DOI: 10.24996/ijs.2021.62.7.22

ISSN: 0067-2904

On the Direct Product of Intuitionistic Fuzzy Topological d-algebra

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Received: 27/7/2020 Accepted: 8/10/2020

ABSTRACT
We applied the direct product concept on the notation of intuitionistic fuzzy semi d-ideals of d-algebra with the investigation of some theorems. Also, we studied the notation of direct product of intuitionistic fuzzy topological d-algebra, with the notation of relatively intuitionistic continuous mapping, on the direct product of intuitionistic fuzzy topological d-algebra.

Keywords: *direct product, topological d-algebra, semi d-ideal, intuitionistic set, d-algebra.*

حول الضرب المباشر على التوبولوجي الحدسي الضبابي على جبر– *d*

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مديرية تربية كربلاء، وزارة التربية، العراق

الخلاصة
طبقنا في هذه الورقة مفهوم الضرب المباشر للمجموعات الحدسية الضبابية على مفهوم شبه مثالي– d الضبابي الحدسي في جبر– d مع دراسة بعض النظريات، وكذلك درسنا مفهوم الضرب المباشر الحدسي الضبابي على التوبولوجي الحدسي الضبابي في جبر– d ، وكذلك درسنا مفهوم الدالة الضبابية المستمرة نسبيا على التوبولوجي الحدسي الضبابي في جبر– d

1. Introduction

A d-algebra is the classes of abstract algebra introduced by Negger and Kim [1] as a useful generalization of BCK-algebra. While the idea of fuzzy set, introduced by Zadeh [2] and Atanassov [3] generalized it to the concept of intuitionistic fuzzy set. Jun *et al.* [4] applied this notion on d-algebra. In another line, Abdullah and Hassan [5] studied the concept of semi d -ideal on d-algebra. After that, Hasan [6] introduced the concept of intuitionistic fuzzy semi d-ideals. Here, we applied the direct product concept on the notation of intuitionistic fuzzy semi d-ideals of d-algebra, with several interesting results. We also studied the notation of the direct product of intuitionistic fuzzy topological d-algebra.

2. Preliminaries

We will offer here some basic concepts which we need for this study.

Definition (2.1): [1] A d-algebra is a non-empty set H with a constant 0 and a binary operation $*$ with the conditions below:

i. $v * v = 0$
ii. $0 * v = 0$
iii. $v * u = 0$ and $u * v = 0$, which implies that $v = u$,

such that $v, u \in H$. We will refer to $v * u$ by vu and $v \leq u$ iff $vu = 0$.

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Every H or G will denote for a d-algebra in this paper.
Definition (2.2) : [5] We define the semi d-ideal of H as a subset $V \neq \emptyset$ of H with :
I) $v, u \in V$ implies $vu \in V$,
II) $vu \in V$ and $u \in V$ implies $v \in V$, $\forall\, v, u \in H$
Definition (2.3): [2] A fuzzy set ω in any set with $H \neq \emptyset$ is a function $\omega: H \to [0,1]$. Also, for all t∈[0,1], the set ω_t={$v \in H, \omega(v) \geq t$} is a ***level subset of*** ω.
Definition (2.4): [7] We define a fuzzy set ω as fuzzy d-subalgebra with the following condition: for any $v, u \in H$, $\omega(vu) \geq \min\{\omega(v), \omega(v)\}$.
Definition (2.5): [6] We call the fuzzy set ω as a fuzzy semi-d-ideal if these conditions hold :
(FI_1) $\omega(vu) \geq \min\{\omega(v), \omega(u)\}$ and (FI_2) $\omega(v) \geq \min\{\omega(vu), \omega(u)\}$, for all $v, u \in H$.
Definition (2.6) [3] : An object S in H is called intuitionistic fuzzy set, with the form $S = \{< x, \alpha_S(v), \beta_S(v) >: v \in H\}$, such that $\alpha_S: H \to [0,1]$, $\beta_S: H \to [0,1]$ is the membership degree $(\alpha_S(v))$ and non-membership degree $(\beta_S(v))$ $\forall\, v \in H$ to the set S, and $0 \leq \alpha_S(v) + \beta_S(v) \leq 1$, $\forall\, v \in H$.

We will use $S = \{< \alpha_S, \beta_S >\}$ instead of $S = \{< v, \alpha_S(v), \beta_S(v) >: v \in H\}$ and call it IFS for short.
Definition (2.7)[8]: Let $f: H \to G$ be a mapping. If $S = \{< u, \alpha_S(u), \beta_S(u) >: u \in G\}$ is an IFS in , then $f^{-1}(S)$ is the IFS in H defined by :

$$f^{-1}(S) = \{< v, f^{-1}(\alpha_S(v)), f^{-1}(\beta_S(v)) >: v \in H\}$$

Also, if $D = \{< v, \alpha_D(v), \beta_D(v) >: v \in H\}$ is an IFS in H, then $f(D)$ is denoted by
$f(D) = \{< u, f_{sup}(\alpha_D(u)), f_{inf}(\beta_D(u)) >: u \in G\}$, where

$$f_{sup}(\alpha_D(u)) = \begin{cases} sup_{v \in f^{-1}(u)} \alpha_D(v) & if\ f^{-1}(u) \neq \emptyset \\ 0 & otherwais \end{cases}$$, and

$$f_{inf}(\beta_D(u)) = \begin{cases} inf_{v \in f^{-1}(u)} \beta_D(v) & if\ f^{-1}(u) \neq \emptyset \\ 0 & otherwais \end{cases}$$, for each $u \in G$.

