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Some Results on the Norm Attainment Set for Bounded Linear Operators on Smooth Banach Spaces

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Abstract

In this paper, we give new results and proofs that include the notion of norm attainment set of bounded linear operators on a smooth Banach spaces and using these results to characterize a bounded linear operators on smooth Banach spaces that preserve of approximate ϵ-$B\mathcal{J}$-orthogonality. Noting that this work takes brief sidetrack in terms of approximate ϵ-$B\mathcal{J}$-orthogonality relations characterizations of a smooth Banach spaces.

Keywords: $\perp_{BJC}^{\epsilon}$-orthogonality, $\perp_{BJC}^{\epsilon}$-symmetric, preserve of $\perp_{BJC}^{\epsilon}$-orthogonality, smoothness of Banach spaces.

بعض النتائج حول مجموعة تحصيل المعيار للمشغلات الخطية المقيدة على فضاءات بناخ الملساء

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الخلاصة

في هذا البحث، نقدم نتائج جديدة وإثباتات تشمل مفهوم مجموعة التحصيل المعياري للمشغلات الخطية المقيدة المعرفة على فضاءات بناخ الملساء واستخدام هذه النتائج لتوصيف المشغلات الخطية المقيدة المعرفة على فضاءات بناخ الملساء التي تحافظ على تعامد$-\epsilon-BJ$ التقريبي. مع ملاحظة أن هذا العمل يأخذ مسارًا جانبيًا موجزًا من حيث توصيفات علاقات تعامد$-\epsilon-BJ$ التقريبي لفضاءات بناخ الملساء.

1. Introduction:

There are various notions of orthogonality in a normed space, which are in general different, if the norm is not induced by an inner product. Among all notions of orthogonality, Robert orthogonality [1] and Birkhoff-James orthogonality [2], plays a very important role in the study of geometry of normed spaces. In [3, 4, 5, 6, 7, 8, 9], James elaborated how the notions Like reflexivity, strict convexity and smoothness of Banach spaces can be studied using Birkhoff-James orthogonality. Recently, Paul et al. [10], obtained a sufficient condition for the smoothness of bounded linear operators using Birkhoff-James orthogonality of bounded linear operator on a Banach spaces. Due to the importance of Birkhoff-James orthogonality it has been generalized by Dragomir [11] and Chmielinski [12] and is known as approximate ϵ-Birkhoff-James orthogonality. In [13], Chmielinski et al. characterized approximate ϵ-Birkhoff-James orthogonality of bounded linear operator on a finite and infinite dimensional Banach spaces respectively.

A part form [14, 15, 16], some results on the characterization of bounded linear operators were also obtained. Very recently, Saied J. and Buthainah A. [17], introduced a complete characterization for

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the norm attainment set of a bounded linear operators on a real Banach spaces at a vector in the unit sphere by using approximate ϵ-Birkhoff-James orthogonality techniques.

Throughout this paper, we only consider Banach spaces $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ are reserved for real Banach spaces if there is no explanation. Without further ado, let us discuss the notions and the terminologies relevant to our study. Let $\mathbb{B}_{\mathbb{X}} = \{x \in \mathbb{X} : \| x \|_{\mathbb{X}} \leq 1\}$ and $\mathbb{S}_{\mathbb{X}} = \{x \in \mathbb{X} : \| x \|_{\mathbb{X}} = 1\}$ be the unite ball and unite sphere of $\mathbb{X}$, respectively. A normed space $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ is smooth, if for any vector x in $\mathbb{S}_{\mathbb{X}}$, there exists a unique hyperplane $x + \mathbb{H}$ supporting $\mathbb{B}_{\mathbb{X}}$ at x, [18]. $\mathbb{H}$ is hyperspace of $\mathbb{X}$ if and only if $x + \mathbb{H}$ is hyperplane. A normed space $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ is strictly convex, if for any two vectors x and y in $\mathbb{B}_{\mathbb{X}}$, the triangle inequality gives $\| x + y \|_{\mathbb{X}} \leq 2$, [19]. We write $\mathcal{B}(\mathbb{X}, \mathbb{Y})$ $(\mathcal{K}(\mathbb{X}, \mathbb{Y}))$, for the Banach space of bounded (compact) linear operators from $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ into $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ equipped with the supermom norm, [20].

Specifically, if $\mathbb{Y}$ is a real field , $\mathcal{B}(\mathbb{X}, \mathbb{R})$ is the dual space of $\mathbb{X}$ and it is denoted by $\mathbb{X}^*$. A hyperspace $\mathbb{H}$ of $\mathbb{X}$ is closed if and only if it is a kernel of $\Theta \neq \psi \in \mathbb{X}^*$, [21]. For any vector x in a normed space $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$, let $\mathcal{J}(x)$ denote the (non-empty) set of its supporting functionals:

$$\mathcal{J}(x) = \{\psi \in \mathbb{X}^* : \| \psi \|_{\mathbb{X}^*} = 1 \text{ and } \psi(x) = \| x \|_{\mathbb{X}}\}, [18].$$

A Banach space $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ is reflexive, if it coincides with bi-dual space. More specifically, if we denote the dual space of $\mathbb{X}$ by $\mathbb{X}^*$, and the bi-dual space by $\mathbb{X}^{**}$; consider for any vector x in $\mathbb{X}$ there exists a linear operator $\varphi: \mathbb{X} \to \mathbb{X}^{**}$ given by $\varphi(x) = \psi_x$, for any $\psi \in \mathbb{X}^*$ such that φ is an isometric isomorphism of normed spaces, [21]. A Banach space $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ is reflexive if and only if for every $\mathcal{T} \in \mathcal{K}(\mathbb{X}, \mathbb{Y})$ attain its norm, where $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ any Banach space, [22].

