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On Centralizers of 2-torsion Free Semiprime Gamma Rings

Abdulkareem T. Mutlak, Abulrahman H. Majeed*
Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq

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Abstract
In this paper, we prove that; Let M be a 2-torsion free semiprime $\Gamma - ring$ which satisfies the condition $x\alpha y\beta z = x\beta y\alpha z$ for all $x, y, z \in M$ and α, β∈ Γ. Consider that $T: M \longrightarrow M$ as an additive mapping such that $2T(x\alpha x) = T(x)\alpha x + x\alpha T(x)$ holds for all $x \in M$ and α ∈ Γ, then T is a left and right centralizer.

Keywords: $\Gamma - ring$, Prime $\Gamma - ring$, Semiprime $\Gamma - ring$, left centralizer, right centralizer, Jordan centralizer.

حول تمركزات لحلقات كاما شبه الاولية طليقة الالتواء من النمط –2

عبد الكريم طلال مطلك ، عبدالرحمن حميد مجيد*
قسم الرياضيات, كلية العلوم, جامعة بغداد, بغداد, العراق

الخلاصة
في هذا البحث سنبرهن الاتي : لتكن M حلقة كاما شبه اولية طليقة الالتواء من النمط –2 تحقق الشرط $x\alpha y\beta z = x\beta y\alpha z$ لكل x و y تنتمي ل M , α و β تنتمي الى Γ . نفرض ان $T: M \longrightarrow M$ دالة جمعية بحيث تحقق الخاصية التالية : $2T(x\alpha x) = T(x)\alpha x + x\alpha T(x)$, فان T هي تمركز أيمن وأ يسر.

1. Introduction

An extensive generalized concept of classical rings was presented by the gamma ring theory. Bernes [1], Luh [2] and Kyuno [3] studied the structure of gamma rings and obtained various generalizations of corresponding parts in the ring theory.

Let M and Γ be additive abelian groups, if there exists a mapping $(x, \alpha, y) \longrightarrow x\alpha y$ of $M \times \Gamma \times M \longrightarrow M$ which satisfies the conditions:

i. $x\alpha y \in M$.
ii. $(x + y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha + \beta)y = x\alpha y + x\beta y$ and $x\alpha(y + z) = x\alpha y + x\alpha z$.
iii. $(x\alpha y)\beta z = x\alpha(y\beta z)$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

Then, M is called a $\Gamma - ring$[1]. A $\Gamma - ring$ M is called prime if $a\Gamma M\Gamma b = 0$ implies $a = 0$ or $b = 0$ where $a, b \in M$. Also, M is called semiprime if $a\Gamma M\Gamma a = 0$ implies $a = 0$ where $a \in M$ [5]. Moreover, M is called 2-torsion free if $2x = 0$ implies $x = 0$.

An additive map $T: M \longrightarrow M$ is called a left (right) centralizer if $T(x\alpha y) = T(x)\alpha y$ $\big(T(x\alpha y) = x\alpha T(y)\big)$ holds for all $x, y \in M$ and α ∈ Γ. A centralizer is an additive mapping which is both a left and right centralizer [5].

Let M be a $\Gamma - ring$, then $[x, y]_\alpha = x\alpha y - y\alpha x$ is known as commutator of x and y with respect to α, where, $x, y \in M$ and α ∈ Γ. In addition, the basic commutator identities are shown below [5]:

i. $[x\alpha y, z]_\beta = [x, z]_\beta\, \alpha y + x[\alpha, \beta]_z y + x\alpha[y, z]_\beta$.

*Email: ahmajeed6@yahoo.com

ii. $[x, y\alpha z]_\beta = [x, y]_\beta\, \alpha z + y[\alpha, \beta]_x z + y\alpha[x, z]_\beta$.

Now, we consider the following assumption,

$x\ \alpha\ y\ \beta\ z = x\ \beta\ y\ \alpha\ z$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$ … (A)

According to this assumption, the above two identities are reduced to:

i. $[x\alpha y, z]_\beta = [x, z]_\beta\, \alpha y + x\alpha[y, z]_\beta$.

ii. $[x, y\alpha z]_\beta = [x, y]_\beta\, \alpha z + y\alpha[x, z]_\beta$.

Hoque and Paul [5] proved that every Jordan centralizer of 2-torsion free semiprime $\Gamma - ring$ is centralizer. Many researchers have proved results on 2-torsion free semiprime $\Gamma - ring$ centralizer. In addition, many researchers worked on centralizer of prime and semiprime ring [6, 7].

Throughout this paper, we use condition (A) and assume that $T: M \longrightarrow M$ is an additive mapping which satisfy the following condition:

$2T(x\alpha x) = T(x)\alpha x + x\alpha T(x)$ for all $x \in M$ and $\alpha \in \Gamma$ … (B)

2. Results

First, we need to prove some lemmas as in the following.

Lemma2.1: Let M be a $\Gamma - ring$ that has an identity element, $T: M \longrightarrow M$ is a left (right) centralizer if and only if there exists $a \in M$and $\alpha \in \Gamma$ such that $T(x) = a\alpha x$ ($T(x) = x\alpha a$) for all $x \in M$.

Proof: ⇐) by assumption we have, $T(x) = a\alpha x$ for some $a \in M$ and $\alpha \in \Gamma$ and all $x \in M$.

$T(x\beta y) = a\alpha x\beta y = T(x)\beta y$ is a left centralizer for all $x, y \in M$ and $\alpha, \beta \in \Gamma$.

By a similar way, we can obtain that if T is satisfying that $T(x) = x\alpha a$ then we get T and is a right centralizer.

⇒) If $T(x\alpha y) = T(x)\alpha y$ for all $x, y \in M$ and $\alpha \in \Gamma$, then $T(1\alpha x) = T(1)\alpha x$.

