figure 12- Closed form of v of Eq.(33) when $\alpha=1$

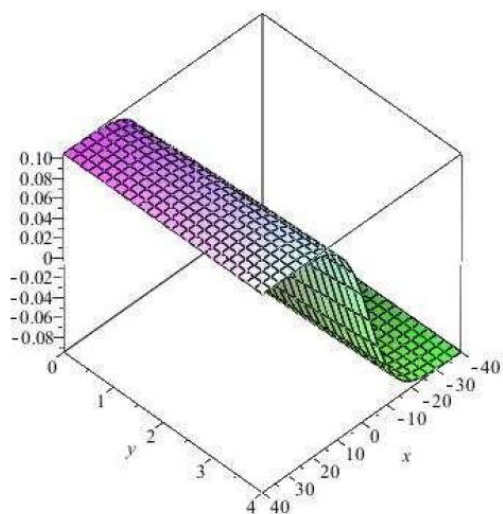


Figure 13-Result of u of Eq.(32) when $\alpha=1/2$

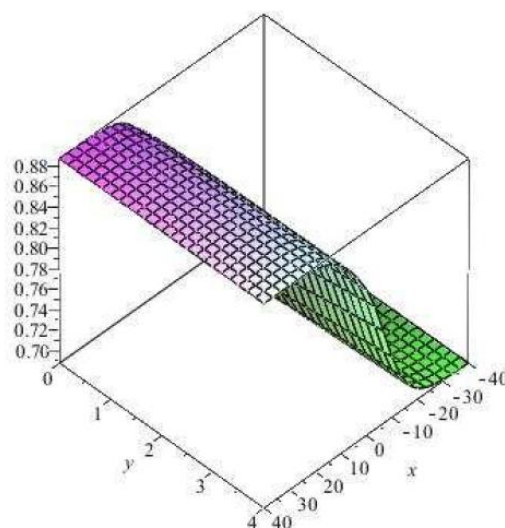


Figure 14-Result of v of Eq.(33) when $\alpha=1/2$

Below, Figures-(15-16) represent the absolute error between the approximate solution and the closed form for different values of x and $y=0.1$.

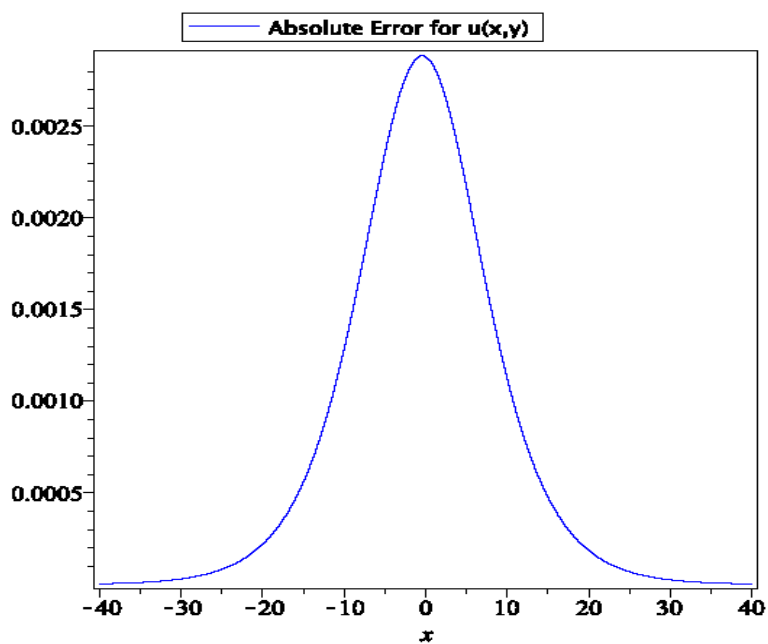


Figure 15-The absolute error between the approximate solution of $u(x,t)$ and the closed form for problems (32)-(35).