Let $\mathcal{T} \in \mathcal{B}(\mathbb{X}, \mathbb{Y})$, $\mathcal{T}$ is said to be attains norm at a vector x in $\mathbb{S}_{\mathbb{X}}$, if $\| \mathcal{T}(x) \|_{\mathbb{Y}} = \| \mathcal{T} \|_{\mathcal{B}(\mathbb{X},\mathbb{Y})}$ and let $M_{\mathcal{T}}$ denote the set of all vectors x in $\mathbb{S}_{\mathbb{X}}$ at which $\mathcal{T}$ attains norm, [10]. i.e.

$$M_{\mathcal{T}} = \{x \in \mathbb{S}_{\mathbb{X}} : \| \mathcal{T}(x) \|_{\mathbb{Y}} = \| \mathcal{T} \|_{\mathcal{B}(\mathbb{X},\mathbb{Y})}\}.$$

The set $M_{\mathcal{T}}$ plays an important role in characterizing approximate ϵ-Birkhoff-James orthogonality of bounded linear operators [17, 23, 24, 25]. Let $\mathbb{K}$ be a locally compact subset in a topological space $(\mathbb{X}, \mathfrak{J})$. As usual $\mathcal{C}(\mathbb{K})$ denotes the linear space of all real continuous mappings defined on $\mathbb{K}$ endowed with the supremum norm. We consider a subspace $\mathcal{C}_0(\mathbb{K})$ of $\mathcal{C}(\mathbb{K})$:

$$\mathcal{C}_0(\mathbb{K}) = \{\psi \in \mathcal{C}(\mathbb{K}) : \forall \epsilon > 0 \text{, the set } \{\eta \in \mathbb{K} : |\psi(\eta)| \geq \epsilon\} \text{ is compact}\}.$$

For every $\psi \in \mathcal{C}_0(\mathbb{K})$, the set M_ψ is non-empty and compact, [13].

Let $x^{\perp_{BJC}^{\epsilon}} = \{ y \in \mathbb{X} : x \perp_{BJC}^{\epsilon} y \}$, [12]. Using this concept, we obtain a necessary condition for $\Theta \neq \mathcal{T} \in \mathcal{B}(\mathbb{X}, \mathbb{Y})$ to attain norm at a vector x in $\mathbb{S}_{\mathbb{X}}$, [10]. $\mathcal{T} \in \mathcal{B}(\mathbb{X})$ is said to be satisfy the Daugavet equation, if $\| \mathcal{I} + \mathcal{T} \|_{\mathcal{B}(\mathbb{X},\mathbb{Y})} = 1 + \| \mathcal{T} \|_{\mathcal{B}(\mathbb{X},\mathbb{Y})}$, where $\mathcal{I}$ is the identity operator on $\mathbb{X}$, [26]. A linear operator $\Theta \neq \mathcal{T} \in \mathcal{B}(\mathbb{X}, \mathbb{Y})$ satisfies Daugavet equation if and only if a linear operator $\eta\mathcal{T}$ satisfies Daugavet equation for all $\eta \geq 0$, [27].

First, we apply Theorem (3.5) in [17], to obtain various interesting properties of a bounded linear operator on a smooth Banach space. Also, we obtain an expression for the kernel of a non-zero bounded linear operator on a smooth , strictly convex of two-dimensional Banach space. Next, we prove that if the underlying Banach space is finite dimensional, strictly convex and smooth, then every non-zero linear operator satisfying Daugavet equation, must have an invariant subspace of co-dimension one. We also prove that in a finite dimensional smooth Banach space $\mathbb{X}$, if the kernel of a non-zero linear operator $\Theta \neq \mathcal{T} \in \mathcal{B}(\mathbb{X})$ contains a non-zero, $\perp_{BJC}^{\epsilon}$-right symmetric vector, then the linearization of $M_{\mathcal{T}}$ is a proper subspace of $\mathbb{X}$. As another theorem, we prove that in a smooth Banach space, image of $\perp_{BJC}^{\epsilon}$-left symmetric vector under an isometry must be $\perp_{BJC}^{\epsilon}$-left symmetric. Finally, it is easy to observe that $\Theta \neq \mathcal{T} \in \mathcal{B}(\mathbb{X})$ preserves the approximate ϵ-Birkhoff-James orthogonality. As potential application, that a non-zero compact linear operator on a smooth reflexive Banach space, preserves approximate ϵ-Birkhoff-James orthogonality at some vectors x in $\mathbb{S}_{\mathbb{X}}$. Motivated by this characterization of isometries on a smooth Banach spaces, must be preserves approximate ϵ-Birkhoff-James orthogonality.

2. Review on some classical basic results:

For more details, we give necessary notions and backgrounds. We first review some various definitions of the orthogonality relation in normed space that are necessary in this paper for the sake of clarity we mention them briefly.

Definition 2.1: [1, 2, 11, 12] For any two vectors x and y in a normed space $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$:
i. x is said to be Roberts-orthogonal to y ($\mathcal{R}$-orthogonality) and written as $x \perp_{\mathcal{R}} y$, if satisfying the following:

$$\| x + \mu y \|_{\mathbb{X}} = \| x - \mu y \|_{\mathbb{X}} \text{ for all } \mu \in \mathbb{R}.$$

ii. x is said to be Birkhohh-James orthogonal to y ($B\mathcal{J}$-orthogonality) and written as $x \perp_{BJ} y$, if satisfying the following:

$$\| x + \mu y \|_{\mathbb{X}} \geq \| x \|_{\mathbb{X}} \text{ for all } \mu \in \mathbb{R}.$$

iii. x is said to be Dragomir-orthogonal to y ($\mathcal{D}$-orthogonality) and written as $x \perp_D^{\epsilon} y$ with $\epsilon \in [0,1)$, if satisfying the following:

$$\| x + \mu y \|_{\mathbb{X}} \geq (1-\epsilon) \ \| x \|_{\mathbb{X}} \text{ for all } \mu \in \mathbb{R}.$$

iv. x is said to be approximate ϵ-Birkhoff-James orthogonal to y (approximate ϵ-$B\mathcal{J}$-orthogonality) and written as $x \perp_{BJC}^{\epsilon} y$ with $\epsilon \in [0,1)$, if satisfying the following:
$\| x + \mu y \|_{\mathbb{X}}^2 \geq \| x \|_{\mathbb{X}}^2 - 2\epsilon \| x \|_{\mathbb{X}} \| \mu y \|_{\mathbb{X}}$ for all $\mu \in \mathbb{R}$.

Remark 2.2: [11, 12]
i. If x is not approximate ϵ-B-$\mathcal{J}$-orthogonal to y, for short the symbol $x \pm_{BJC}^{\epsilon} y$.
ii. The notion "$\perp_{\mathcal{R}}$" has been generalized by "$\perp_{BJ}$" and "$\perp_{BJ}$" has been generalized by "$\perp_{BJC}^{\epsilon}$".

Proposition 2.3: [12] Let $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ be a normed space:
i. $x \perp_{BJC}^{\epsilon} \Theta$ and $\Theta \perp_{BJC}^{\epsilon} x$ for any vector x in $\mathbb{X}$.
ii. $x \perp_{BJC}^{\epsilon} x$, if and only if $x = \Theta$ for any vector x in $\mathbb{X}$.
iii. If $x \perp_{BJC}^{\epsilon} y$, implies that $\eta x \perp_{BJC}^{\epsilon} \gamma y$ for any two vectors x and y in $\mathbb{X}$ with $\eta, \gamma \in \mathbb{R}$.
iv. For any non-zero vectors x and y in $\mathbb{X}$. If $x \perp_{BJC}^{\epsilon} y$, then x and y are linearly independent.