If we take $T(1) = a$ then we obtain, $T(x) = T(1\alpha x) = a\alpha x$ for all $a, x \in M$ and $\alpha \in \Gamma$.

By a similar way, we can prove whether T is a right centralizer $T(x) = T(1\alpha x) = x\alpha T(1) = x\alpha a$.

Lemma2.2: - Let M be 2-torsion free semiprime $\Gamma - ring$ satisfying condition (A) and $T: M \longrightarrow M$ satisfying condition (B), then $8T(x\alpha y\beta x) = T(x)\,\alpha(x\beta y + 3y\beta x) + (y\beta x + 3x\beta y)\,\alpha T(x) + 2x\alpha T(y)\beta x - x\alpha x\beta T(y) - T(y)\beta x\alpha x$ where, $x, y \in M$ and $\alpha, \beta \in \Gamma$.

Proof: By replacing x in equation (B) by $x + y$ we get,

$2T((x + y)\alpha(x + y)) = T(x + y)\alpha(x + y) + (x + y)\alpha T(x + y)$

$2T(x\alpha x) + 2T(x\alpha y) + 2T(y\alpha x) + 2T(y\alpha y) = T(x)\alpha x + T(x)\alpha y + T(y)\alpha x + T(y)\alpha y + x\alpha T(x) + x\alpha T(y) + y\alpha T(x) + y\alpha T(y)$. Then, for all $x, y \in M$ and $\alpha \in \Gamma$ we obtain,

$$2T(x\alpha y + y\alpha x) = T(x)\alpha y + x\alpha T(y) + T(y)\alpha x + y\alpha T(x) \quad \text{... (1)}$$

Now, replacing y in equation (1) by $2(x\beta y + y\beta x)$ and then using equation (1) implies,

$$\begin{aligned} 4T(x\alpha(x\beta y + y\beta x) + (x\beta y + y\beta x)\alpha x) &= 2T(x)\alpha(x\beta y + y\beta x) + 2x\alpha T(x\beta y + y\beta x) + 2T(x\beta y + y\beta x)\alpha x \\ &\quad + 2(x\beta y + y\beta x)\alpha T(x) \\ = 2T(x)\alpha(x\beta y + y\beta x) &+ x\alpha T(x)\beta y + x\alpha x\beta T(y) + x\alpha T(y)\beta x + x\alpha y\beta T(x) + T(x)\beta y\alpha x \\ &\quad + x\beta T(y)\alpha x + T(y)\beta x\alpha x + y\beta T(x)\alpha x + 2(x\beta y + y\beta x)\alpha T(x). \end{aligned}$$

By simplifying the above equation, we obtain,

$$4T(x\alpha(x\beta y + y\beta x) + (x\beta y + y\beta x)\,\alpha x) = T(x)\alpha(2x\beta y + 3y\beta x) + (3x\beta y + 2y\beta x)\,\alpha T(x) + x\alpha T(x)\beta y + y\beta T(x)\alpha x + 2x\alpha T(y)\beta x + x\alpha x\beta T(y) + T(y)\beta x\alpha x \quad \text{... (2)}$$

On the other hand, by using equation (1) and equation (B) we get,

$$\begin{aligned} 4T(x\alpha(x\beta y + y\beta x) + (x\beta y + y\beta x)\alpha x) &= 4T(x\alpha x\beta y) + 4T(x\alpha y\beta x) + 4T(x\beta y\alpha x) + 4T(y\beta x\alpha x) \\ &= 4T(x\alpha x\beta y) + 4T(y\beta x\alpha x) + 8T(x\alpha y\beta x) \\ &= 2T(2x\alpha x\beta y + 2y\beta x\alpha x) + 8T(x\alpha y\beta x) \\ &= 2T(x\alpha x)\beta y + 2x\alpha x\beta T(y) + 2T(y)\beta x\alpha x + 2y\beta T(x\alpha x) + 8T(x\alpha y\beta x). \end{aligned}$$

Hence, for all $x, y \in M$ and $\alpha, \beta \in \Gamma$ we have,

$$4T(x\alpha(x\beta y + y\beta x) + (x\beta y + y\beta x)\alpha x) = T(x)\alpha x\beta y + y\beta x\alpha T(x) + x\alpha T(x)\beta y + y\beta T(x)\alpha x + 2x\alpha x\beta T(y) + 2T(y)\beta x\alpha x + + 8T(x\alpha y\beta x) \quad \text{... (3)}$$

By comparing equation (2) with equation (3) we get,

$$8T(x\alpha y\beta x) = T(x)\alpha(x\beta y + 3y\beta x) + (y\beta x + 3x\beta y)\alpha T(x) + 2x\alpha T(y)\beta x - x\alpha x\beta T(y) - T(y)\beta x\alpha x \text{ for all } x, y \in M \text{ and } \alpha, \beta \in \Gamma \quad \text{... (4)}$$

Lemma2.3: - Let M be 2-torsion free semiprime $\Gamma - ring$ satisfying condition (A) and $T: M \longrightarrow M$ satisfying condition (B), then for all $x, y \in M$ and $\alpha, \beta \in \Gamma$ we have,

$T(x)\alpha(x\alpha y\beta x-2y\beta x\alpha x-2x\alpha x\beta y)+(x\alpha y\beta x-2x\alpha x\beta y-2y\beta x\alpha x)\alpha T(x)+x\beta T(x)\alpha(x\beta y+y\beta x)+(x\beta y+y\beta x)\alpha T(x)\beta x+x\beta x\alpha T(x)\beta y+y\beta T(x)\alpha x\beta x=0$ …(5)

