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The Numerical Solutions of Nonlinear Time-Fractional Differential Equations by LMADM

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Abstract

This paper presents a numerical scheme for solving nonlinear time-fractional differential equations in the sense of Caputo. This method relies on the Laplace transform together with the modified Adomian method (LMADM), compared with the Laplace transform combined with the standard Adomian Method (LADM). Furthermore, for the comparison purpose, we applied LMADM and LADM for solving nonlinear time-fractional differential equations to identify the differences and similarities. Finally, we provided two examples regarding the nonlinear time-fractional differential equations, which showed that the convergence of the current scheme results in high accuracy and small frequency to solve this type of equations**.**

Keywords: Fractional order differential equations, Caputo fractional derivative, Laplace Transform, Adomian decomposition methods.

الحلول العددية للمعادلات التفاضلية الغير خطية الكسورية الزمن بأستخدام *LMADM*

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الخلاصة

قدمت هذه الورقة صيغة عددية لحل المعادلات التفاضلية الغير خطية الكسورية الزمن بمفهوم Caputo. تعتمد هذه الطريقة على تحويل لابلاس وطريقة Adomian المحسنة (LMADM) معاً مقارنةً مع تحويل لابلاس وطريقة Adomian القياسية (LADM) ، أيضًا ؛ لغرض المقارنة ، قمنا بتطبيق LMADM و LADM لحل المعادلات التفاضلية الكسورية الزمن الغير خطية لمعرفة الاختلافات والتشابهات. وأخيرًا ، قدمنا مثالين فيما يتعلق بالمعادلات التفاضلية الكسورية الزمن الغير خطية والتي أظهرت أن التقارب للصيغة الحالية مع الدقة عالية والتكرارت الصغير لحل هذا النوع من المعادلات.

1. Introduction

Fractional calculus is a branch of mathematics that deals with real or complex number powers of the differential and integral operators. On the basis that the idea of fractional calculus was born more than three decades ago, serious efforts have been devoted to its modern study. Fractional differential equations (FDEs) are the generalization of the differential equations of integer order, studied through the theory of fractional calculus. In recent years, studies have been extensive about FDEs, due to their applications in many areas of vital research, such as physics, medicine, and engineering.

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Moreover, fractional calculus studies can allow the understanding of many fractal phenomena, which cannot be studied by ordinary means. There are many applications for solving FDEs, such as the fundamental function solutions to the fractional development advection-dispersion equation [1], the implicit difference approximation [2], the introduction of stable numerical schemes by replacing fractional quadrature rules to obtain the general solution for stability problems [3], the homotopy analysis method as an approximate technique to find Solitary wave solutions [4], and the Laplace and Fourier transforms [5]. In order to better understand the phenomenon described by a given nonlinear FDEs, the solutions of differential equations for the fractional order are largely involved. Fractional derivatives provide more accurate models of real world problems than integer order derivatives. Due to many applications in scientific fields, FDEs has been found to be an effective tool to describe some physical phenomena, such as propagation processes [6], properties of electrical and rheological materials, and viscosity theories [3]. It is important to resolve time FDEs. It was found that fractional time derivatives generally originate as infinitesimal generators of the time evolution when taken along the time scale boundaries. Thus, the importance of investigating FDEs arises from the need to refine the concepts of equilibrium, stability states, and evolution of time in a long time limit. In general, no method provides an accurate solution for non-linear FDEs. Several different and robust methods have been proposed to resolve FDEs to obtain approximate solutions.

In this paper, we will apply a combination of Laplace transform with MADM [1][7] to solve the general form of non-linear FDEs. [8]

$$D_t^{\alpha} u(x,t) = Lu(x,t) + Nu(x,t) + g(x,t) \qquad (1.1)$$

with, $n-1 < \alpha \leq n$, and subject to the initial condition,

$$\frac{\partial^r}{\partial t^r} u(x,0) = u^{(r)}(x,0) = f_r(x), r = 0(1)(n-1) \qquad (1.2)$$

where, $D_t^{\alpha} u(x,t) = L_t^{\alpha} u(x,t)$ is the Caputo fractional derivative, $g(x,t)$ is the source term, L is the linear operator, and N is the general nonlinear operator.

2. Definitions

We will adopt the Caputo definition for the concept of the fractional derivative, which is a modification of the Riemann–Liouville, there are many papers are recommended for more details on the geometric and physical interpretations for fractional derivatives of both Riemann-Liouville and Caputo types [9, 10]. The Caputo definition has the advantage of dealing properly with the initial value problems, in which the initial conditions are given in terms of the field variables and their integer-order, which is the case in most physical processes.