Definition (2.8) [9] : If D is an IFS in H, then
(i) $\Box D = \{< v, \alpha_D(v): v \in H >\} = \{< v, \alpha_D(v), 1 - \alpha_D(v) : v \in H >\} = \{< v, \alpha_D(v), \overline{\alpha_D}(v) >\}$
(ii) $\Diamond D = \{< v, 1 - \beta_D(v) >: v \in H\} = \{< v, 1 - \beta_D(v), \beta_D(v): v \in H\} = \{< v, \overline{\beta_D}(v), \beta_D(v) >\}$
Definition (2.9) [3] : Let $C =< \alpha_C, \beta_C >$ and $S =< \alpha_S, \beta_S >$ are IFS of H, then the cartesian product
$C \times S =< \alpha_C \times \alpha_S, \beta_C \times \beta_S >$ of $H \times H$ is define by the following :
$(\alpha_C \times \alpha_S)(a, b) = \min\{\alpha_C(a), \alpha_S(b)\}$ and $(\beta_C \times \beta_S)(a, b) = \max\{\beta_C(a), \beta_S(b)\}$,
where $(\alpha_C \times \alpha_S)(a, b): H \times H \to [0,1]$ and $(\beta_C \times \beta_S)(a, b): H \times H \to [0,1]$.
Definition (2.10) [3]: Let $C =< \alpha_C, \beta_C >$ and $S =< \alpha_S, \beta_S >$ are IFS of H , for any r, t ∈ [0,1]. The set $U(\alpha_C \times \alpha_S, t) = \{(v, u) \in H \times H, (\alpha_C \times \alpha_S)(v, u) \geq t\}$ is called the upper level of $(\alpha_C \times \alpha_S)(v, u)$ and the set $L(\beta_C \times \beta_S, r) = \{(v, u) \in H \times H, (\beta_C \times \beta_S)(v, u) \geq r\}$ is the lower level of $(\beta_C \times \beta_S)(v, u)$.
Definition (2.11) [4]: An IFS $D =< \alpha_D, \beta_D >$ in H is called intuitionistic fuzzy d-algebra with the conditions $\alpha_D(vu) \geq \min\{\alpha_D(v), \alpha_D(u)\}$ and $\beta_D(vu) \leq \max\{\beta_D(v), \beta_D(u)\}$, for all $v, u \in H$.
Definition(2.12) [10] : An intuitionistic fuzzy semi d-ideal of H , shortly $IFSd - ideal$, is an IFS , where
$D =< \alpha_D, \beta_D >$ in H satisfies the following inequalities :
$(IFSd_1)$ $\alpha_D(v) \geq \min\{\alpha_D(vu), \alpha_D(u)\}$ and $(IFSd_2)$ $\beta_D(v) \leq \max\{\beta_D(vu), \beta_D(u)\}$
$(IFSd_3)$ $\alpha_D(vu) \geq \min\{\alpha_D(v), \alpha_D(u)\}$ and $(IFSd_4)$ $\beta_D(vu) \leq \max\{\beta_D(v), \beta_D(u)\}$, for all $v, u \in H$.
Proposition(2.13) [4] : Every IFS d-algebra ($IFSd - ideal$) $D =< \alpha_D, \beta_D >$ of H satisfies the inequalities $\alpha_D(0) \geq \alpha_D(v)$ and $\beta_D(0) \leq \beta_D(v)$,$\forall\, v \in H$.

3. Direct product of IFS d-ideal

We apply here the notation of direct product for intuitionistic set on intuitionistic fuzzy d-algebra and intuitionistic semi d-ideal.

Proposition (3.1) *: Let* $C =< \alpha_C, \beta_C >$ and $S =< \alpha_S, \beta_S >$ are $IFSd - ideal$ of H, then $C \times S =< \alpha_C \times \alpha_S, \beta_C \times \beta_S >$ is $IFSd - ideal$ of $H \times H$.
proof: We know that for any (a_1, b_1) , $(a_2, b_2) \in H \times H$, we have

$(\alpha_C \times \alpha_S)(a_1, b_1) = \min\{\alpha_C(a_1), \alpha_S(b_1)\} \geq \min\{\min\{\alpha_C(a_1a_2), \alpha_C(a_2)\}, \{\min\{\alpha_S(b_1b_2), \alpha_S(b_2)\}\}$

$= \min\{\min\{\alpha_C(a_1a_2), \alpha_S(b_1b_2)\}, \{\min\{\alpha_C(a_2), \alpha_S(b_2)\}\}$
$= \min\{(\alpha_C \times \alpha_S)(a_1a_2, b_1b_2), (\alpha_C \times \alpha_S)\big((a_2, b_2)\big\}$
$= \min\{(\alpha_C \times \alpha_S)\big((a_1, b_1), (a_2, b_2)\big), (\alpha_C \times \alpha_S)\big((a_2, b_2)\big\}$
and
$(\beta_C \times \beta_S)(a_1, b_1) = \max\{\beta_C(a_1), \beta_S(b_1)\}$
$\leq \max\{\max\{\beta_C(a_1a_2), \beta_C(a_2)\}, \{\max\{\beta_S(b_1b_2), \beta_S(b_2)\}\}$

$= \max\{\max\{\beta_C(a_1a_2), \beta_S(b_1b_2)\}, \{\max\{\beta_C(a_2), \beta_S(b_2)\}\}$
$= \max\{(\beta_C \times \beta_S)(a_1a_2, b_1b_2), (\beta_C \times \beta_S)\big((a_2, b_2)\big\}$
$= \max\{(\beta_C \times \beta_S)((a_1, b_1), (a_2, b_2)), (\beta_C \times \beta_S)((a_2, b_2)\}$
Also, we have
$(\alpha_C \times \alpha_S)\big((a_1, b_1), (a_2, b_2)\big) = \min\{\alpha_C(a_1, a_2), \alpha_C(b_1, b_2)\}$
$\geq \min\{\min\{\alpha_C(a_1), \alpha_C(a_2)\}, \{\min\{\alpha_S(b_1), \alpha_S(b_2)\}\}$
$= \min\{\min\{\alpha_C(a_1), \alpha_S(b_1)\}, \{\min\{\alpha_C(a_2), \alpha_S(b_2)\}\}$
$= \min\{(\alpha_C \times \alpha_S)(a_1, b_1), (\alpha_C \times \alpha_S)\big((a_2, b_2)\big\}$
and, $(\beta_C \times \beta_S)\big((a_1, b_1), (a_2, b_2)\big) = \max\{\beta_C(a_1, a_2), \beta_C(b_1, b_2)\}$
$\leq \max\{\max\{\beta_C(a_1), \beta_C(a_2)\}, \{\max\{\beta_S(b_1), \beta_S(b_2)\}\}$
$= \max\{\max\{\beta_C(a_1), \beta_S(b_1)\}, \{\max\{\beta_C(a_2), \beta_S(b_2)\}\}$
$= \max\{(\beta_C \times \beta_S)(a_1, b_1), (\beta_C \times \beta_S)((a_2, b_2)\}$

Proposition (3.2) *:* Let $C \times S =< \alpha_C \times \alpha_S, \beta_C \times \beta_S >$ be an $IFSd-ideal$ of $H \times H$, then $(\alpha_C \times \alpha_S)(0,0) \geq (\alpha_C \times \alpha_S)(a, b)$ and $(\beta_C \times \beta_S)(0,0) \leq (\beta_C \times \beta_S)(a, b)$.

Proof : we know that $(\alpha_C \times \alpha_S)(0,0) = \min\{\alpha_C(0), \alpha_S(0)\} \geq \min\{\alpha_C(a), \alpha_S(b)\} = (\alpha_C \times \alpha_S)(a, b)$ and $(\beta_C \times \beta_S)(0,0) = \max\{\beta_C(0), \beta_S(0)\} \leq \max\{\beta_C(a), \beta_S(b)\} = (\beta_C \times \beta_S)(a, b)$.

Proposition (3.3) *:* Let $C \times S =< \alpha_C \times \alpha_S, \beta_C \times \beta_S >$ be an $IFSd-ideal$ of $H \times H$. If $(a, b) \leq (x, y)$, then $(\alpha_C \times \alpha_S)(a, b) \geq (\alpha_C \times \alpha_S)(x, y)$ $(\beta_C \times \beta_S)(a, b) \leq (\beta_C \times \beta_S)(x, y)$.