Proof: By using equation (1) with replacing y by $8x\alpha y\beta x$ and then using equation (4) we get,

$$16T(x\alpha x\alpha y\beta x+x\alpha y\beta x\alpha x)=8T(x)\alpha x\alpha y\beta x+8x\alpha T(x\alpha y\beta x)+8T(x\alpha y\beta x)\alpha x+8x\alpha y\beta x\alpha T(x)$$
$$=8T(x)\alpha x\alpha y\beta x+x\alpha T(x)\alpha(x\beta y+3y\beta x)+(x\alpha y\beta x+3x\alpha x\beta y)\alpha T(x)$$
$$+2x\alpha x\alpha T(y)\beta x-x\alpha x\alpha x\beta T(y)-x\alpha T(y)\beta x\alpha x+T(x)\alpha(x\beta y\alpha x+y\beta x\alpha x)$$
$$+(y\beta x+3x\beta y)\alpha T(x)\alpha x+2x\alpha T(y)\beta x\alpha x-x\alpha x\beta T(y)\alpha x-T(y)\beta\alpha x\alpha x$$
$$+8x\alpha y\beta xT(x)=0.$$

Therefore, for all $x,y\in M$ and $\alpha,\beta\in\Gamma$ we can get,

$16T(x\alpha x\alpha y\beta x+x\alpha y\beta x\alpha x)=T(x)\alpha(9x\alpha y\beta x+3y\beta x\alpha x)+(9x\alpha y\beta x+3x\alpha x\beta y)\alpha T(x)+x\alpha T(x)\alpha(x\beta y+3y\beta x)+(y\beta x+3x\beta y)\alpha T(x)\alpha x+x\alpha x\alpha T(y)\beta x+x\alpha T(y)\beta x\alpha x-T(y)\beta x\alpha x\alpha x-x\alpha x\alpha x\beta T(y)$ ... (6)

We can obtain the other hand by using equation (4) and then, after collecting some terms, using equation (1), as follows:

$$16T(x\alpha x\alpha y\beta x+x\beta y\alpha x\alpha x)=16(x\alpha(x\alpha y)\beta x)+16T(x\alpha(y\alpha x)\beta x)$$
$$=2T(x)\alpha(x\beta x\alpha y+3x\alpha y\beta x)+2(x\alpha y\beta x+3x\beta x\alpha y)\alpha T(x)+4x\alpha T(x\alpha y)\beta x-2x\alpha x\beta T(x\alpha y)$$
$$-2T(x\alpha y)\beta x\alpha x+2T(x)\alpha(x\alpha y\beta x+3y\beta x\alpha x)+2(y\beta x\alpha x+3x\alpha y\beta x)\alpha T(x)$$
$$+4x\alpha T(y\alpha x)\beta x-2x\alpha x\beta T(y\alpha x)-2T(y\alpha x)\beta x\alpha x.$$
$$=T(x)\alpha(2x\alpha x\beta y+6y\beta x\alpha x+8x\alpha y\beta x)+(8x\alpha y\beta x+2y\beta x\alpha x+6x\alpha x\beta y)\alpha T(x)$$
$$+4x\alpha T(x\beta y+y\beta x)\alpha x-2x\alpha x\alpha T(x\beta y+y\beta x)-2T(x\beta y+y\beta x)\alpha x\alpha x.$$

$=$
$T(x)\alpha(2x\alpha x\beta y+6y\beta x\alpha x+8x\alpha y\beta x)+(8x\alpha y\beta x+2y\beta x\alpha x+6x\alpha x\beta y)\alpha T(x)+2x\alpha T(x)\beta y\alpha x+2x\alpha x\beta T(y)\alpha x+2x\alpha T(y)\beta x\alpha x+2x\alpha y\beta T(x)\alpha x-x\alpha x\alpha T(x)\beta y-x\alpha x\alpha x\beta T(y)-x\alpha x\beta T(y)\alpha x-x\alpha x\beta y\alpha T(x)-x\alpha T(y)\beta x\alpha x-T(y)\beta x\alpha x\alpha x-y\beta T(x)\alpha x\alpha x.$

Hence, for all $x,y\in M$ and $\alpha,\beta\in\Gamma$ we have,

$$16T(x\alpha x\alpha y\beta x+x\beta y\alpha x\alpha x)$$
$$=T(x)\alpha(2x\alpha x\beta y+5y\beta x\alpha x+8x\alpha y\beta x)+(2y\beta x\alpha x+5x\alpha x\beta y+8x\alpha y\beta x)\alpha T(x)$$
$$+2x\alpha T(x)\beta y\alpha x+2x\alpha y\beta T(x)\alpha x+x\alpha x\beta T(y)\alpha x+x\alpha T(y)\beta x\alpha x-x\alpha x\alpha T(x)\beta y$$
$$-y\beta T(x)\alpha x\alpha x-x\alpha x\alpha x\beta T(y)$$
$$-T(y)\beta x\alpha x\alpha x \quad \ldots(7)$$

By comparing equation (6) with equation (7), we obtain equation (5) which is the result.

Lemma2.4: - Let M be 2-torsion free Semiprime $\Gamma-ring$ satisfying condition (A) and $T:M\longrightarrow M$ satisfying condition (B), then $[T(x),x\alpha x]_{\alpha}=0$, for all $x\in M$ and $\alpha\in\Gamma$.