Remark 2.4: [12]
i. If $x \perp_{BJC}^{\epsilon} y$, then need not to be $y \perp_{BJC}^{\epsilon} x$.
ii. If $x \perp_{BJC}^{\epsilon} y$ and $\perp_{BJC}^{\epsilon} z$, then need not to be $x \perp_{BJC}^{\epsilon} y + z$.

Definition 2.5: [17] For any two vectors x and y in $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ with $\epsilon \in [0,1)$, $y \in x^{\perp_{BJC}^{\epsilon}}$. The following two subsets $x^{+(\epsilon)}$ and $x^{-(\epsilon)}$ of $\mathbb{X}$ defined as:

$$y \in x^{+(\epsilon)}, \text{ if } \| x + \mu y \|_{\mathbb{X}}^2 \geq \| x \|_{\mathbb{X}}^2 - 2\epsilon \| x \|_{\mathbb{X}} \| \mu y \|_{\mathbb{X}} \text{ for all } \mu \geq 0;$$

and

$$y \in x^{-(\epsilon)}, \text{ if } \| x + \mu y \|_{\mathbb{X}}^2 \geq \| x \|_{\mathbb{X}}^2 - 2\epsilon \| x \|_{\mathbb{X}} \| \mu y \|_{\mathbb{X}} \text{ for all } \mu \leq 0.$$

Proposition 2.6: [17] Let $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ be a normed space, x and y any two vectors in $\mathbb{X}$. Then the following statements are satisfying:
i. If $y \in x^{+(\epsilon)}$ and $y \in x^{-(\epsilon)}$, then $y \in x^{\perp_{BJC}^{\epsilon}}$. This means that $x^{\perp_{BJC}^{\epsilon}} = x^{+(\epsilon)} \cap x^{-(\epsilon)}$.
ii. If $y \in x^{+(\epsilon)}$ ($y \in x^{-(\epsilon)}$), then $\delta y \in (\eta x)^{+(\epsilon)}$ ($\delta y \in (\eta x)^{-(\epsilon)}$) for all $\delta, \eta > 0$.
iii. If $y \in x^{+(\epsilon)}$ ($y \in x^{-(\epsilon)}$), then $-y \in x^{-(\epsilon)}$ ($-y \in (x)^{+(\epsilon)}$) and $y \in (-x)^{-(\epsilon)}$ ($y \in (-x)^{+(\epsilon)}$).

Theorem 2.7: [13] For any two vectors x and y in a normed space $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$:
i. $x \perp_{BJC}^{\epsilon} y$ if and only if there exists $\psi \in \mathcal{J}(x)$ with $|\psi(y)| \leq \epsilon \| y \|_{\mathbb{X}}$.
ii. $x \perp_{BJC}^{\epsilon} y$ if and only if there exists $z \in Lin\{x, y\}$ with $x \perp_{BJ} z$ and $\| y - z \|_{\mathbb{X}} \leq \epsilon \| y \|_{\mathbb{X}}$.

Theorem 2.8: [29] A Banach space $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ is strictly convex if and only if $B\mathcal{J}$-orthogonality is left symmetric.

Remark 2.9: [17] The notation $x \perp\!\!\!\perp y$ signals to the following cases $x \perp_{BJC}^{\epsilon} y$ and $x \perp_R y$ for any two vectors x and y in $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$, which we will use in the following theorem which plays an important role in this work.

Theorem 2.10: [17] Let $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ and $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ be two Banach spaces and $\Theta \neq \mathcal{T} \in \mathcal{B}(\mathbb{X}, \mathbb{Y})$ be such that there exists $x \in M_{\mathcal{T}}$ with $\mathcal{T}(x) \perp\!\!\!\perp \mathcal{T}(y)$ for any vector y in $\mathbb{X}$. Then:

i . $\mathcal{T}(x^{+(\epsilon)} \backslash x^{\perp_{BJC}^{\epsilon}}) \subseteq (\mathcal{T}x)^{+(\epsilon)} \backslash (\mathcal{T}x)^{\perp_{BJC}^{\epsilon}}$.

ii. $\mathcal{T}(x^{-(\epsilon)} \backslash x^{\perp_{BJC}^{\epsilon}}) \subseteq (\mathcal{T}x)^{-(\epsilon)} \backslash (\mathcal{T}x)^{\perp_{BJC}^{\epsilon}}$.

Theorem 2.11: [17] Let $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ and $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ be two Banach spaces with $\Theta \neq \mathcal{T} \in \mathcal{B}(\mathbb{X}, \mathbb{Y})$, be such that there exists $x \in M_{\mathcal{T}}$ such that $\mathcal{T}(x) \perp\!\!\!\perp \mathcal{T}(y)$. Then:

i. For any vector y in $\mathbb{X}$, implies that $x \perp_{BJC}^{\epsilon} y$.

ii. $ker\, \mathcal{T} \subseteq \bigcap_{x \in M_{\mathcal{T}}} x^{\perp_{BJC}^{\epsilon}}$.

Theorem 2.12: [7] A Banach space $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ is revlexive if and only if for any closed hyperspace $\mathbb{H}$ of $\mathbb{X}$, there exists a vector x in $\mathbb{S}_{\mathbb{X}}$ with $x \perp_{BJ} \mathbb{H}$.

3. Main Results:

We would like to remark that such a study was initiated by Saied J. and Buthainah A. in [24], for bounded linear operators $\mathcal{T} \in \mathcal{B}(\mathbb{X}, \mathbb{Y})$ between two Banach spaces. However, our study has little intersection with the above study, moreover, we are also exploring the problems for bounded linear operators between two Banach spaces. Our first results of this section do not require the limitation mentioned in Theorem (3.5) of [17]. We start with the first theorem connects smoothness of $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ and $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ with $\perp_{BJC}^{\epsilon}$ -orthogonality.

Theorem 3.1: Let $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$, $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ be two smooth Banach spaces and $\Theta \neq \mathcal{T} \in \mathcal{B}(\mathbb{X}, \mathbb{Y})$ with $x \in M_{\mathcal{T}}$. Then $\mathcal{T}(x^{\perp_{BJC}^{\epsilon}}) \subseteq (\mathcal{T}x)^{\perp_{BJC}^{\epsilon}}$.