Proof : Let $(a, b), (x, y) \in H \times H$ such that $(a, b) \leq (x, y)$. This implies that $(a, b)(x, y) = (0,0)$.
Now, $(\alpha_C \times \alpha_S)\big((a, b)\big) \geq \min\{(\alpha_C \times \alpha_S)(a, b)(x, y), (\alpha_C \times \alpha_S)\big((a_2, b_2)\big\}$
$\geq \min\{(\alpha_C \times \alpha_S)(0,0), (\alpha_C \times \alpha_S)(x, y)\}$
$= (\alpha_C \times \alpha_S)(x, y)$
and $(\beta_C \times \beta_S)\big((a, b)\big) \leq \max\{(\beta_C \times \beta_S)(a, b)(x, y), (\beta_C \times \beta_S)\big((a_2, b_2)\big\}$
$\leq \max\{(\beta_C \times \beta_S)(0,0), (\beta_C \times \beta_S)(x, y)\}$
$= (\beta_C \times \beta_S)(x, y)$

Lemma (3.4) : Let $C \times S =< \alpha_C \times \alpha_S, \beta_C \times \beta_S >$ be an $IFS\ d-ideal$ of $H \times H$. If $(a, b)(c, d) \leq (e, f)$ holds in $H \times H$, then $(\alpha_C \times \alpha_S)(a, b) \geq \min\{(\alpha_C \times \alpha_S)(c, d), (\alpha_C \times \alpha_S)(e, f)\}$ and
$(\beta_C \times \beta_S)(a, b) \leq \max\{(\beta_C \times \beta_S)(c, d), (\beta_C \times \beta_S)(e, f)\}$.

Proof : Let $(a, b), (c, d), (e, f) \in H \times H$ with $(a, b)(c, d) \leq (e, f)$. Then, $\big((a, b)(c, d)\big)(e, f) = (0,0)$
$(\alpha_C \times \alpha_S)(a, b) \geq \min\{(\alpha_C \times \alpha_S)(a, b)(c, d), (\alpha_C \times \alpha_S)(c, d)\}$
$\geq \min\Big\{\min\Big\{(\alpha_C \times \alpha_S)\Big(\big((a, b)(c, d)\big)(e, f)\Big), (\alpha_C \times \alpha_S)(e, f)\Big\}, (\alpha_C \times \alpha_S)(c, d)\Big\}$
$\geq \min\{\min\{(\alpha_C \times \alpha_S)(0,0), (\alpha_C \times \alpha_S)(e, f)\}, (\alpha_C \times \alpha_S)(c, d)\}$
$\geq \min\{(\alpha_C \times \alpha_S)(e, f), (\alpha_C \times \alpha_S)(c, d)\}$.
$(\beta_C \times \beta_S)(a, b) \leq \max\{(\beta_C \times \beta_S)(a, b)(c, d), (\beta_C \times \beta_S)(c, d)\}$
$\leq \max\{\max\{\beta_C \times \beta_S\Big(\big((a, b)(c, d)\big)(e, f)\Big), \beta_C \times \beta_S(e, f)\}, \beta_C \times \beta_S(c, d)\}$
$\leq \max\{\max\{\beta_C \times \beta_S(0,0), \beta_C \times \beta_S(e, f)\}, \beta_C \times \beta_S(c, d)\}$
$= \max\{\beta_C \times \beta_S(e, f), \beta_C \times \beta_S(c, d)\}$. The proof is completed.

Theorem (3.5) : Let $C\times S=<\alpha_C\times\alpha_S,\beta_C\times\beta_S>$ be an $IFSd-ideal$ of $H\times H$, then for any $(a,b),(v_1,u),(v_2,u_2)\ \dots,(v_n,u_n)\in H\times H$, such that $\left(\dots\left(\left((a,b)(v_1,u_1)\right)(v_2,u_2)\right)\dots\right)(v_n,u_n)=(0,0)$, which implies that $(\alpha_C\times\alpha_S)(a,b)\geq\min\{(\alpha_C\times\alpha_S)(v_1,u_1),(\alpha_C\times\alpha_S)(v_2,u_2),\dots,(\alpha_C\times\alpha_S)(v_n,u_n)\}$, and $(\beta_C\times\beta_S)(a,b)\leq\max\{(\beta_C\times\beta_S)(v_1,u_1),(\beta_C\times\beta_S)(v_2,u_2),\dots,(\beta_C\times\beta_S)(v_n,u_n)\}$.

Proof**:** We can obtain this directly from lemma 3.4 and theorem 3.5.

Lemma (3.6) : Let $C\times S=<\alpha_C\times\alpha_S,\beta_C\times\beta_S>$ be an $IFSd-ideal$ of $H\times H$, then $\square\,(C\times S)=\{<\alpha_C\times\alpha_S,\overline{\alpha_C}\times\overline{\alpha_S}>\}$ is an $IFSd-ideal$ of $H\times H$.

Proof: We know that $(\alpha_C\times\alpha_S)(a,b)=\min\{\alpha_C(a),\alpha_S(b)\}$, therefore $1-(\overline{\alpha_C}\times\overline{\alpha_S})(a,b)=\min\{1-\overline{\alpha_C}(a),1-\overline{\alpha_S}(b)\}$. Thus, $(\overline{\alpha_C}\times\overline{\alpha_S})(a,b)=1-\min\{\overline{\alpha_C}(a),\overline{\alpha_S}(b)\}$, moreover we get $(\overline{\alpha_C}\times\overline{\alpha_S})(a,b)=\max\{\overline{\alpha_C}(a),\overline{\alpha_S}(b)\}$. Hence, $\square\,(C\times S)=\{<\alpha_C\times\alpha_S,\overline{\alpha_C}\times\overline{\alpha_S}>\}$ is an $IFSd-ideal$ of $H\times H$.

Lemma (3.7) : Let $C\times S=<\alpha_C\times\alpha_S,\beta_C\times\beta_S>$ be an $IFSd-ideal$ of $H\times H$, then $\lozenge\,(C\times S)=\left\{<\overline{\beta_C}\times\overline{\beta_S},\beta_C\times\beta_S>\right\}$ is an $IFSd-ideal$ of $H\times H$.

Proof: We know that $(\beta_C\times\beta_S)(a,b)=\max\{\beta_C(a),\beta_S(b)\}$, therefore $1-\left(\overline{\beta_C}\times\overline{\beta_S}\right)(a,b)=\max\left\{1-\overline{\beta_C}(a),1-\overline{\beta_S}(b)\right\}$. Thus, $\left(\overline{\beta_C}\times\overline{\beta_S}\right)(a,b)=1-\max\left\{\overline{\beta_C}(a),\overline{\beta_S}(b)\right\}$. Moreover, we get $\left(\overline{\beta_C}\times\overline{\beta_S}\right)(a,b)=\min\left\{\overline{\overline{\beta_C}}(a),\overline{\beta_S}(b)\right\}$. Hence, $\lozenge\,(C\times S)=\left\{<\overline{\beta_C}\times\overline{\beta_S},\beta_C\times\beta_S>\right\}$ is an $IFSd-ideal$ of $H\times H$.