Proof: By putting $y\alpha x$ instead of y in equation (5) we obtain,

$$T(x)\alpha(x\alpha y\alpha x\beta x-2y\alpha x\beta x\alpha x-2x\alpha x\beta y\alpha x)+(x\alpha y\alpha x\beta x-2x\alpha x\beta y\alpha x-2y\alpha x\beta x\alpha x)\alpha T(x)$$
$$+x\beta T(x)\alpha(x\beta y\alpha x+y\alpha x\beta x)+(x\beta y\alpha x+y\alpha x\beta x)\alpha T(x)\beta x+x\beta x\alpha T(x)\beta y\alpha x$$
$$+y\alpha x\beta T(x)\alpha x\beta x=0 \quad ...(8)$$

Right multiplication of equation (5) by x gives for all $x,y\in M$ and $\alpha,\beta\in\Gamma$ the equation,

$$T(x)\alpha(x\alpha y\alpha x\beta x-2y\alpha x\beta x\alpha x-2x\alpha x\beta y\alpha x)+(x\alpha y\beta x-2x\alpha x\beta y-2y\beta x\alpha x)\alpha T(x)\alpha x$$
$$+x\beta T(x)\alpha(x\beta y\alpha x+y\alpha x\beta x)+(x\beta y-2y\beta x\alpha x)\alpha T(x)\beta x\alpha x+x\beta x\alpha T(x)\beta y\alpha x$$
$$+y\beta T(x)\alpha x\beta x\alpha x=0 \quad ...(9)$$

Now, by subtracting equation (9) from equation (8) we get,

$$(x\alpha y\alpha x\beta x-2x\alpha x\beta y\alpha x-2y\alpha x\beta x\alpha x)\alpha T(x)+(x\beta y\alpha x+y\alpha x\beta x)\alpha T(x)\beta x+y\alpha x\beta T(x)\alpha x\beta x$$
$$-(x\alpha y\beta x-2x\alpha x\beta y-2y\beta x\alpha x)\alpha T(x)\alpha x-(x\beta y+y\beta x)\alpha T(x)\beta x\alpha x$$
$$-(y\beta T(x)\alpha x\beta x\alpha x=0 \quad \text{for all } x,y\in M \text{ and } \alpha,\beta\in\Gamma.$$

$x\alpha y\beta x\alpha[x,T(x)]_{\alpha}+2x\alpha x\beta y\alpha[T(x),x]_{\alpha}+2y\beta x\alpha x\alpha[T(x),x]_{\alpha}+x\beta y\alpha[x,T(x)]_{\alpha}\beta x+y\beta x\alpha[x,T(x)]_{\alpha}\beta x+y\beta[x,T(x)]_{\alpha}\beta x\alpha x=0$ for all $x,y\in M$ and $\alpha,\beta\in\Gamma$.

Let $\alpha=\beta$ and by collecting the first and the fourth terms together we get, for all $x,y\in M$ and $\alpha\in\Gamma$, the following equation:

$x\alpha y\alpha[x\alpha x,T(x)]_{\alpha}+2x\alpha x\alpha y\alpha[T(x),x]_{\alpha}+2y\alpha x\alpha x\alpha[T(x),x]_{\alpha}+y\alpha x\alpha[x,T(x)]_{\alpha}\alpha x+y\alpha[x,T(x)]_{\alpha}\alpha x\alpha x=0$ ... (10)

Substituting $T(x)\alpha y$ for y in the equation (10) implies,

$x\alpha T(x)\alpha y\alpha[x\alpha x,T(x)]_{\alpha}+2x\alpha x\alpha T(x)\alpha y\alpha[T(x),x]_{\alpha}+2T(x)\alpha y\alpha x\alpha x\alpha[T(x),x]_{\alpha}+T(x)\alpha y\alpha x\alpha[x,T(x)]_{\alpha}\alpha x+T(x)\alpha y\alpha[x,T(x)]_{\alpha}\alpha x\alpha x=0$ ... (11)