Proof: From smoothness of , there exists a unique hyperplane $x + \mathbb{H}$ supporting $\mathbb{B}_{\mathbb{X}}$ at x, where $\mathbb{H}$ is a subspace of $\mathbb{X}$ having co-dimension one. It is clear that $\mathbb{H} = x^{\perp_{BJC}^{\epsilon}}$ and $\mathbb{H}$ divides into two closed half-planes whose intersection is $\mathbb{H}$. Let $\mathbb{W}_1$ denote the closed half-plane containing x and $\mathbb{W}_2$ denote the other closed half-plane. Deduce that $\mathbb{W}_1 = x^{+(\epsilon)}$ and $\mathbb{W}_2 = x^{-(\epsilon)}$. Also, every vector in $\mathbb{H}$ can be expressed by a vector from either of the sets $\mathbb{W}_1 \backslash \mathbb{H}$ and $\mathbb{W}_2 \backslash \mathbb{H}$. Let $u \in x^{\perp_{BJC}^{\epsilon}}$ and $(u_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{W}_1 \backslash \mathbb{H}$ with $u_n \longrightarrow u$.

It now follows from Theorem (2.10. i), that $\mathcal{T}(u_n) \in (\mathcal{T}x)^{+(\epsilon)} \backslash (\mathcal{T}x)^{\perp_{BJC}^{\epsilon}}$. Now, from continuity of $\mathcal{T}$ and convergence of $(u_n)_{n \in \mathbb{N}}$, we must have $\mathcal{T}(u) \in cl((\mathcal{T}x)^{+(\epsilon)} \backslash (\mathcal{T}x)^{\perp_{BJC}^{\epsilon}})$.

Also, we considering $(u_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{W}_2 \backslash \mathbb{H}$ such that $u_n \longrightarrow u$. In the same manner as above, it yields that $\mathcal{T}(u) \in cl((\mathcal{T}x)^{-(\epsilon)} \backslash (\mathcal{T}x)^{\perp_{BJC}^{\epsilon}})$.

From smoothness of $\mathbb{Y}$, we must have:

$$\mathcal{T}(u) \in cl((\mathcal{T}x)^{+(\epsilon)} \backslash (\mathcal{T}x)^{\perp_{BJC}^{\epsilon}}) \cap cl((\mathcal{T}x)^{-(\epsilon)} \backslash (\mathcal{T}x)^{\perp_{BJC}^{\epsilon}}) = (\mathcal{T}x)^{\perp_{BJC}^{\epsilon}}.$$

This proves that $\mathcal{T}(u) \in (\mathcal{T}x)^{\perp_{BJC}^{\epsilon}}$ for each $u \in x^{\perp_{BJC}^{\epsilon}}$.

Remark 3.2: The smoothness assumption in the above theorem is necessary condition. The following example to negate $\mathcal{T}(x^{\perp_{BJC}^{\epsilon}}) \subseteq (\mathcal{T}x)^{\perp_{BJC}^{\epsilon}}$.

Let $\Theta \neq \mathcal{T} \in \mathcal{B}(l_\infty(\mathbb{R}^3))$,which is defined by $\mathcal{T}(1, 1, 0) = (0, 1, 0)$ and $\mathcal{T}(-1, 1, 0) = (-1, 0, 0)$. It is clear that $M_{\mathcal{T}} = \{(1, 1, 0), (-1, 1, 0), (-1, -1, 0), (1, -1, 0)\}$ with $(1, 1, 0) \perp_{BJC}^{\epsilon} (0, 1, 0)$, but $\mathcal{T}(1, 1, 0) \not\perp_{BJC}^{\epsilon} \mathcal{T}(0, 1, 0)$.

Remark 3.3:
i. Theorem (3.1) can be interpreted geometrically in a new way. When $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ be a smooth normed space and $\Theta \neq \mathcal{T} \in \mathcal{B}(\mathbb{X})$. If $x \in M_{\mathcal{T}}$, then the image of the hyperplane $x^{\perp^{\epsilon}_{BJC}}$ under $\mathcal{T}$ is a subset of the hyperplane $(\mathcal{T}x)^{\perp^{\epsilon}_{BJC}}$.
ii. In general, if $\mathcal{T}(x) \perp^{\epsilon}_{BJC} \mathcal{T}(y)$ for some $x \in M_{\mathcal{T}}$ and $\Theta \neq \mathcal{T} \in \mathcal{B}(\mathbb{X}, \mathbb{Y})$ we may not obtain $x \perp^{\epsilon}_{BJC} y$, as the example shows:
Let $\Theta \neq \mathcal{T} \in \mathcal{B}(l_{\infty}(\mathbb{R}^2))$, which is defined by $\mathcal{T}(1,1) = (0,1)$ and $\mathcal{T}(-1,1) = (-1,0)$. It is clear that $M_{\mathcal{T}} = \{(1,1), (-1,1), (-1,-1), (1,-1)\}$,we have $(1,1) \in M_{\mathcal{T}}$ with $(1,1) \perp^{\epsilon}_{BJC} (0,1)$, $\mathcal{T}(1,1) = (0,1) \not\perp^{\epsilon}_{BJC} \mathcal{T}(0,1) = (-\frac{1}{2}, \frac{1}{2})$.
iii. In fact, Theorem (2.11.i), is also true if both $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ and $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ are smooth Banach spaces.
iv. In particular, some information regarding $M_{\mathcal{T}}$ may be obtained, even without knowing the action of $\mathcal{T}$ on $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$. For instance, if x and y in $\mathbb{S}_{\mathbb{X}}$ are such that $x \perp^{\epsilon}_{BJC} y$ and $\mathcal{T}(x) \not\perp^{\epsilon}_{BJC} \mathcal{T}(y)$, then ensure that $x \notin M_{\mathcal{T}}$.
Remark 3.4: Let us pictorially illustrate the necessary condition for $M_{\mathcal{T}}$ Banach space $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ with two dimensional smooth Banach spaces of two dimension. That is for any $\Theta \neq \mathcal{T} \in \mathcal{B}(\mathbb{X}, \mathbb{Y})$ and $x \in M_{\mathcal{T}}$. Without loss of generality, we assume that $\| \mathcal{T} \|_{\mathcal{B}(\mathbb{X},\mathbb{Y})} = 1$. In the following diagram, Theorem (3.1) states that, if $x \in M_{\mathcal{T}}$, then $\mathcal{T}(x^{\perp^{\epsilon}_{BJC}}) \subseteq (\mathcal{T}x)^{\perp^{\epsilon}_{BJC}}$.