From these two lemmas, it is not difficult to verify that the following theorem is valid.

Theorem (3.8) : If $C=<\alpha_C,\beta_C>$ and $S=<\alpha_S,\beta_S>$ is an $IFSd-ideal$ of H , then $\square\,(C\times S)$ and $\lozenge\,(C\times S)$ are $IFSd-ideal$ of $H\times H$.

Theorem (3.9) : Let $C=<\alpha_C,\beta_C>$ and $S=<\alpha_S,\beta_S>$ are IFS of H, then $C\times S$ is $IFSd-ideal$ of $H\times H$ if and only if $\forall\,\mathrm{r,t}\in[0,1]$, $\mathrm{U}(\alpha_C\times\alpha_S,\mathrm{t})$, and $\mathrm{L}(\beta_C\times\beta_S,\mathrm{r})$ are empty or semi d-ideal of $H\times H$.

Proof : For $C\times S=<\alpha_C\times\alpha_S,\beta_C\times\beta_S>$ is an $IFSd-ideal$ of $H\times H$ and $\mathrm{U}(\alpha_C\times\alpha_S,\mathrm{t})\neq\emptyset$, $\mathrm{L}(\beta_C\times\beta_S,\mathrm{r})\neq\emptyset$ for any $r,t\in[0,1]$. Let $(v_1,u_1)\,,(v_2,u_2)\in H\times H$ such that $(v_1,u_1)(v_2,u_2)\in\mathrm{U}(\alpha_C\times\alpha_S,\mathrm{t})$ and $(v_2,u_2)\in\mathrm{U}(\alpha_C\times\alpha_S,\mathrm{t})$, then $(\alpha_C\times\alpha_S)\left((v_1,u_1)(v_2,u_2)\right)\geq\mathrm{t}$ and $(\alpha_C\times\alpha_S)\left((v_2,u_2)\right)\geq\mathrm{t}$, which implies that
$(\alpha_C\times\alpha_S)\left((v_1,u_1)\right)\geq\min\{(\alpha_C\times\alpha_S)\left((v_1,u_1)(v_2,u_2)\right),(\alpha_C\times\alpha_S)\left((v_2,u_2)\right)\}\geq t$,
so that $(v_1,u_1)\in\mathrm{U}(\alpha_C\times\alpha_S,\mathrm{t})$. Also, let $(v_1,u_1)\,,(v_2,u_2)\in\mathrm{U}(\alpha_C\times\alpha_S,\mathrm{t})$.
Then $(\alpha_C\times\alpha_S)\left((v_1,u_1)\right)\geq\mathrm{t}$ and $(\alpha_C\times\alpha_S)\left((v_2,u_2)\right)\geq\mathrm{t}$,
But $(\alpha_C\times\alpha_S)\left((v_1,u_1)(v_2,u_2)\right)\geq\min\{(\alpha_C\times\alpha_S)\left((v_1,u_1)\right),(\alpha_C\times\alpha_S)\left((v_2,u_2)\right)\}\geq\mathrm{t}$,
so $(v_1,u_1)(v_2,u_2)\in\mathrm{U}(\alpha_C\times\alpha_S,\mathrm{t})$. Therefore, $\mathrm{U}(\alpha_C\times\alpha_S,\mathrm{t})$ is semi d-ideal in $H\times H$.
Let $(v_1,u_1)\,,(v_2,u_2)\in H\times H$ such that $(v_1,u_1)(v_2,u_2)\in\mathrm{L}(\beta_C\times\beta_S,\mathrm{r})$ and $(v_2,u_2)\in\mathrm{L}(\beta_C\times\beta_S,\mathrm{r})$, then $(\beta_C\times\beta_S)\left((v_1,u_1)(v_2,u_2)\right)\leq\mathrm{r}$ and $(\beta_C\times\beta_S)\left((v_2,u_2)\right)\leq\mathrm{r}$,
Then $(\beta_C\times\beta_S)\left((v_2,u_2)\right)\leq\max\{(\beta_C\times\beta_S)\left((v_1,u_1)(v_2,u_2)\right),(\beta_C\times\beta_S)\left((v_2,u_2)\right)\}\leq r$,
so that $(v_1,u_1)\in\mathrm{L}(\beta_C\times\beta_S,\mathrm{r})$. Also, let $(v_1,u_1)\,,(v_2,u_2)\in\mathrm{L}(\beta_C\times\beta_S,\mathrm{r})$. Then, $(\beta_C\times\beta_S)\left((v_1,u_1)\right)\leq\mathrm{r}$ and $(\beta_C\times\beta_S)\left((v_2,u_2)\right)\leq\mathrm{r}$, so
$(\beta_C\times\beta_S)\left((v_1,u_1)(v_2,u_2)\right)\leq\max\left\{(\beta_C\times\beta_S)\left((v_1,u_1)\right),(\beta_C\times\beta_S)\left((v_2,u_2)\right)\right\}\leq\mathrm{r}$.
Then $(v_1,u_1)(v_2,u_2)\in\mathrm{L}(\beta_C\times\beta_S,\mathrm{r})$. Hence, $\mathrm{L}(\beta_C\times\beta_S,\mathrm{r})$ is semi d-ideal in $H\times H$.
In a converse way, assume that for any $r,t\in[0,1]$, $\mathrm{U}(\alpha_C\times\alpha_S,\mathrm{t})$ and $\mathrm{L}(\beta_C\times\beta_S,\mathrm{r})$ are empty or semi d-ideal of $H\times H$. $\forall\,(v_1,u_1)\in H\times H$ Let $(\alpha_C\times\alpha_S)\left((v_1,u_1)\right)=\mathrm{t}$ and $(\beta_C\times\beta_S)\left((v_1,u_1)\right)=r$. Then, $(v_1,u_1)\in\mathrm{U}(\alpha_C\times\alpha_S,\mathrm{t})\cap\mathrm{L}(\beta_C\times\beta_S,\mathrm{r})$ and so $\mathrm{U}(\alpha_C\times\alpha_S,\mathrm{t})\neq\emptyset\neq\mathrm{L}(\beta_C\times\beta_S,\mathrm{r})$.
Since $\mathrm{U}(\alpha_C\times\alpha_S,\mathrm{t})$ and $\mathrm{L}(\beta_C\times\beta_S,\mathrm{r})$ are semi d-ideal, if there exist $(p_1,q_1)\,,(p_2,q_2)\in H\times H$ such that $(\alpha_C\times\alpha_S)\left((p_1,q_1)\right)<\min\{(\alpha_C\times\alpha_S)\left((p_1,q_1)(p_2,q_2)\right),(\alpha_C\times\alpha_S)\left((p_2,q_2)\right)\}$, then by taking $t_0=\frac{1}{2}\left((\alpha_C\times\alpha_S)\left((p_1,q_1)\right)+\min\left\{(\alpha_C\times\alpha_S)\left((p_1,q_1)(p_2,q_2)\right),(\alpha_C\times\alpha_S)\left((p_2,q_2)\right)\right\}\right)$,
we have $(\alpha_C\times\alpha_S)\left((p_1,q_1)\right)<t_0<\min\left\{(\alpha_C\times\alpha_S)\left((p_1,q_1)(p_2,q_2)\right),(\alpha_C\times\alpha_S)\left((p_2,q_2)\right)\right\}$.
Hence, $(p_1,q_1)\notin\mathrm{U}(\alpha_C\times\alpha_S,t_0)$, $(p_1,q_1)(p_2,q_2)\in\mathrm{U}(\alpha_C\times\alpha_S,t_0)$ and $(p_2,q_2)\in\mathrm{U}(\alpha_C\times\alpha_S,t_0)$. That is, $\mathrm{U}(\alpha_C\times\alpha_S,t_0)$ is not semi d-ideal, which is a contradiction.