Left multiplication of equation (10) by $T(x)$ leads to,
$T(x)\alpha x\alpha y\alpha[x\alpha x,T(x)]_\alpha + 2T(x)\alpha x\alpha x\alpha y\alpha[T(x),x]_\alpha + 2T(x)\alpha y\alpha x\alpha x\alpha[T(x),x]_\alpha +$
$T(x)\alpha y\alpha x\alpha[x,T(x)]_\alpha \alpha x + T(x)\alpha y\alpha[x,T(x)]_\alpha \alpha x\alpha x = 0$ ... (12)
By subtracting equation (12) from equation (11) we obtain,
$x\alpha T(x)\alpha y\alpha[x\alpha x,T(x)]_\alpha + 2x\alpha x\alpha T(x)\alpha y\alpha[T(x),x]_\alpha - T(x)\alpha x\alpha y\alpha[x\alpha x,T(x)]_\alpha -$
$2T(x)\alpha x\alpha x\alpha y\alpha[\,T(x),x]_\alpha = 0$ for all $x,y\in M$ and $\alpha\in\Gamma$.
Now, for all $x,y\in M$ and $\alpha\in\Gamma$ we have,
$[T(x),x]_\alpha \alpha y\alpha[T(x),x\alpha x]_\alpha - 2[T(x),x\alpha x]_\alpha \alpha y\alpha[T(x),x]_\alpha = 0$ ... (13)
By putting $y\alpha[T(x),x]_\alpha \alpha z$ instead of y in equation (13) we get for all $x,y,z\in M$ and $\alpha\in\Gamma$,
$[T(x),x]_\alpha \alpha y\alpha[T(x),x]_\alpha \alpha z\alpha[T(x),x\alpha x]_\alpha - 2[T(x),x\alpha x]_\alpha \alpha y\alpha[T(x),x]_\alpha \alpha z\alpha[T(x),x]_\alpha = 0$ ... (14)
Left multiplication of equation (13) by $[T(x),x]_\alpha \alpha y$ implies,
$$[T(x),x]_\alpha \alpha y\alpha[T(x),x]_\alpha \alpha z\alpha[T(x),x\alpha x]_\alpha + [T(x),x]_\alpha \alpha y\alpha[-2T(x),x\alpha x]_\alpha \alpha z\alpha[T(x),x]_\alpha = 0 \quad ...(15)$$
By subtracting equation (15) from equation (14) for all $x,y,z\in M$ and $\alpha\in\Gamma$ we have,
$$([T(x),x]_\alpha \alpha y\alpha[-2T(x),x\alpha x]_\alpha + [2T(x),x\alpha x]_\alpha \alpha y\alpha[T(x),x]_\alpha)\alpha z\alpha[T(x),x]_\alpha = 0 \quad ...(16)$$
Let, in equation (16), z be $z\alpha[-2T(x),x\alpha x]_\alpha \alpha y$, then we obtain for all $x,y,z\in M$ and $\alpha\in\Gamma$,
$([T(x),x]_\alpha \alpha y\alpha[-2T(x),x\alpha x]_\alpha +$
$[2T(x),x\alpha x]_\alpha \alpha y\alpha[T(x),x]_\alpha)\,\alpha z\alpha[-2T(x),x\alpha x]_\alpha \alpha y\alpha[T(x),x]_\alpha = 0$ ... (17)
Right multiplication of equation (16) by $y\alpha[-2T(x),x\alpha x]_\alpha$ gives,
$([T(x),x]_\alpha \alpha y\alpha[-2T(x),x\alpha x]_\alpha +$
$[2T(x),x\alpha x]_\alpha \alpha y\alpha[T(x),x]_\alpha)\,\alpha z\alpha[T(x),x]_\alpha \alpha y\alpha[-2T(x),x\alpha x]_\alpha = 0$ ... (18)
By subtracting equation (17) from equation (18) we obtain,
$([T(x),x]_\alpha \alpha y\alpha[-2T(x),x\alpha x]_\alpha +$
$[2T(x),x\alpha x]_\alpha \alpha y\alpha[T(x),x]_\alpha)\,\alpha z\alpha([T(x),x]_\alpha \alpha y\alpha[-2T(x),x\alpha x]_\alpha +$
$[2T(x),x\alpha x]_\alpha \alpha y\alpha[T(x),x]_\alpha) = 0$ for all $x,y,z\in M$ and $\alpha\in\Gamma$.
$[T(x),x]_\alpha \alpha y\alpha[2T(x),x\alpha x]_\alpha = [2T(x),x\alpha x]_\alpha \alpha y\alpha[T(x),x]_\alpha$ ... (19)
Combining equation (13) with equation (19) leads to,
$[T(x),x]_\alpha \alpha y\alpha([T(x),x\alpha x]_\alpha - 2[T(x),x\alpha x]_\alpha)$ for all $x,y\in M$ and $\alpha\in\Gamma$.
This implies that $[T(x),x]_\alpha \alpha y\alpha[T(x),x\alpha x]_\alpha = 0$ for all $x,y\in M$ and $\alpha\in\Gamma$ ...(20)
By left multiplying equation (20) by x we obtain,
$x\alpha[T(x),x]_\alpha \alpha y\alpha[T(x),x\alpha x]_\alpha = 0$ for all $x,y\in M$ and $\alpha\in\Gamma$ …..(21)
By replacing $x\alpha y$ for y in equation (21) we get,
$[T(x),x]_\alpha \alpha x\alpha y\alpha[T(x),x\alpha x]_\alpha = 0$ for all $x,y\in M$ and $\alpha\in\Gamma$ ...(22)
We combine equation (22) with equation (23) and the result is,
$([T(x),x]_\alpha \alpha x + x\alpha[T(x),x]_\alpha)\alpha y\alpha[T(x),x\alpha x]_\alpha = 0$ for all $x,y\in M$ and $\alpha\in\Gamma$.
$[T(x),x\alpha x]_\alpha \alpha y\alpha[T(x),x\alpha x]_\alpha = 0$ for all $x,y\in M$ and $\alpha\in\Gamma$ … (23)
By semiprimness, we have, $[T(x),x\alpha x]_\alpha = 0$ for all $x\in M$ and $\alpha\in\Gamma$ …(24)

Lemma 2.5: - Let M be 2-torsion free Semiprime Γ-ring satisfying condition (A) and $T: M\longrightarrow M$ satisfying condition (B), then, $[T(x),x]_\alpha \alpha x\alpha x = 0$, $x\alpha x\alpha[T(x),x]_\alpha = 0$ and $x\alpha[T(x),x]_\alpha \alpha x = 0$ for all $x\in M$ and $\alpha\in\Gamma$.

Proof: For Lemma 2.4, we have equation (24). By the substitution of $x+y$ for x in equation (24) we obtain,
$$[T(x+y),(x+y)\alpha(x+y)]_\alpha = [T(x),x\alpha x+y\alpha y+x\alpha y+y\alpha x]_\alpha + [T(y),x\alpha x+y\alpha y+x\alpha y+y\alpha x]_\alpha$$
Hence, for all $x,y\in M$ and $\alpha\in\Gamma$,
$$[T(x),y\alpha y]_\alpha + [T(y),x\alpha x]_\alpha + [T(x),x\alpha y+y\alpha x]_\alpha + [T(y),x\alpha y+y\alpha x]_\alpha = 0.$$
Putting, in the above equation, $-x$ for x implies for all $x,y\in M$ and $\alpha\in\Gamma$,
$[-T(x),y\alpha y]_\alpha + [T(y),x\alpha x]_\alpha + [-T(x),-x\alpha y-y\alpha x]_\alpha + [T(y),-x\alpha y-y\alpha x]_\alpha = 0$
By comparing the above two equations we have,
$[T(x),x\alpha y+y\alpha x]_\alpha + [T(y),x\alpha x]_\alpha = 0$ for all $x,y\in M$ and $\alpha\in\Gamma$ …..(25)
By putting $2(x\alpha y+y\alpha x)$ in equation (25) for y then according to equations (2) and (16) we obtain,