2.1 The fractional derivative of $f(x)$ in the Caputo sense is defined as [1][9]

$$D^{\alpha} f(x) = I^{n-\alpha} D^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} f^{(n)}(t) dt \qquad (2.1),$$

for $n-1 < \alpha \leq n$, $n \in \mathbb{N}$, $x > 0$, $f \in C_{-1}^{n}$.

In addition, we need here two of its basic properties.

Lemma 2.1: If $n-1 < \alpha \leq n$, $n \in \mathbb{N}$, $f \in C_{\mu}^{n}$ and $\mu \geq -1$, then [1][9]

$$D^{\alpha} I^{n} f(x) = f(x) \qquad (2.2),$$

$$I^{n} D^{\alpha} f(x) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(0) \frac{x^k}{k!} \quad , \quad x > 0 \quad (2.3).$$

3. An Analysis of the ADM and Modified ADM

Now consider the general non-linear FDEs written in an operator's form (1.1). By applying the inverse operator to both sides of (1.1), we get:

$$u(x,t) = f_r(x) + L_t^{-\alpha}[Lu(x,t) + Nu(x,t) + g(x,t)] \qquad (3.1)$$

Then the recursive relation according to the standard Adomian decomposition method (ADM) [7][11] is expressed as follows

$$u_0(x,t) = f_r(x) + L_t^{-\alpha}[g(x,t)],$$

$$u_{n+1}(x,t) = L_t^{-\alpha}[Lu(x,t) + A_n] \qquad (3.2),$$

where $$A_n(u_0,\dots,u_n)=\frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{n}\lambda^i u_i\right)\right]_{\lambda=0} \qquad (3.3).$$

We present MADM that will facilitate the calculations and accelerate the convergence by decomposing the nonlinear term into two parts, taking into consideration not to repeat the term for more than one time in computing the polynomials. Therefore, the polynomials for the nonlinear term $N(u)=\bar{N}(m_1,m_2)$ can be obtained as follows: [12]

$\bar{A}_0(u_0)=\bar{N}\left(m_{1_0},m_{2_0}\right)$

$\bar{A}_1(u_0,u_1)=\bar{N}\left(m_{1_0},m_{2_1}\right)+\bar{N}\left(m_{1_1},m_{2_0}\right)+\bar{N}\left(m_{1_1},m_{2_1}\right)$

$\bar{A}_2(u_0,u_1,u_2)=\bar{N}\left(m_{1_0},m_{2_2}\right)+\bar{N}\left(m_{1_2},m_{2_0}\right)+\bar{N}\left(m_{1_1},m_{2_2}\right)+\bar{N}\left(m_{1_2},m_{2_1}\right)+\bar{N}\left(m_{1_2},m_{2_2}\right)\vdots$

$\bar{A}_n$ can be finally written as in the following convenient relation

$$\bar{A}_i=\sum_{j=i}^{2i}\rho^j(\bar{N}(m_1,m_2))=\sum_{\substack{j=i\\k+h=j}}^{2i}\bar{N}\left(m_{1_k},m_{2_h}\right),\quad i,k,h=0,1,2,\dots \qquad (3.4),$$

where ρ is a decompose of the nonlinear term and $m_1,\ m_2$ represent the dependent variable u.

4. Laplace operation

The Laplace transform (LT) is a powerful tool in applied mathematics and engineering. It will allow us to transform FDEs into algebraic equations and then, by solving these algebraic equations, we can obtain the unknown function by using the inverse Laplace transform.

4.1 Laplace transform

Given a function $f(x)$ defined for $0<x<\infty$, the LT $F(s)$ is defined by

$$F(s)=\mathcal{L}[f(x)]=\int_0^\infty f(x)\,e^{-sx}dx \qquad (4.1),$$

at least for those s for which the integral converges.

4.2 Laplace Transform Properties [6]

1. $\mathcal{L}[f(x)\pm g(x)]=F(s)\pm G(s)$,
2. $\mathcal{L}[x^\alpha]=\frac{\Gamma(\alpha+1)}{s^\alpha}$,
3. $\mathcal{L}\left[f^{(n)}(x)\right]=s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-\cdots f^{(n-1)}(0)$,
4. $\mathcal{L}[x^nf(x)]=(-1)^nF^{(n)}(s)$,
5. $\mathcal{L}\left[\int_0^x f(t)\,dt\right]=\frac{F(s)}{s}$,
6. $\mathcal{L}\left[\int_0^x f(x-t)\,g(t)dt\right]=F(s)G(s)$.