Now, suppose that $(\alpha_C \times \alpha_S)\big((p_1,q_1)(p_2,q_2)\big) < \min\{(\alpha_C \times \alpha_S)((p_1,q_1)),(\alpha_C \times \alpha_S)\big((p_2,q_2)\big)\}$.
Then, by taking :
$t_0 = \frac{1}{2}((\alpha_C \times \alpha_S)\big((p_1,q_1)(p_2,q_2)\big) + \min\{(\alpha_C \times \alpha_S)\big((p_1,q_1)\big),(\alpha_C \times \alpha_S)\big((p_2,q_2)\big)\})$, we have
$(\alpha_C \times \alpha_S)\big((p_1,q_1)(p_2,q_2)\big) < t_0 < \min\{(\alpha_C \times \alpha_S)\big((p_1,q_1)\big),(\alpha_C \times \alpha_S)\big((p_2,q_2)\big)$.
Hence, $(p_1,q_1),(p_2,q_2) \in \mathrm{U}(\alpha_C \times \alpha_S,t_0)$, but $(p_1,q_1)(p_2,q_2) \notin \mathrm{U}(\alpha_C \times \alpha_S,t_0)$.
That is, $\mathrm{U}(\alpha_C \times \alpha_S,t_0)$ is not semi d-ideal, which is a contradiction.
Now, assume that $(p_1,q_1)(p_2,q_2) \in H \times H$ such that :
$\beta_C \times \beta_S\big((p_2,q_2)\big) > \max\{\beta_C \times \beta_S\big((p_1,q_1)(p_2,q_2)\big),\beta_C \times \beta_S\big((p_2,q_2)\big)\}$.
By taking $r_0 = \frac{1}{2}(\beta_C \times \beta_S\big((p_1,q_1)\big) + \max\{\beta_C \times \beta_S\big((p_1,q_1)(p_2,q_2)\big),\beta_C \times \beta_S\big((p_2,q_2)\big)\})$,
then $\max\{\beta_C \times \beta_S\big((p_1,q_1)(p_2,q_2)\big),\beta_C \times \beta_S\big((p_2,q_2)\big)\} < r_0 < \beta_C \times \beta_S\big((p_1,q_1)\big)$ and there are $(p_1,q_1)(p_2,q_2) \in \mathrm{L}(\beta_C \times \beta_S,r_0)$ and $(p_2,q_2) \in \mathrm{L}(\beta_C \times \beta_S,r_0)$, but $(p_1,q_1) \notin \mathrm{L}(\beta_C \times \beta_S,r_0)$, and this is a contradiction.
Also, if we take $(p_1,q_1),(p_2,q_2) \in H \times H$ such that
$\beta_C \times \beta_S\big((p_1,q_1)(p_2,q_2)\big) > \max\{\beta_C \times \beta_S\big((p_1,q_1)\big),\beta_C \times \beta_S\big((p_2,q_2)\big)\}$,
then, by taking $r_0 = \frac{1}{2}(\beta_C \times \beta_S\big((p_1,q_1)(p_2,q_2)\big) + \max\{\beta_C \times \beta_S\big((p_1,q_1)\big),\beta_C \times \beta_S\big((p_2,q_2)\big)\})$,
we have $\max\{\beta_C \times \beta_S\big((p_1,q_1)\big),\beta_C \times \beta_S\big((p_2,q_2)\big)\} < s_0 < \beta_C \times \beta_S\big((p_1,q_1)(p_2,q_2)\big)$. Therefore $(p_1,q_1),(p_2,q_2) \in \mathrm{L}(\beta_C \times \beta_S,r_0)$, but $(p_1,q_1)(p_2,q_2) \notin \mathrm{L}(\beta_C \times \beta_S,r_0)$, and this is a contradiction.

Theorem (3.10) : *Let* $C \times S =< \alpha_C \times \alpha_S, \beta_C \times \beta_S >$ be an $IFSd-ideal$ of $H \times H$, then the sets $H_{\alpha_{C\times S}} = \{(a,b) \in H \times H: \alpha_{C\times S}(a,b) = \alpha_{C\times S}(0,0)$ and $H_{\beta_{C\times S}} = \{(a,b) \in H \times H: \beta_{C\times S}(a,b) = \beta_{C\times S}(0,0)$ are semi d-ideal in $H \times H$.

Proof: If we take $(a,b),(x,y) \in H \times H$, let $(a,b)(x,y) \in H_{\alpha_{C\times S}}$ and $(x,y) \in H_{\alpha_{C\times S}}$. Then, $\alpha_{C\times S}\big((a,b)(x,y)\big) = \alpha_{C\times S}(0,0) = \alpha_{C\times S}(x,y)$, so $\alpha_{C\times S}(a,b) \geq \min\{\alpha_{C\times S}\big((a,b)(x,y)\big),\alpha_{C\times S}(x,y)\} = \alpha_{C\times S}(0,0)$. Knowing that $\alpha_{C\times S}(a,b) = \alpha_{C\times S}(0,0)$ (proposition (3.3)), thus $(a,b) \in H_{\alpha_{C\times S}}$.
Let $(a,b),(x,y) \in H_{\alpha_{C\times S}}$. Then, $\alpha_{C\times S}(a,b) = \alpha_{C\times S}(x,y) = \alpha_{C\times S}(0,0)$, so $\alpha_{C\times S}\big((a,b)(x,y)\big) \geq \min\{\alpha_{C\times S}(a,b),\alpha_{C\times S}(x,y)\} = \alpha_{C\times S}(0,0)$. Knowing that $\alpha_{C\times S}\big((a,b)(x,y)\big) = \alpha_{C\times S}(0,0)$ (proposition (3.3)), thus $(a,b)(x,y) \in H_{\alpha_{C\times S}}$.
Also, let $(a,b)(x,y) \in H_{\beta_{C\times S}}$ and $(x,y) \in H_{\beta_{C\times S}}$. Then, $\beta_{C\times S}\big((a,b)(x,y)\big) = \beta_{C\times S}(x,y) = \beta_{C\times S}(0,0)$, so $\beta_{C\times S}(a,b) \leq \max\{\beta_{C\times S}\big((a,b)(x,y)\big),\beta_{C\times S}(x,y)\} = \beta_{C\times S}(0,0)$. Knowing that $\beta_{C\times S}(a,b) = \beta_{C\times S}(0,0)$ (proposition (3.3)), so we get $(a,b) \in H_{\beta_{C\times S}}$.
Let $(a,b),(x,y) \in H_{\beta_{C\times S}}$. Then, $\beta_{C\times S}(a,b) = \beta_{C\times S}(x,y) = \beta_{C\times S}(0,0)$, so $\beta_{C\times S}\big((a,b)(x,y)\big) \leq \max\{\beta_{C\times S}(a,b),\beta_{C\times S}(x,y)\} = \beta_{C\times S}(0,0)$. Hence, from proposition (3.3), we get $\beta_{C\times S}\big((a,b)(x,y)\big) = \beta_{C\times S}(0,0)$. Then, $(a,b)(x,y) \in H_{\beta_{C\times S}}$. Thus, $\beta_{C\times S}$ is semi d-ideal.
The next theorems are easy to prove.