$$2[T(x), x\alpha(x\alpha y + y\alpha x) + (x\alpha y + y\alpha x)]_\alpha + [2T(x\alpha y + y\alpha x), x\alpha x]_\alpha = 0$$
$$2[T(x), x\alpha x\alpha y + 2x\alpha y\alpha x + y\alpha x\alpha x]_\alpha + [T(x)\alpha y + x\alpha T(y) + T(y)\alpha x + y\alpha T(x), x\alpha x]_\alpha = 0$$
$$2[T(x), x\alpha x\alpha y]_\alpha + 2[T(x), y\alpha x\alpha x]_\alpha + 4[T(x), x\alpha y\alpha x]_\alpha + [T(x)y\alpha, x\alpha x]_\alpha + [x\alpha T(y), x\alpha x]_\alpha + [T(y)\alpha x, x\alpha x]_\alpha + [y\alpha T(x), x\alpha x]_\alpha = 0$$
$2x\alpha x\alpha[T(x), y]_\alpha + 2[T(x), y]_\alpha\alpha x\alpha x + 4[T(x), x\alpha y\alpha x]_\alpha + T(x)\alpha[y, x\alpha x]_\alpha + x\alpha[T(y), x\alpha x]_\alpha +$
$[T(y), x\alpha x]_\alpha\alpha x + [y, x\alpha x]_\alpha\alpha T(x) = 0$ … (26)

Thus, for all $x, y \in M$ and $\alpha \in \Gamma$ we have,
$$2x\alpha x\alpha[T(x), y]_\alpha + 2[T(x), y]_\alpha\alpha x\alpha x + 4[T(x), x\alpha y\alpha x]_\alpha + T(x)\alpha[y, x\alpha x]_\alpha + [y, x\alpha x]_\alpha\alpha T(x) + x\alpha[T(y), x\alpha x]_\alpha + [T(y), x\alpha x]_\alpha\alpha x = 0$$
For $y = x$, equation (27) reduces to,
$$x\alpha x\alpha[T(x), x]_\alpha + [T(x), x]_\alpha\alpha x\alpha x + 2[T(x), x\alpha x\alpha x]_\alpha = 0$$
$$x\alpha x\alpha[T(x), x]_\alpha + [T(x), x]_\alpha\alpha x\alpha x + 2[T(x), x]_\alpha\alpha x\alpha x + 2x\alpha[T(x), x\alpha x]_\alpha = 0$$
Which gives, $x\alpha x\alpha[T(x), x]_\alpha + 3[T(x), x]_\alpha\alpha x\alpha x = 0$ for all $x \in M$ and $\alpha \in \Gamma$ …(27)

From equation (25) we get, $[T(x), x]_\alpha\alpha x + x\alpha[T(x), x]_\alpha = 0$ for all $x \in M$ and $\alpha \in \Gamma$.

By right multiplication of the above relation by x we get,

$[T(x), x]_\alpha\alpha x\alpha x + x\alpha[T(x), x]_\alpha\alpha x = 0$ for all $x \in M$ and $\alpha \in \Gamma$ ...(28)

Now, by left multiplication of the above relation by x we have,

$x\alpha[T(x), x]_\alpha\alpha x + x\alpha x\alpha[T(x), x]_\alpha = 0$ for all $x \in M$ and $\alpha \in \Gamma$ ...(29)

Comparing equation (28) with equation (29) gives,

$[T(x), x]_\alpha\alpha x\alpha x = x\alpha x\alpha[T(x), x]_\alpha$ for all $x \in M$ and $\alpha \in \Gamma$ ...(30)

From equation (27) and equation (30) we obtain, $4x\alpha x\alpha[T(x), x]_\alpha = 0$ for all $x \in M$ and $\alpha \in \Gamma$, implies, $x\alpha x\alpha[T(x), x]_\alpha = 0$ for all $x \in M$ and $\alpha \in \Gamma$ ...(31)

$[T(x), x]_\alpha\alpha x\alpha x = 0$ for all $x \in M$ and $\alpha \in \Gamma$ ...(32)

In addition, from equation (31) we have, $x\alpha[T(x), x]_\alpha\alpha x = 0$ for all $x \in M$ and $\alpha \in \Gamma$ ...(33)

Lemma 2.6: Let M be 2-torsion free semiprime Γ-ring satisfying condition (A) and let $T: M \longrightarrow M$ satisfying condition (B), then
$[T(x), x]_\alpha\alpha x = 0$ and $x\alpha[T(x), x]_\alpha = 0$ for all $x \in M$ and $\alpha \in \Gamma$.

Proof: From equation (25) we get, $[T(y), x\alpha x]_\alpha = -[T(x), x\alpha y + y\alpha x]_\alpha$.

Left multiplication of the above equation by x gives, $x\alpha[T(y), x\alpha x]_\alpha = -x\alpha[T(x), x\alpha y + y\alpha x]_\alpha$.

Similarly, right multiplication by x gives, $x\alpha[T(y), x\alpha x]_\alpha = -x\alpha[T(x), x\alpha y + y\alpha x]_\alpha$.

$[T(y), x\alpha x]_\alpha\alpha x = -[T(x), x\alpha y + y\alpha x]_\alpha\alpha x$ for all $x, y \in M$ and $\alpha \in \Gamma$.