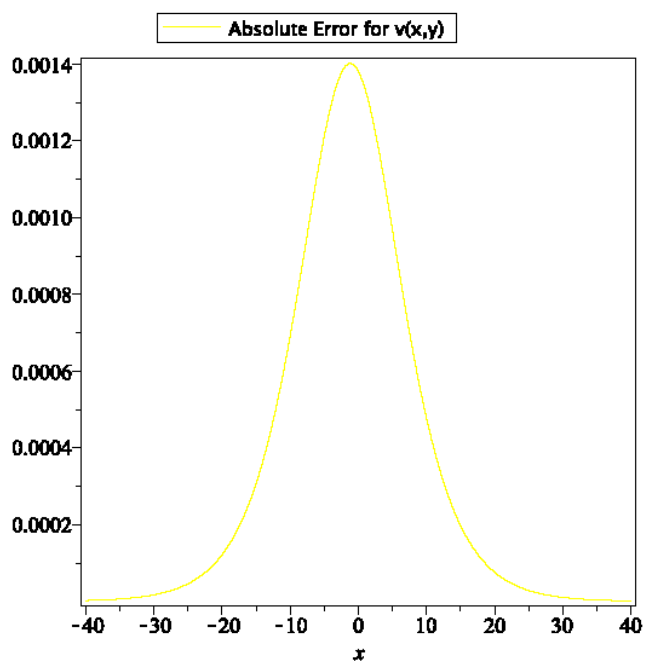


Figure 16-The absolute error between the approximate solution of $v(x,t)$ and the closed form for problems (32)-(35)

6. Numerical Simulation

In the previous sections the accuracy and efficiency of the proposed method were verified. The actions of the approximate solution $u(x,y)$ and $v(x,y)$ for problems (13)-(16) and (32)-(35) for different values of α and x at $t=0.2$ are given in Figures (17)-(20).

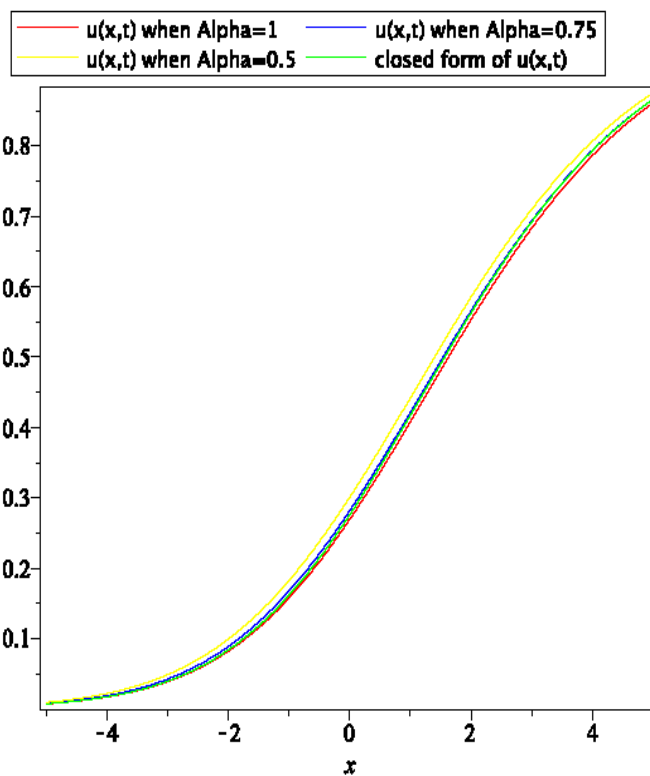


Figure 17-The action of $u(x,t)$ of problems (13)-(16).

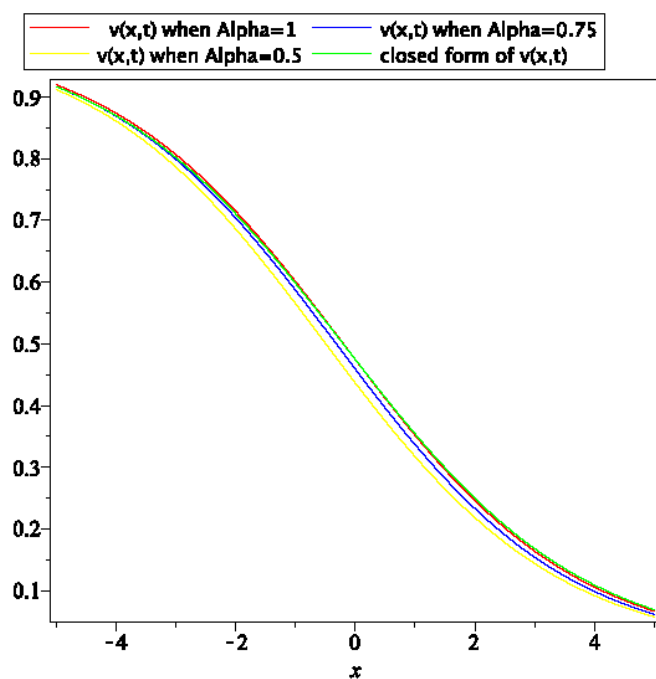


Figure 18-The action of $v(x,t)$ of problems (13)-(16).