Theorem (3.11): In a d-homorphism $f: H \times H \to G \times G$, if $C \times S$ ise an $IFSd-ideal$ of $G \times G$, then $f^{-1}(C \times S)$ is an $IFSd-ideal$ of $H \times H$.

Theorem (3.12): Let $f: H \times H \to G \times G$ be an d-homomorphism and let $C \times S$ be a direct product of $IFS\ C$ and S in $G \times G$. If $f^{-1}(C \times S) =< \alpha_{f^{-1}(C\times S)}, \beta_{f^{-1}(C\times S)} >$ is an $IFSd-ideal$ in $H \times H$, then $C \times S =< \alpha_{C\times S}, \beta_{C\times S} >$ is an $IFSd-ideal$ of $G \times G$.

4. Direct product of Intuitionistic fuzzy topological d-algebra

In this section, we apply the concept of direct product for intuitionistic set on the notation of **intuitionistic fuzzy topological d-algebra** with some theorems of continues maps.

Definition (4.1) [3] : An intuitionistic fuzzy topology (IFT shortly) on a non-empty set H is a family $\mathfrak{H}$ of IFSs in H that satisfies :
(IFT_1) $0_\sim, 1_\sim \in \mathfrak{H}$,
(IFT_2) $\aleph_1 \cap \aleph_2 \in \mathfrak{H}$ for any $\aleph_1, \aleph_2 \in \mathfrak{H}$,
(IFT_1) $\cup_{i\in\Delta} \aleph_i \in \mathfrak{H}$ for any family $\{\aleph_i, i \in \Delta\} \subseteq \mathfrak{H}$.
So, we call the pair $(H, \mathfrak{H})$ as an intuitionistic fuzzy topological space (IFTS shortly) and the IFS in $\mathfrak{H}$ as an intuitionistic fuzzy open (shortly IFOS) .

If we have a map $f: H \to G$ such that $(H, \mathfrak{H})$, (Y, ϑ) are two IFTS, then f is called intuitionistic fuzzy continuous (IFC) if the inverse image for any IFS in ϑ being IFS in $\mathfrak{H}$. Also, if the image for any IFS in $\mathfrak{H}$ is an IFS in ϑ, then we call f as an intuitionistic fuzzy open (IFO). **[1]**

Definition (4.2) [10] : For an IFS $\aleph$ in an IFTS $(H, \mathfrak{H})$, we say that the induced intuitionistic fuzzy topology (shortly IIFT) on $\aleph$ is a family of IFSs in $\aleph$ such that the intersection of it with $\aleph$ is an IFOS in H . The IIFTS is denoted by $\mathfrak{H}_\aleph$ and $(\aleph\, \mathfrak{H}_\aleph)$ is an intuitionistic fuzzy subspace (IFS ub) of $(H, \mathfrak{H})$.

Definition (4.3) [10] : Take $(\aleph, \mathfrak{H}_\aleph)$ and $(\mathcal{M}, \vartheta_\mathcal{M})$ as IFSub of IFTSs $(H, \mathfrak{H})$ and (G, ϑ), respectively, with the mapping $f: H \to G$ be a mapping. Then, f is a mapping $\aleph$ into $\mathcal{M}$ if $f(\aleph) \subset \mathcal{M}$. Also f is called *relatively intuitionistic fuzzy continuous (RIFC)* if, for any IFS $V_\mathcal{M}$ in $\vartheta_\mathcal{M}$, the intersection $f^{-1}(V_\mathcal{M}) \cap \aleph$ is an IFS in $\mathfrak{H}_\aleph$; and f is called *relatively intuitionistic fuzzy open (RIFO)* if, for any IFS $U_\aleph$ in $\mathfrak{H}_\aleph$, the image $f(U_\aleph)$ is IFS in $\vartheta_\mathcal{M}$.

Proposition (4.4) : Let $(\aleph \times \mathcal{M}, \mathfrak{H}_{\aleph\times\mathcal{M}})$ and $(\mathrm{F} \times \mathcal{L}, \vartheta_{\mathrm{F}\times\mathcal{L}})$ be direct products of IFSub of direct product of IFTSs $(H \times \mathrm{H}, \mathfrak{H})$ and $(G \times G, \vartheta)$, respectively, and let $f: H \times H \to G \times G$ be an *intuitionistic fuzzy continuous* mapping, such that $f(\aleph \times \mathcal{M}) \subset (\mathrm{F} \times \mathcal{L})$. Then, f is *RIFC* mapping of $(\aleph \times \mathcal{M})$ into $(\mathrm{F} \times \mathcal{L})$.

Proof: Let $(U_2 \times V_2)_{(\mathrm{F}\times\mathcal{L})}$ be IFS in $\vartheta_{(\mathrm{F}\times\mathcal{L})}$, then there exists $U \times \mathrm{V} \in \vartheta$, such that

$$(U_2 \times V_2)_{(\mathrm{F}\times\mathcal{L})} = (U \times \mathrm{V}) \cap (\mathrm{F} \times \mathcal{L})$$

Since f is *IFC*, so it follows that $f^{-1}(U \times \mathrm{V})$ is an IFS in $\mathfrak{H}$. So

$$\begin{aligned} f^{-1}\big((U_2 \times V_2)_{(\mathrm{F}\times\mathcal{L})}\big) \cap (\aleph \times \mathcal{M}) &= f^{-1}\big((U \times \mathrm{V}) \cap (\mathrm{F} \times \mathcal{L})\big) \cap (\aleph \times \mathcal{M}) \\ &= f^{-1}\big((U \times \mathrm{V})\big) \cap f^{-1}(\, (\mathrm{F} \times \mathcal{L})) \cap (\aleph \times \mathcal{M}) \\ &= f^{-1}\big((U \times \mathrm{V})\big) \cap (\aleph \times \mathcal{M}) \end{aligned}$$

is IFS in $\mathfrak{H}_{\aleph\times\mathcal{M}}$. This completes the proof.