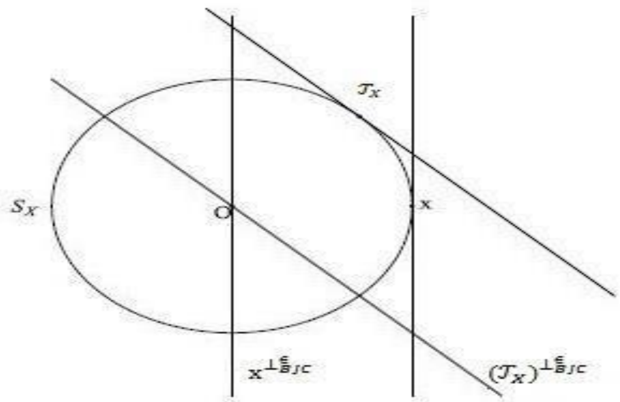

Figure 1- $\mathcal{T}(x^{\perp^{\epsilon}_{BJC}}) \subseteq (\mathcal{T}x)^{\perp^{\epsilon}_{BJC}}$ in $l_2(\mathbb{R}^2)$.

Now, by using Theorem (3.1) we can get an expression for the kernel of a bounded linear operators defined on a Banach space, in terms of $M_{\mathcal{T}}$.
Theorem 3.5: Let $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ be a smooth and strictly convex Banach spaces of two dimension. Then for any $\Theta \neq \mathcal{T} \in \mathcal{B}(\mathbb{X})$ attaining norm at more than one pair of points must be an invertible.

Proof: Let $x_1, x_2 \in M_{\mathcal{T}}$ such that $x_1 \neq \pm x_2$. Claim that $x_1^{\perp^{\epsilon}_{BJC}} \cap x_2^{\perp^{\epsilon}_{BJC}} = \{\Theta\}$. Let $y \in x_1^{\perp^{\epsilon}_{BJC}} \cap x_2^{\perp^{\epsilon}_{BJC}}$, implies that $y \in x_1^{\perp^{\epsilon}_{BJC}}$ and $y \in x_2^{\perp^{\epsilon}_{BJC}}$. Now, applying Theorem (2.7.ii), Proposition (2.3.iii) and from strict convexity of $\mathbb{X}$, we can apply Theorem (2.8) to get $y \perp_{BJ} \eta_1 x_1 + \eta_2 x_2$, for any scalars η_1, η_2. In particular, $y = \eta_1 x_1 + \eta_2 x_2$. This implies that $y \perp_{BJ} y$. From Proposition (2.3.ii), we must have $y = \Theta$.
Now, the second part proof of the theorem follows directly from Theorem (3.1) and Theorem (2.11.ii), $ker\mathcal{T} \subseteq x_1^{\perp^{\epsilon}_{BJC}} \cap x_2^{\perp^{\epsilon}_{BJC}} = \{\Theta\} \subseteq ker\mathcal{T}$. We note that $\mathcal{T}$ is an invertible.

Theorem 3.6: Let $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ be a finite dimensional smooth and strictly convex Banach space and let $\mathcal{I} \in \mathcal{B}(\mathbb{X})$ be a spear operator. Then for every $\Theta \neq \mathcal{T} \in \mathcal{B}(\mathbb{X})$, $\mathbb{X}$ has $\mathcal{T}$-invariant subspace of co-dimension one.

Proof: Let us assume that $\mathcal{T}$ satisfies Daugavet equation and deduce $\eta\mathcal{T}$ satisfies Daugavet equiation, for any scalar $\eta > 0$. Since $\mathbb{X}$ be a finite dimension, then there exists a vector x_0 in $\mathbb{S}_{\mathbb{X}}$ such that:

$\| \mathcal{I} + \mathcal{T} \|_{\mathcal{B}(\mathbb{X})} = \| (\mathcal{I} + \mathcal{T})(x_0) \|_{\mathbb{X}}$.
We claim that $\mathcal{T}(x_0) = x_0$. Indeed,

$$2 = \| \mathcal{I} + \mathcal{T} \|_{\mathcal{B}(\mathbb{X})} = \| (\mathcal{I} + \mathcal{T})(x_0) \|_{\mathbb{X}} = \| x_0 + \mathcal{T}(x_0) \|_{\mathbb{X}}$$
$$\leq \| x_0 \|_{\mathbb{X}} + \| \mathcal{T}(x_0) \|_{\mathbb{X}} \leq 1 + \| \mathcal{T}(x_0) \|_{\mathbb{X}} \leq 1 + \| \mathcal{T} \|_{\mathcal{B}(\mathbb{X})} = 2 \;\ldots.(*)$$

From strict convexity of $\mathbb{X}$, we must have $\mathcal{T}(x_0) = t_0 x_0$, for some $t_0 \geq 0$. On the other hand, from $(*)$, implies that $t_0 = 1$. This proves our claim. Thus, we have $1 = \| \mathcal{T} \|_{\mathcal{B}(\mathbb{X})} = \| \mathcal{T}(x_0) \|_{\mathbb{X}}$. This proves that $x_0 \in M_{\mathcal{T}}$. Now, from smoothness of $\mathbb{X}$, $x_0^{\perp^{\epsilon}_{BJC}}$ is a subspace having co-dimension one. Thus, $x_0^{\perp^{\epsilon}_{BJC}}$ is a $\mathcal{T}$-invarent subspace having co-dimension one.

Remark 3.7:
i. An invariant vector x_0 in a linear space $\mathbb{X}$ other than Θ is a linearization of an invariant subspace of dimension one. An invariant subspace of dimension one will be acted on $\mathbb{X}$ by a scalar and consists of invariant vectors if and only if that scalar is 1.
ii. In particular, it follows from the method used in the proof of Theorem (3.6), that if $\Theta \neq \mathcal{T} \in \mathcal{B}(\mathbb{X})$ satisfies Daugavet equation on a finite dimensional smooth and strictly convex Banach space, then $\mathbb{X}$ has a fixed point on $\mathbb{S}_{\mathbb{X}}$.

Definition 3.8: Let $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ be a normed space. Then:
i. A vector x in $\mathbb{X}$ is said to be a $\perp^{\epsilon}_{BJC}$-left-symmetric and denoted by $x \dashv^{\epsilon}_{BJC} y$, if $x \perp^{\epsilon}_{BJC} y$, implies that $y \perp^{\epsilon}_{BJC} x$ for any vector y in $\mathbb{X}$.
ii. A vector x in $\mathbb{X}$ is said to be a $\perp^{\epsilon}_{BJC}$-right-symmetric and denoted by $y \vdash^{\epsilon}_{BJC} x$, if $y \perp^{\epsilon}_{BJC} x$, implies that $x \perp^{\epsilon}_{BJC} y$ for any vector y in $\mathbb{X}$.