Lemma 4.1: The Laplace transform of Caputo fractional derivative for $n-1<\alpha\le n,\ n\in\mathbb{N}$ can be obtained in the form of [11] [6]

$$\mathcal{L}[D^\alpha f(x)]=\frac{s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-\cdots f^{(n-1)}(0)}{s^{n-\alpha}} \qquad (4.2).$$

5. Applications

In this section, we consider two examples that demonstrate the performance and efficiency of the present algorithms for solving nonlinear FDEs. By the comparison with the exact solution, we report the absolute error which is defined by **Abs.error1**=|exact solution – LMADM| and **Abs.error2**=|exact solution – LADM|. All our calculations were achieved by using MAPLE software.

Example 1: Consider the following time-fractional nonlinear dispersive KdV equation:

$$\frac{\partial^\alpha u}{\partial t^\alpha}+a(u^2)_x+(u(u)_{xx})_x+(u(u)_x)_{xx}=0 \qquad (5.1),$$

with the initial condition

$$u(x,0)=\frac{1}{2}\left(1+cosh(Bx)\right), \qquad (5.2),$$

and the exact solution is

$$u(x,t)=\frac{1}{2}\left(1+cosh(B(x-\lambda_1t^\alpha)\right), \qquad (5.3),$$

where a is real valued constant and $(u(u)_{xx})_x$ and $(u(u)_x)_{xx}$ are dispersive terms. $\frac{\partial^\alpha u}{\partial t^\alpha} = D_t^\alpha u$ is applied to study the behaviour of weakly nonlinear ion acoustic waves in a plasma comprised of cold ions and hot isothermal electrons in the presence of a uniform magnetic field. The LT of (5.1) is converted to

$$\frac{sU(x,s) - u(x,0)}{s^{1-\alpha}} = -\mathcal{L}[a(u^2)_x + (u(u)_{xx})_x + (u(u)_x)_{xx}],$$

We shall apply (3.2) and then substitute the initial conditions (5.2).

$$U(x,s) = \frac{1}{s}u(x,0) - \frac{1}{s^\alpha}\mathcal{L}[2a\,A_n + 2B_n + 3C_n],$$

$N(u) = uu_x = \bar{N}(m_1, m_2)$ $\bar{A}_0 = u_0 u_{0_x}$, $\bar{A}_1 = u_0 u_{1_x} + u_1 u_{0_x} + u_1 u_{1_x}$, $\bar{A}_2 = u_0 u_{2_x} + u_2 u_{0_x} + u_1 u_{2_x} + u_2 u_{1_x} + u_2 u_{2_x}$, $\bar{A}_3 = u_0 u_{3_x} + u_3 u_{0_x} + u_1 u_{3_x} + u_3 u_{1_x} + u_2 u_{3_x} + u_3 u_{2_x} + u_3 u_{3_x}$, $\vdots$	While with the standard ADM, we get the following components: $A_0 = u_0 u_{0_x}$, $A_1 = u_0 u_{1_x} + u_1 u_{0_x}$, $A_2 = u_0 u_{2_x} + u_2 u_{0_x} + u_1 u_{1_x}$, $A_3 = u_0 u_{3_x} + u_3 u_{0_x} + u_1 u_{2_x} + u_2 u_{1_x}$, $\vdots$
and , $\bar{B}_1 = u_0 u_{1_{xxx}} + u_1 u_{0_{xxx}} + u_1 u_{1_{xxx}}$, $\bar{B}_2 = u_0 u_{2_{xxx}} + u_2 u_{0_{xxx}} + u_1 u_{2_{xxx}} + u_2 u_{1_{xxx}} + u_2 u_{2_{xxx}}$, $\bar{B}_3 = u_0 u_{3_{xxx}} + u_3 u_{0_{xxx}} + u_1 u_{3_{xxx}} + u_3 u_{1_{xxx}} + u_2 u_{3_{xxx}} + u_3 u_{2_{xxx}} + u_3 u_{3_{xxx}}$, $\vdots$	Also , $B_0 = u_0 u_{0_{xxx}}$, $B_1 = u_0 u_{1_{xxx}} + u_1 u_{0_{xxx}}$, $B_2 = u_0 u_{2_{xxx}} + u_2 u_{0_{xxx}} + u_1 u_{1_{xxx}}$, $B_3 = u_0 u_{3_{xxx}} + u_3 u_{0_{xxx}} + u_1 u_{2_{xxx}} + u_2 u_{1_{xxx}}$, $\vdots$