Putting the above two equations in equation (26) gives,
$$\begin{aligned}0 &= 2x\alpha x\alpha[T(x), y]_\alpha + 2[T(x), y]_\alpha\alpha x\alpha x + 4[T(x), x\alpha y\alpha x]_\alpha + T(x)\alpha[y, x\alpha x]_\alpha + [y, x\alpha x]_\alpha\alpha T(x) \\ &\quad - x\alpha[T(x), x\alpha y + y\alpha x]_\alpha - [T(x), x\alpha y + y\alpha x]_\alpha\alpha x \\ &= 2x\alpha x\alpha[T(x), y]_\alpha + 2[T(x), y]_\alpha\alpha x\alpha x + 4[T(x), x]_\alpha\alpha y\alpha x + 4x\alpha[T(x), y]_\alpha\alpha x \\ &\quad + 4x\alpha y\alpha[T(x), x]_\alpha + T(x)\alpha[y, x\alpha x]_\alpha + [y, x\alpha x]_\alpha\alpha T(x) - x\alpha[T(x), x]_\alpha\alpha y \\ &\quad - x\alpha x\alpha[T(x), y]_\alpha - x\alpha y\alpha[T(x), x]_\alpha - x\alpha[T(x), y]_\alpha\alpha x - x\alpha[T(x), y]_\alpha\alpha x \\ &\quad - [T(x), x]_\alpha\alpha y\alpha x - y\alpha[T(x), x]_\alpha\alpha x - [T(x), y]_\alpha\alpha x\alpha x.\end{aligned}$$
Therefore, for all $x, y \in M$ and $\alpha \in \Gamma$,
$x\alpha x\alpha[T(x), y]_\alpha + [T(x), y]_\alpha\alpha x\alpha x + 3[T(x), x]_\alpha\alpha y\alpha x + 2x\alpha[T(x), y]_\alpha\alpha x + 3x\alpha y\alpha[T(x), x]_\alpha +$
$T(x)\alpha[y, x\alpha x]_\alpha + [y, x\alpha x]_\alpha\alpha T(x) - x\alpha[T(x), x]_\alpha\alpha y - y\alpha[T(x), x]_\alpha\alpha x = 0$ ...(34)

By the substitution of $y\alpha x$ for y in equation (34) we get,
$x\alpha x\alpha[T(x), y\alpha x]_\alpha + [T(x), y\alpha x]_\alpha\alpha x\alpha x + 3[T(x), x]_\alpha\alpha y\alpha x\alpha x + 2x\alpha[T(x), y\alpha x]_\alpha\alpha x +$
$3x\alpha y\alpha x\alpha[T(x), x]_\alpha + T(x)\alpha[y\alpha x, x\alpha x]_\alpha + [y\alpha x, x\alpha x]_\alpha\alpha T(x) - x\alpha[T(x), x]_\alpha\alpha y\alpha x -$
$y\alpha x\alpha[T(x), x]_\alpha\alpha x = 0$
$x\alpha x\alpha[T(x), y]_\alpha\alpha x + x\alpha x\alpha y\alpha[T(x), x]_\alpha + [T(x), y]_\alpha\alpha x\alpha x\alpha x + y\alpha[T(x), x]_\alpha\alpha x\alpha x +$
$3[T(x), x]_\alpha\alpha y\alpha x\alpha x + 2x\alpha[T(x), y]_\alpha\alpha x\alpha x + 2x\alpha y\alpha[T(x), x]_\alpha\alpha x + 3x\alpha y\alpha x\alpha[T(x), x]_\alpha +$
$T(x)\alpha[y, x\alpha x]_\alpha\alpha x + [y, x\alpha x]_\alpha\alpha x\alpha T(x) - x\alpha[T(x), x]_\alpha\alpha y\alpha x - y\alpha x\alpha[T(x), x]_\alpha\alpha x = 0$

This reduces equation (32) and equation (33) to,
$x\alpha x\alpha[T(x), y]_\alpha\alpha x + x\alpha x\alpha y\alpha[T(x), x]_\alpha + [T(x), y]_\alpha\alpha x\alpha x\alpha x + 3x\alpha y\alpha x\alpha[T(x), x]_\alpha +$
$2x\alpha[T(x), y]_\alpha\alpha x\alpha x + 2x\alpha y\alpha[T(x), x]_\alpha\alpha x + T(x)\alpha[y, x\alpha x]_\alpha\alpha x + [y, x\alpha x]_\alpha\alpha x\alpha T(x) -$
$x\alpha[T(x), x]_\alpha\alpha y\alpha x = 0$ for all $x, y \in M$ and $\alpha \in \Gamma$ ...(35)

Right multiplication of equation (34) by x gives, for all $x, y \in M$ and $\alpha \in \Gamma$

$x\alpha x\alpha[T(x),y]_\alpha \alpha x+[T(x),y]_\alpha \alpha x\alpha x\alpha x+3[T(x),x]_\alpha \alpha y\alpha x\alpha x+3x\alpha y\alpha[T(x),x]_\alpha \alpha x+$
$2x\alpha[T(x),y]_\alpha \alpha x\alpha x+T(x)\alpha[y,x\alpha x]_\alpha \alpha x+[y,x\alpha x]_\alpha \alpha T(x)\alpha x-x\alpha[T(x),x]_\alpha \alpha y\alpha x=0$ ... (36)
Subtracting equation (36) from equation (35) implies,

$$x\alpha x\alpha y\alpha[T(x),x]_\alpha+3x\alpha y\alpha x\alpha[T(x),x]_\alpha-3x\alpha y[T(x),x]_\alpha \alpha x+2x\alpha y\alpha[T(x),x]_\alpha \alpha x+[y,x\alpha x]_\alpha \alpha x\alpha T(x)-[y,x\alpha x]_\alpha \alpha T(x)\alpha x$$
$$=x\alpha x\alpha y\alpha[T(x),x]_\alpha+3x\alpha y[[x,T(x),x]_\alpha]_\alpha+2x\alpha y[T(x),x]_\alpha \alpha x+[y,x\alpha x]_\alpha \alpha[x,T(x)]_\alpha=0$$

which reduces equation (31) to,
$2x\alpha x\alpha y\alpha[T(x),x]_\alpha+3x\alpha y\alpha x\alpha[T(x),x]_\alpha-x\alpha y\alpha[T(x),x]_\alpha=0$ for all $x,y\in M$ and $\alpha\in\Gamma$.
By replacing $-[T(x),x]_\alpha \alpha x$ by $x\alpha[T(x),x]_\alpha$ in the above equation, and by 2-torsion free Γ-ring, we get,