Remark 3.9: Geometrically, x is an $\perp^{\epsilon}_{BJC}$-left-symmetric to y, if the line μy intersects with $\mathbb{B}_{\mathbb{X}}$ at x, (see Figure-2). Likewise, we describe x is an $\perp^{\epsilon}_{BJC}$-right-symmetric to .

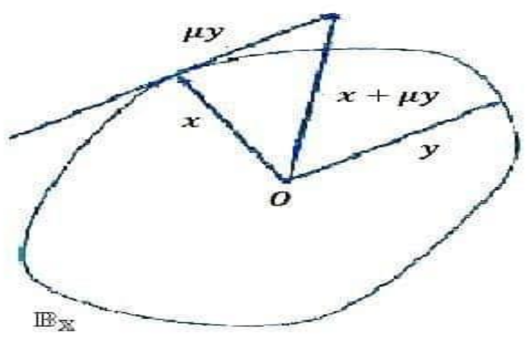

Figure 2- $x \dashv^{\epsilon}_{BJC} y$,

Example 3.10:

i. In $l_2(c_o)$, if $y \in x^{\perp_{BJC}^{\epsilon}}$ $(x \in y^{\perp_{BJC}^{\epsilon}})$, then for any vector y in c_o, it follows that $x \dashv_{BJC}^{\epsilon} y$ $(y \vdash_{BJC}^{\epsilon} x)$.

ii. In $l_\infty(\mathbb{R}^2)$, $(0,1) \in (1,1)^{\perp_{BJC}^{\epsilon}}$, but does not satisfy $(1,1) \dashv_{BJC}^{\epsilon} (0,1)$.

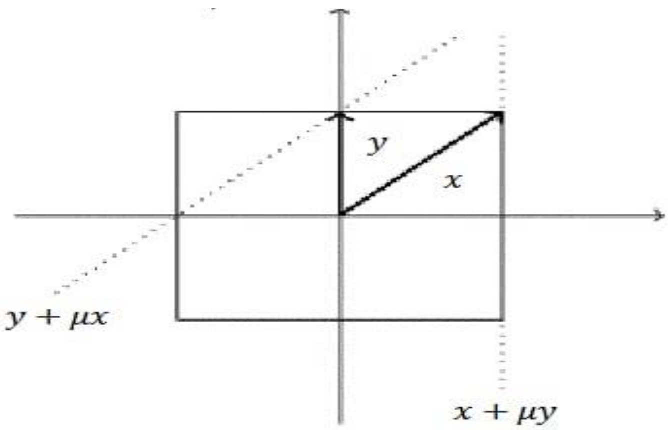

Figure 3- is not $x \dashv_{BJC}^{\epsilon} y$ in $l_\infty(\mathbb{R}^2)$.

Theorem 3.11: Let $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ be a finite dimensional smooth Banach space and let $\Theta \neq \mathcal{T} \in \mathcal{B}(\mathbb{X})$ such that $ker\mathcal{T}$ contains a non-zero $\vdash_{BJC}^{\epsilon}$-symmetric. Then $Lin\, M_{\mathcal{T}}$ is a proper subspace of $\mathbb{X}$.

Proof: Suppose on the contrary, $Lin\, M_{\mathcal{T}} = \mathbb{X}$. Then $M_{\mathcal{T}}$ has a basis $\{x_1, x_2, \dots, x_n\}$ of $\mathbb{X}$. Let $\Theta \neq w \in ker\mathcal{T}$, where w is $\vdash_{BJC}^{\epsilon}$-symmetric. Without loss of generality, we assume that $\| w \|_{\mathbb{X}} = 1$. Let $w = \sum_{i=1}^{n} \eta_i x_i$, for some scalars $\eta_1, \eta_2, \dots, \eta_n$. Applying Theorem (3.1) and Theorem (2.11.ii), we see that $x_i \perp_{BJC}^{\epsilon} w$, for all $i = 1, 2, \dots, n$.
Now, applying Theorem (2.7.i), there exists $\psi \in \mathbb{X}^*$ supporting at x_i with $|\psi(w)| \leq \epsilon \| w \|_{\mathbb{X}}$ for all $i = 1, 2, \dots, n$. We must have $1 = \| \psi \|_{\mathbb{X}^*} = \sup\{|\psi(w)| : \| w \|_{\mathbb{X}} = 1\} < 1$, a contradiction.

In the next theorem, we prove that in smooth Banach space, image of a $\dashv_{BJC}^{\epsilon}$-symmetric vector under an isometry must be a $\dashv_{BJC}^{\epsilon}$-symmetric vector.

Theorem 3.12: In $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ be a smooth Banach space with $\Theta \neq \mathcal{T} \in \mathcal{B}(\mathbb{X})$ be an isometry. If a vector x in $\mathbb{X}$ is $\dashv_{BJC}^{\epsilon}$-symmetric, then $\mathcal{T}(x)$ is also $\dashv_{BJC}^{\epsilon}$-symmetric.

Proof: We first note that $M_{\mathcal{T}} = \mathbb{S}_{\mathbb{X}}$, as $\mathcal{T}$ is an isometry. Let $\mathcal{T}(x) \perp_{BJC}^{\epsilon} y$ for some vector y in $\mathbb{S}_{\mathbb{X}}$. If $y = \Theta$, then $y \perp_{BJC}^{\epsilon} \mathcal{T}(x)$. Let $y \neq \Theta$. Since $\mathcal{T}$ is an isometry. It follows that $\mathcal{T}$ invarent. There exists a vector $\Theta \neq z$ in $\mathbb{X}$, such that $\mathcal{T}(z) = y$, i.e. $\mathcal{T}(x) \perp_{BJC}^{\epsilon} \mathcal{T}(z)$ and $x \in M_{\mathcal{T}}$. As stated in Remark (3.3.ii), it becomes clear that $x \perp_{BJC}^{\epsilon} z$ as x is $\dashv_{BJC}^{\epsilon}$-symmetric in $\mathbb{X}$ with $\frac{z}{\|z\|_{\mathbb{X}}} \in M_{\mathcal{T}}$ and by using Theorem (3.1), and using the homogeneity of $\perp_{BJC}^{\epsilon}$, we must have $\mathcal{T}(z) \perp_{BJC}^{\epsilon} \mathcal{T}(x)$.