Definition (4.5) : For any H and any order pair (a, b) of $H \times H$, we define the self-map $(a, b)_r$ of $H \times H$ by $(a, b)_r((x, y)) = (x, y)(a, b)$ for all $(x, y) \in H \times H$.

Definition (4.6) [10] : For an IFT $\mathfrak{H}$ on H, if $\aleph$ is an IFd-algebra with IIFT $\mathfrak{H}_\aleph$, then we say that $\aleph$ intuitionistic fuzzy topological d-algebra (IFTd-algebra shortly), if for any $\hbar \in H$, the mapping $\hbar_r: (\aleph, \mathfrak{H}_\aleph) \to (\aleph, \mathfrak{H}_\aleph)$, $x \to x\hbar$ is relatively intuitionistic fuzzy continuous.

Definition (4.7): For an IFT $\mathfrak{H}$ on H, if $\aleph, \mathcal{M}$ are IFd-algebras with IIFTs $\mathfrak{H}_\aleph, \mathfrak{H}_\mathcal{M}$, respectively. Then, $\aleph \times \mathcal{M}$ is called a direct product of IFTd-algebra if for any $(a, b) \in H \times H$ the mapping $(a, b)_r: (\aleph \times \mathcal{M}, \varphi_{\aleph\times\mathcal{M}}) \to (\aleph \times \mathcal{M}, \varphi_{\aleph\times\mathcal{M}})$, $(x, y) \to (x, y)(a, b)$ is relatively intuitionistic fuzzy continuous .

Theorem (4.8): Let $\delta: H \to G$ be a d-homorphism and $\mathfrak{H}$, ϑ be IFTs on H and G, respectively, such that $\mathfrak{H} = \delta^{-1}(\vartheta)$. If $\aleph \times \mathcal{M}$ is a direct product of IFTd-algebra in $G \times G$, then $\delta^{-1}(\aleph \times \mathcal{M})$ is an IFTd-algebra in $H \times H$.

Proof: Suppose that $(a, b) \in H \times H$ and let $U_1 \times V_1$ be IFS in $\mathfrak{H}_{\delta^{-1}(\aleph\times\mathcal{M})}$. We know that δ^{-1} is an IFC mapping of $(H \times H, \mathfrak{H})$ into $(G \times G, \vartheta)$, so we have from (4.4) that δ is a IRFC mapping of $(\delta^{-1}(\aleph \times \mathcal{M}), \mathfrak{H}_{\delta^{-1}(\aleph\times\mathcal{M})})$ into $(\aleph \times \mathcal{M}, \vartheta_{\aleph\times\mathcal{M}})$. Note that there exists an IFS $U_2 \times V_2$ in $\vartheta_{\aleph\times\mathcal{M}}$ such that $\delta^{-1}(U_2 \times V_2\,) = U_1 \times V_1$. Then

$$\begin{aligned} \alpha_{(a,b)_r^{-1}((U_1\times V_1))}\big((x, y)\big) &= \alpha_{U_1\times V_1}\Big((a, b)_r\big((x, y)\big)\Big) \\ &= \alpha_{U_1\times V_1}\big((x, y)(a, b)\big) \\ &= \alpha_{\delta^{-1}(U_2\times V_2\,)}\big((x, y)(a, b)\big) \\ &= \alpha_{U_2\times V_2}\Big(\delta\big((x, y)(a, b)\big)\Big) \\ &= \alpha_{U_2\times V_2}(\delta((x, y))\delta((a, b))) \end{aligned}$$

and
$$\begin{aligned} \beta_{(a,b)_r^{-1}((U_1\times V_1))}\big((x, y)\big) &= \beta_{U_1\times V_1}\Big((a, b)_r\big((x, y)\big)\Big) \\ &= \beta_{U_1\times V_1}\big((x, y)(a, b)\big) \\ &= \beta_{\delta^{-1}(U_2\times V_2\,)}\big((x, y)(a, b)\big) \\ &= \beta_{U_2\times V_2}\Big(\delta\big((x, y)(a, b)\big)\Big) \\ &= \beta_{U_2\times V_2}(\delta((x, y))\delta((a, b))). \end{aligned}$$

Since $\aleph \times \mathcal{M}$ is a direct product of IFTd-algebra in $\times G$, then we have the RIFC mapping $(b_1, b_2)_r: (\aleph \times \mathcal{M}, \vartheta_{\aleph\times\mathcal{M}}) \to (\aleph \times \mathcal{M}, \vartheta_{\aleph\times\mathcal{M}})$, $(y_1, y_2) \to (y_1, y_2)(b_1, b_2)$, for every (b_1, b_2) in $G \times G$. Hence,

$$\begin{aligned}\alpha_{(a,b)_r^{-1}((U_1\times V_1))}\big((x,y)\big) &= \alpha_{U_2\times V_2}\Big(\delta\big((x,y)\big)\delta\big((a,b)\big)\Big)\\ &= \alpha_{U_2\times V_2}\Big(\delta(a,b)_r\Big(\delta\big((x,y)\big)\Big)\Big)\\ &= \alpha_{\delta((a,b)_r^{-1}((U_2\times V_2)))}\Big(\delta\big((x,y)\big)\Big)\\ &= \alpha_{\delta^{-1}(\delta((a,b)_r^{-1}((U_2\times V_2))))}\big((x,y)\big).\end{aligned}$$

and
$$\begin{aligned}\beta_{(a,b)_r^{-1}((U_1\times V_1))}\big((x,y)\big) &= \beta_{U_2\times V_2}\Big(\delta\big((x,y)\big) * \delta\big((a,b)\big)\Big)\\ &= \beta_{U_2\times V_2}\Big(\delta(a,b)_r\Big(\delta\big((x,y)\big)\Big)\Big)\\ &= \beta_{\delta((a,b)_r^{-1}((U_2\times V_2)))}\Big(\delta\big((x,y)\big)\Big)\\ &= \beta_{\delta^{-1}(\delta((a,b)_r^{-1}((U_2\times V_2))))}\big((x,y)\big).\end{aligned}$$

Therefore, $(a,b)_r^{-1}\big((U_1\times V_1)\big) = \delta^{-1}\big(\delta((a,b)_r^{-1}\big((U_2\times V_2)\big))\big)$.

So, $(a,b)_r^{-1}\big((U_1\times V_1)\big) \cap \delta^{-1}(\aleph\times\mathcal{M}) = \delta^{-1}\big(\delta((a,b)_r^{-1}\big((U_2\times V_2)\big))\big) \cap \delta^{-1}(\aleph\times\mathcal{M})$ is an IFS in $\varphi_{\delta^{-1}(\aleph\times\mathcal{H})}$.

Theorem (4.9): For a d-homorphism $\delta: H \to G$ and $\mathfrak{H}$, ϑ being IFTs on H and G, respectively, such that $(\mathfrak{H}) = \vartheta$. If $D \times C$ is a direct product of IFTd-algebra in $H \times H$, then $\delta(D \times C)$ is an IFTd-algebra in $G \times G$.