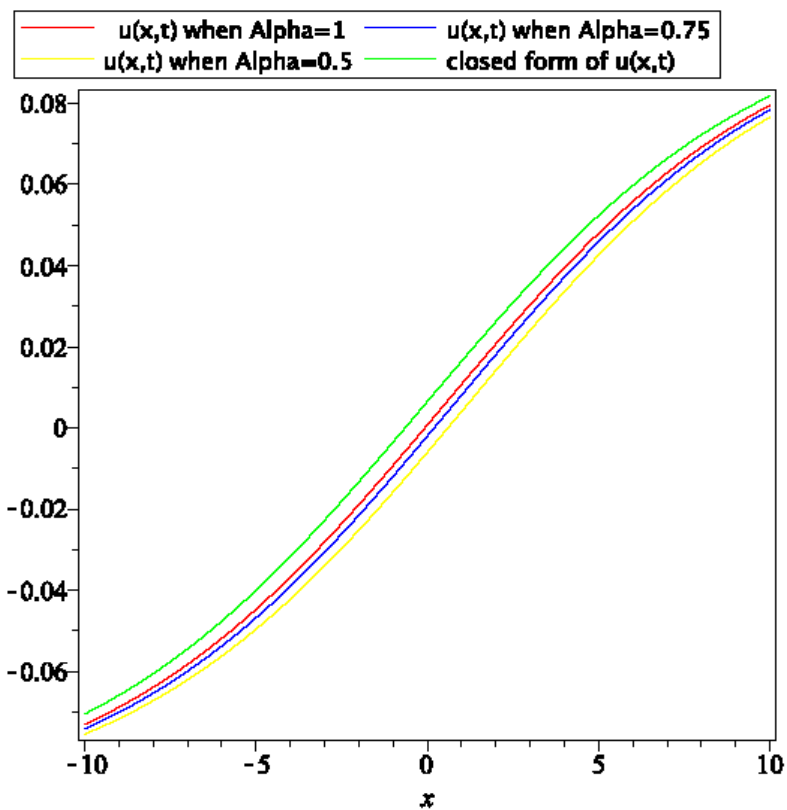


Figure 19- The action of $u(x,t)$ of problems (32)-(35).

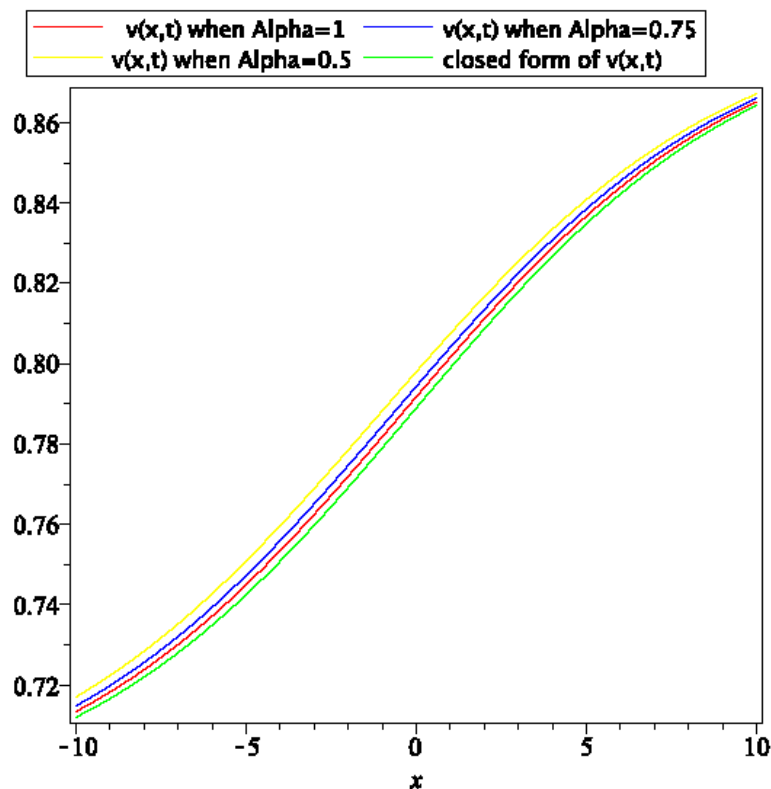


Figure 20-The action of $v(x,t)$ of problems (32)-(35).

7. Conclusions

In this study, RPSM was implemented to find the solutions of CD-REs and CMKdv. The approximate solution was given as an infinite FPS. The suggested method introduced an easy manner to find the coefficients of the solution, which converges quickly to the closed form. The numerical results demonstrate the significant feature, efficiency, and reliability of the proposed method for solving CD-REs and CMKdv.

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