The next definition is a generalization of the definition preserve of Birkhoff -James orthogonality.
Definition 3.13: Let $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ and $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ be two Banach spaces. Then $\Theta \neq \mathcal{T} \in \mathcal{B}(\mathbb{X}, \mathbb{Y})$ is said to be a preserve of $\perp_{BJC}^{\epsilon}$ -orthogonality (in short $\mathcal{P}(\perp_{BJC}^{\epsilon})$), if $\mathcal{T}(x) \perp_{BJC}^{\epsilon} \mathcal{T}(y)$ whenever $x \perp_{BJC}^{\epsilon} y$, for any two vectors x and y in $\mathbb{X}$.

Example 3.14:

i. Consider $\mathcal{T} \in \mathcal{B}(l_\infty(\mathbb{R}^2))$, which is defined by:
$\mathcal{T}(1,0) = (0,1)$ and $\mathcal{T}(0,1) = (-1,0)$.
Then $(0,1) \in (1,1)^{\perp_{BJC}^{\epsilon}}$, but $\mathcal{T}(0,1) \notin (\mathcal{T}((1,1)))^{\perp_{BJC}^{\epsilon}}$ and hence $\mathcal{T}$ is $\mathcal{P}(\perp_{BJC}^{\epsilon})$.
ii. A linear operator $\mathcal{T} \in \mathcal{B}(l_\infty(\mathbb{R}^2))$ which is defined by $\mathcal{T}(x) = \mathcal{T}(x_1, x_2) = (x_1 - x_2, x_1 + x_2)$ for any vector x in $\mathbb{R}^2$. Let $x = (1,1)$ and $y = (1,0)$, with $\mathcal{T}(x) = (0,2)$ and $\mathcal{T}(y) = (1,1)$. Then $x \perp_{BJC}^{\epsilon} y$, but $\mathcal{T}(x) \not\perp_{BJC}^{\epsilon} \mathcal{T}(y)$. Thus $\mathcal{T}$ is not $\mathcal{P}(\perp_{BJC}^{\epsilon})$.

Theorem 3.15: Let x and y any two vectors in a normed space $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$. Then $x \perp_{BJC}^{\epsilon} y$, if and only if, there exists $\varphi \in \mathcal{J}(x)$ and $\varphi(y) = 0$.

Proof: For the proof of " if " part, we see that $\| x + \mu y \|_{\mathbb{X}}^2 \geq |\varphi(x + \mu y)|^2 = \| x \|_{\mathbb{X}}^2$ for all $\mu \in \mathbb{R}$.
This means, $\| x + \mu y \|_{\mathbb{X}}^2 \geq \| x \|_{\mathbb{X}}^2 - 2\epsilon \| x \|_{\mathbb{X}} \| \mu y \|_{\mathbb{X}}$ for all $\mu \in \mathbb{R}$. Implies that $x \perp_{BJC}^{\epsilon} y$.
For "only if " part, we may assume that for any non-zero vectors x and y. Also, it is enough to show only for case $\| x \|_{\mathbb{X}} = 1$, we have $x \perp_{BJC}^{\epsilon} y$ if and only if $\frac{x}{\|x\|_{\mathbb{X}}} \perp_{BJC}^{\epsilon} y$.
Now applying Theorem (2.7. ii), there exists $z \in Lin\{x, y\}$ with $x \perp_{BJ} z$ and $\| y - z \|_{\mathbb{X}} \leq \epsilon \| y \|_{\mathbb{X}}$. From the well-known separation theorem, there exists $\Theta \neq \varphi \in \mathbb{X}^*$ such that $\varphi(x) = \| x \|_{\mathbb{X}}$ with $\| \varphi \|_{\mathbb{X}^*} = 1$, whose $ker\varphi = Lin\{y\}$ (i.e. $\varphi(y) \neq 0$) . Let $0 < \varphi(x) < 1$ and $\| z \|_{\mathbb{X}} = 1$. Since $\varphi(x - \varphi(x)z) = 0$ implies that $x - \varphi(x)z = \eta y$ for some $0 \neq \eta \in \mathbb{R}$. This shows that:
$|\varphi(x)|^2 = \| \varphi(x)z \|_{\mathbb{X}}^2 = \| x - \eta y \|_{\mathbb{X}}^2 = \| x \|_{\mathbb{X}}{}^2 - 2\epsilon \| x \|_{\mathbb{X}} \| \eta y \|_{\mathbb{X}} < 1$;
Which contradicts to $x \perp_{BJC}^{\epsilon} y$.

Definition 3.16: Let $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ and $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ be two Banach spaces. Then $\Theta \neq \mathcal{T} \in \mathcal{B}(\mathbb{X}, \mathbb{Y})$ is said to be a preserve of $\perp_{BJC}^{\epsilon}$-orthogonality at x for some vectors x in $\mathbb{X}$ (in short $\mathcal{P}_x(\perp_{BJC}^{\epsilon})$), if $\mathcal{T}(x) \perp_{BJC}^{\epsilon} \mathcal{T}(y)$ whenever $x \perp_{BJC}^{\epsilon} y$, for any vector y in $\mathbb{X}$.

Theorem 3.17: Let $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ be a smooth Banach space, $\Theta \neq \mathcal{T} \in \mathcal{B}(\mathbb{X})$ and $x \in M_{\mathcal{T}}$. Then $\mathcal{T}$ is a $\mathcal{P}_x(\perp_{BJC}^{\epsilon})$.

Proof: If in addition, both x and $\mathcal{T}(x)$ are smooth points in $\mathbb{X}$, then for any vector y in $\mathbb{X}$, we must have $\mathcal{T}(x) \perp_{BJC}^{\epsilon} \mathcal{T}(y)$, whenever $x \perp_{BJC}^{\epsilon} y$. Without any loss of generally we can assume that $\| \mathcal{T} \|_{\mathcal{B}(\mathbb{X})} = 1$. From smoothness of x, there exists a unique support functional $\varphi \in \mathbb{X}^*$ at x. Again, from smoothness of $\mathcal{T}(x)$, there exists a unique support functional $\psi \in \mathbb{X}^*$ at $\mathcal{T}(x)$.
Now, $\psi(\mathcal{T}(x)) = \| \mathcal{T}(x) \|_{\mathbb{X}} = \| \mathcal{T} \|_{\mathcal{B}(\mathbb{X})} \| x \|_{\mathbb{X}} = 1$.
It is clear that $\psi o\mathcal{T} \in \mathbb{X}^*$ and $\| \psi o\mathcal{T} \|_{\mathbb{X}^*} \leq \| \psi \|_{\mathbb{X}^*} \| \mathcal{T} \|_{\mathcal{B}(\mathbb{X})} = 1$. So $\| \psi o\mathcal{T} \|_{\mathbb{X}^*} = 1$. From uniqueness of φ, we get $\varphi = \psi o\mathcal{T}$. Applying Theorem (3.15) as $x \perp_{BJC}^{\epsilon} y$, we must have $\varphi(y) = 0$. i.e. $\psi(\mathcal{T}(y)) = 0$. However, this equivalent to $\mathcal{T}(x) \perp_{BJC}^{\epsilon} \mathcal{T}(y)$.