Proof : We need to show that the mapping $(b_1, b_2)_r: (\delta(D\times C), \vartheta_{\delta(D\times C)}) \to (\delta(D\times C), \vartheta_{\delta(D\times C)})$, $(y_1, y_2) \to (y_1, y_2)(b_1, b_2)$ is relatively intuitionistic fuzzy continuous for every (b_1, b_2) in $H \times H$. Let $(U_1 \times V_1)_{D\times C}$ be IFS in $\mathfrak{H}_{D\times C}$.

Then, there exists an IFS $U_2 \times V_2$ in φ such that $(U_1\times V_1)_{D\times C} = (U\times V) \cap D \times C$.

Since δ is one-one, it follows that $\delta((U_1\times V_1)_{D\times C}) = \delta\big((U\times V)\cap D\times C\big) = \delta\big((U\times V)\big) \cap \delta(D\times C)$, which is an IFS in $\vartheta_{\delta(D\times C)}$. This shows that δ is RIFO.

Let $(U_1 \times V_1)_{D\times C}$ be an IFS in $\vartheta_{\delta(D\times C)}$. Since δ is surjective , so we have for every (b_1, b_2) in $G \times G$, there exists (a_1, a_2) in $H \times H$ such that $(b_1, b_2) = \delta\big((a_1, a_2)\big)$. Hence,

$$\begin{aligned}\alpha_{\delta^{-1}\left((b_1,b_2)_r^{-1}((U_2\times V_2)_{\delta(D\times C)})\right)}\big((x,y)\big) &= \alpha_{\delta^{-1}(\delta((a_1,a_2))_r^{-1}((U_2\times V_2)_{\delta(D\times C)}))}\big((x,y)\big)\\ &= \alpha_{\delta((a_1,a_2))_r^{-1}((U_2\times V_2)_{\delta(D\times C)})}\Big(\delta\big((x,y)\big)\Big)\\ &= \alpha_{(U_2\times V_2)_{\delta(D\times C)}}\Big(\delta\big((a_1,a_2)\big)_r\Big(\delta\big((x,y)\big)\Big)\Big)\\ &= \alpha_{(U_2\times V_2)_{\delta(D\times C)}}\Big(\delta\big((x,y)\big) * \delta\big((a_1,a_2)\big)\Big)\\ &= \alpha_{(U_2\times V_2)_{\delta(D\times C)}}\Big(\delta\big((x,y) * (a_1,a_2)\big)\Big)\\ &= \alpha_{\delta^{-1}((U_2\times V_2)_{\delta(D\times C)})}\big((x,y) * (a_1,a_2)\big)\\ &= \alpha_{\delta^{-1}((U_2\times V_2)_{\delta(D\times C)})}\Big((a_1,a_2)_r\big((x,y)\big)\Big)\\ &= \alpha_{(a_1,a_2)_r^{-1}\left(\delta^{-1}((U_2\times V_2)_{\delta(D\times C)})\right)}\big((x,y)\big)\end{aligned}$$

and
$$\begin{aligned}\beta_{\delta^{-1}\left((b_1,b_2)_r^{-1}((U_2\times V_2)_{\delta(D\times C)})\right)}\big((x,y)\big) &= \beta_{\delta^{-1}(\delta((a_1,a_2))_r^{-1}((U_2\times V_2)_{\delta(D\times C)}))}\big((x,y)\big)\\ &= \beta_{\delta((a_1,a_2))_r^{-1}((U_2\times V_2)_{\delta(D\times C)})}\Big(\delta\big((x,y)\big)\Big)\\ &= \beta_{(U_2\times V_2)_{\delta(D\times C)}}\Big(\delta\big((a_1,a_2)\big)_r\Big(\delta\big((x,y)\big)\Big)\Big)\\ &= \beta_{(U_2\times V_2)_{\delta(D\times C)}}\Big(\delta\big((x,y)\big) * \delta\big((a_1,a_2)\big)\Big)\\ &= \beta_{(U_2\times V_2)_{\delta(D\times C)}}\Big(\delta\big((x,y) * (a_1,a_2)\big)\Big)\\ &= \beta_{\delta^{-1}((U_2\times V_2)_{\delta(D\times C)})}\big((x,y) * (a_1,a_2)\big)\end{aligned}$$

$$= \beta_{\delta^{-1}((U_2 \times V_2)_{\delta(D\times C)})}\left((a_1, a_2)_r((x, y))\right)$$
$$= \beta_{(a_1,a_2)_r^{-1}\left(\delta^{-1}((U_2\times V_2)_{\delta(D\times C)})\right)}((x, y)).$$

Therefore, $\delta^{-1}\left((b_1, b_2)_r^{-1}\left((U_2 \times V_2)_{\delta(D\times C}\right)\right) = (a_1, a_2)_r^{-1}\left(\delta^{-1}\left((U_2 \times V_2)_{\delta(D\times C)}\right)\right)$.

By hypothesis, the mapping $(a_1, a_2)_r : (D \times C, \mathfrak{H}_{D\times C}) \to (\delta(D \times C), \vartheta_{\delta(D\times C)})$, $(x, y) \to (x, y)(a_1, a_2)$ is RIFC and δ is RIFC map such that $\delta : (D \times C, \mathfrak{H}_{D\times C}) \to (\delta(D \times C), \vartheta_{\delta(D\times C)})$. Thus,

$\delta^{-1}\left((b_1, b_2)_r^{-1}\left((U_2 \times V_2)_{\delta(D\times C}\right)\right) \cap (D \times C) = (a_1, a_2)_r^{-1}\left(\delta^{-1}\left((U_2 \times V_2)_{\delta(D\times C)}\right)\right) \cap (D \times C)$ is an IFS in $\mathfrak{H}_{D\times C}$.

Since δ is RIFO, then

$$\delta\left(\delta^{-1}\left((b_1, b_2)_r^{-1}\left((U_2 \times V_2)_{\delta(D\times C}\right)\right) \cap (D \times C)\right) = (b_1, b_2)_r^{-1}\left((U_2 \times V_2)_{\delta(D\times C}\right) \cap \delta\left((D \times C)\right)$$

is IFS in $\vartheta_{D\times C}$. This completes the proof.

Conclusions

We showed in this paper that the definition of relatively intuitionistic fuzzy continuous has led us to define the notation of the direct product of intuitionistic fuzzy topological d-algebra. We also found that the homomorphism map δ provides the notion that the primage for the direct product of intuitionistic fuzzy topological d-algebra is also a direct product of intuitionistic fuzzy topological d-algebra. Also, the image for the direct product of intuitionistic fuzzy topological d-algebra is a direct product of intuitionistic fuzzy topological d-algebra.

We believe that this work can enhance further studies in this field for the generation of direct products of finite and infinite intuitionistic fuzzy semi d-ideals on d-algebra as well as intuitionistic topological d-algebra. We hope that this work can impact upcoming research in this field or in other algebraic structures.

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