$x\alpha x\alpha y\alpha[T(x),x]_\alpha+2x\alpha y\alpha x\alpha[T(x),x]_\alpha=0$ for all $x,y\in M$ and $\alpha\in\Gamma$ ...(37)

Recall equation (5) and Lemma 2.5, so, for all $x,y\in M$ and $\alpha\in\Gamma$ we have,
$x\alpha y\alpha[x\alpha x,T(x)]_\alpha+2x\alpha x\alpha y\alpha[T(x),x]_\alpha+2y\alpha x\alpha x\alpha[T(x),x]_\alpha+y\alpha x\alpha[x,T(x)]_\alpha \alpha x+$
$y\alpha[x,T(x)]_\alpha \alpha x\alpha x=0$
Using equation (24) leads to, $x\alpha x\alpha y\alpha[T(x),x]_\alpha=0$ for all $x,y\in M$ and $\alpha\in\Gamma$.
Which gives, together with equation (37), $x\alpha y\alpha x\alpha[T(x),x]_\alpha=0$ ... (38)
Left multiplication of equation (38) by $T(x)$ gives,

$T(x)\alpha x\alpha y\alpha x\alpha[T(x),x]_\alpha=0$ for all $x,y\in M$ and $\alpha\in\Gamma$ ...(39)

Replacing, in equation (38), $T(x)\alpha y$ by y we obtain,

$x\alpha T(x)\alpha y\alpha x\alpha[T(x),x]_\alpha=0$ for all $x,y\in M$ and $\alpha\in\Gamma$ ...(40)

Subtracting (40) from (39) implies, $[T(x),x]_\alpha\ \alpha y\alpha x\alpha[T(x),x]_\alpha=0$ for all $x,y\in M$ and $\alpha\in\Gamma$.
Thus, $x\alpha[T(x),x]_\alpha=0$ for all $x,y\in M$ and $\alpha\in\Gamma$ ...(41)
In addition, $[T(x),x]_\alpha \alpha x=0$ for all $x,y\in M$ and $\alpha\in\Gamma$ ...(42)

Theorem 2.7: Let M be a 2-torsion free semiprime Γ-ring satisfying condition (A) and $T: M\longrightarrow M$ satisfying condition (B), then T is a left and right centralizer.

Proof: We take equation(25), then, for all $x,y\in M$ and $\alpha\in\Gamma$ we get,
$[T(x),x]_\alpha \alpha y+x\alpha[T(x),y]_\alpha+[T(x),y]_\alpha \alpha x+y\alpha[T(x),x]_\alpha+x\alpha[T(y),x]_\alpha+[T(y),x]_\alpha \alpha x=0.$
From equation (37) with the above equation we obtain,
$x\alpha[T(x),y]_\alpha+y\alpha[T(x),x]_\alpha+x\alpha[T(y),x]_\alpha=0$ for all $x,y\in M$ and $\alpha\in\Gamma$.
Left multiplication of the above equation by $[T(x),x]_\alpha$, and by equation (38), implies,
$[T(x),x]_\alpha \alpha y\alpha[T(x),x]_\alpha=0$ for all $x,y\in M$ and $\alpha\in\Gamma$.
From the semiprime we have, $[T(x),x]_\alpha=0$ for all $x\in M$ and $\alpha\in\Gamma$ ...(43)
Combining equation (41) with equation (1) gives $T(x\alpha x)=T(x)\alpha x$ for all $x\in M$ and $\alpha\in\Gamma$.
Also, $T(x\alpha x)=x\alpha T(x)$ for all $x\in M$ and $\alpha\in\Gamma$.
This implies that T is a left and right Jordon centralizer, and by a previous work [8; $Theorem$ 3.1], the result is that T is both a left and right centralizer.

References

1. Barnes, W. E. **1966**. On the Γ-ring of Nobusawa, *Pacific J. Math*., **18**: 411-422.
2. Luh, J. **1970**. A Note on Commuting Automorphism of Rings, *Amer. Math. Monthly*, **16**: 65-75.
3. Kyuno, S. **1978**. On Prime Gamma Ring, *Pacific J. Math*., **75**: 185-190.
4. Nobusawa, N. **1964**. On the Generalization of the Ring Theory, *Osaka J. Math*, **1**: 81-89.
5. Hoque, Md. F. and Paul, A. C. **2011**. On Centralizer of Semiprime Gamma Ring, *International Mathematical Forum*, **13**: 627-638.
6. Shaheen, R. C. **2007**. On Centralizer of Some Gamma Ring, *Journal Al-Qadisiya for Pure Science*, **12**(2).
7. Hoque, Md. F. and Paul, A. C. **2013**. Centralizer on Semiprime Gamma Ring, *Italian J. of Pure and Applied Mathematics*, **30**: 289-302.
8. Hoque, Md. F. and Paul, A. C. **2011**. On Centralizer of Semiprime Gamma Ring, *International Mathematical Forum*, **6**(13): 627-638.