Remark 3.18: It is interesting to observe that the smoothness condition in the last theorem is indeed required. We give the next example to illustrate our point.
Consider a linear operator $\mathcal{T} \in \mathcal{B}(l_\infty(\mathbb{R}^2))$ given by $\mathcal{T}(1,1) = (1,0)$ and $\mathcal{T}(1,-1) = (\frac{1}{2}, \frac{1}{2})$. It easy to check that $(1,1) \in M_{\mathcal{T}}$ with $(1,1) \perp_{BJC}^{\epsilon} (\frac{-1}{2}, 1)$. But $\mathcal{T}(\frac{-1}{2}, 1) = (\frac{-1}{8}, \frac{-3}{8}) \notin (\mathcal{T}(1,1))^{\perp_{BJC}^{\epsilon}}$. Hence $\mathcal{T}$ is not a $\mathcal{P}_{(1,1)}(\perp_{BJC}^{\epsilon})$.

Theorem 3.19: Let $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ be a reflexive and smooth Banach space. Then for any $\Theta \neq \mathcal{T} \in \mathcal{K}(\mathbb{X})$, there exists x in $\mathbb{S}_{\mathbb{X}}$ such that $\mathcal{T}$ is a $\mathcal{P}(\perp_{BJC}^{\epsilon})$.

Proof: Follows from Theorems (2.12), (3.1) and (3.17).

Recall that, in a smooth Banach space $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ both two types of orthognality $\perp_D^{\epsilon}$ and $\perp_{BJC}^{\epsilon}$ are equivalent, [13].

Theorem 3.20: Let $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ and $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ be two smooth Banach spaces with $\Theta \neq \mathcal{T} \in \mathcal{B}(\mathbb{X}, \mathbb{Y})$ be a $\mathcal{P}(\perp_{BJC}^{\epsilon})$, where $\mathcal{T}$ is an isometry multiplied by a positive constant. Assume that $\Theta \neq \mathcal{S} \in \mathcal{B}(\mathbb{X}, \mathbb{Y})$ such that $\| \mathcal{T} - \mathcal{S} \|_{\mathcal{B}(\mathbb{X},\mathbb{Y})} \leq \frac{\epsilon}{2-\epsilon} \| T \|_{\mathcal{B}(\mathbb{X},\mathbb{Y})}$. Then $\mathcal{S}$ is a $\mathcal{P}(\perp_{BJC}^{\epsilon})$.

Proof: For any two vectors x and y in $\mathbb{X}$ with $x \perp_{BJC}^{\epsilon} y$. Setting $\eta = \| \mathcal{T} \|_{\mathcal{B}(\mathbb{X},\mathbb{Y})}$ and $\delta = \frac{\epsilon\,\eta}{2-\epsilon} < \eta$, we have:

$\| \mathcal{T}(x) - \mathcal{S}(x) \|_{\mathbb{Y}} \leq \| \mathcal{T} - \mathcal{S} \|_{\mathcal{B}(\mathbb{X},\mathbb{Y})} \| x \|_{\mathbb{X}} \leq \delta \| x \|_{\mathbb{X}}, x \in \mathbb{X}$.

Now, from our hypothesis, we must have $\| \mathcal{T}(x) \|_{\mathbb{Y}} = \eta \| x \|_{\mathbb{X}}$ and hence:

$| \eta \| x \|_{\mathbb{X}} - \| \mathcal{S}(x) \|_{\mathbb{Y}} | = | \| \mathcal{T}(x) \|_{\mathbb{Y}} - \| \mathcal{S}(x) \|_{\mathbb{Y}} | \leq \| \mathcal{T}(x) - \mathcal{S}(x) \|_{\mathbb{Y}} \leq \delta \| x \|_{\mathbb{X}}, x \in \mathbb{X}$.

$\Rightarrow (\eta - \delta) \| x \|_{\mathbb{X}} \leq \| \mathcal{S}(x) \|_{\mathbb{Y}} \leq (\eta + \delta) \| x \|_{\mathbb{X}}$, for all $\in \mathbb{X}$.

$\Rightarrow \frac{\|\mathcal{S}(x)\|_{\mathbb{Y}}}{(\eta+\delta)} \leq \| x \|_{\mathbb{X}} \leq \frac{\|\mathcal{S}(x)\|_{\mathbb{Y}}}{(\eta-\delta)}$, for all $\in \mathbb{X}$.

Let $x \perp_{BJC}^{\epsilon} y$. Then for all $\mu \in \mathbb{R}$, we have $\| x + \mu y \|_{\mathbb{X}}^2 \geq \| x \|_{\mathbb{X}}^2 - 2\epsilon \| x \|_{\mathbb{X}} \| \mu y \|_{\mathbb{X}}$. Thus:

$$\| \mathcal{S}(x) + \mu \mathcal{S}(x) \|_{\mathbb{Y}}^2 = \| \mathcal{S}(x + \mu y) \|_{\mathbb{Y}}^2 \geq (\eta - \delta)^2 \| x + \mu y \|_{\mathbb{X}}^2$$
$$\geq (\eta - \delta)^2 \| x \|_{\mathbb{X}}^2 \geq \frac{(\eta-\delta)^2}{(\eta+\delta)^2} \| \mathcal{S}(x) \|_{\mathbb{Y}}^2 = \left(1 - \epsilon\right)^2 \| \mathcal{S}(x) \|_{\mathbb{Y}}^2.$$

This implies that $\mathcal{S}(x) \perp_{\mathcal{D}}^{\epsilon} \mathcal{S}(y)$. It follows that $\mathcal{S}(x) \perp_{BJC}^{\epsilon} \mathcal{S}(y)$. i.e. $\mathcal{S}$ is a $\mathcal{P}(\perp_{\mathrm{BJC}}^{\epsilon})$.

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