where, $uu_x = \sum_{n=0}^{\infty} A_n$, $uu_{xxx} = \sum_{n=0}^{\infty} B_n$, $u_x u_{xx} = \sum_{n=0}^{\infty} C_n$, and $n \geq 0$. Hence, by using the relations (3.4) and (3.3) for nonlinear terms, we can obtain the following forms of $\bar{A}_n$, $\bar{B}_n$, A_n and B_n, respectively:

Thus, in the same way, we calculate the other polynomials of $\bar{C}_n$ and C_n, using the inverse LT and, according to (3.2), the zeroth component u_0 is written as follows:

$$u_0(x,t) = \mathcal{L}^{-1}\left[\frac{1}{s}u(x,0)\right],$$

and the recursive relation can be written as follows:

$$u_0 = \frac{1}{2}(1 + \cosh(Bx)), \qquad u_{n+1}(x,t) = -\mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}[2a\,A_n + 2B_n + 3C_n]\right].$$

Both zeroth and the first components are similar in ADM and MADM, but the other components are different, which is why the approximate solution has higher accuracy and faster approximation to the exact solution than LADM, where the other components can be written as follows:

$$u_1 = -\frac{1}{4}\frac{\sinh(Bx)(2aB + 2B^3 + \cosh(\mathrm{B}x)\,B(5B^2 + 2a))t^\alpha}{\Gamma(1+\alpha)}$$

$$u_2 = \frac{1}{8}\,\frac{\mathbf{1}}{\Gamma(1+3\alpha)\Gamma(1+\alpha)^2}\Big(\big(2a^2 - 6\cosh(Bx)^2\,(a + 6B^2)(a + B^2) - 2cosh(Bx)^3(a + 10B^2)(5B^2 + 2a) + B^4\big(11 + 48cosh(Bx)\big) + aB^2\big(13 + 21cosh(Bx)\big)\big)\,t^{3\alpha}\Gamma(2\alpha+1)sinh(Bx)(5B^2 + 2a)B^3\Big) + 2a + \frac{1}{8}\frac{1}{\Gamma(2\alpha+1)}(B^2(-50B^4 - 40aB^2 - 8a^2))$$

$$\vdots$$

By collecting these components, we obtain the following numerical solutions:

$$u(x,t)=\frac{1}{2}+\frac{1}{2}\cosh(Bx)-\frac{1}{4}\frac{\sinh(Bx)\left(2aB+2B^3+\cosh(\mathrm{B}x)\,B(5B^2+2a)\right)t^{\alpha}}{\Gamma(1+\alpha)}$$
$$+\frac{1}{8}\frac{\mathbf{1}}{\Gamma(1+3\alpha)\Gamma(1+\alpha)^2}\Big(\Big(2a^2-6\,cosh(Bx)^2(a+6B^2)(a+B^2)$$
$$-2cosh(Bx)^3(a+10B^2)(5B^2+2a)+B^4\big(11+48cosh(Bx)\big)$$
$$+aB^2\big(13+21cosh(Bx)\big)\Big)\,t^{3\alpha}\Gamma(2\alpha+1)sinh(Bx)(5B^2+2a)B^3\Big)$$
$$+\frac{1}{8}\frac{1}{\Gamma(2\alpha+1)}\big(B^2\big(-50B^4-40aB^2-8a^2+4cosh(Bx)^2(5B^2+2a)^2$$
$$+6cosh(Bx)^3(a+6B^2)(5B^2+2a)-cosh(Bx)(121B^4+62aB^2+4a^2)\big)t^{2\alpha}\big),$$

which provides us with the closed form solutions (5.3) as in Table-1, where $\beta=\frac{k}{\sqrt{\frac{-3k}{a}}}$, $\lambda_1=\frac{2k^2a_1}{b_0\Gamma(1+\alpha)}$, $k=-0.5,\ a=1,\ a_1=2.5,\ b_0=0.5.$

Example 5.2: Consider the solution of generalized Hirota–Satsuma coupled KdV of time-fractional order[4]

$$\left.\begin{aligned} u_t^{\alpha}&=\frac{1}{2}u_{xxx}-3uu_x+3(vw)_x\,,\\ v_t^{\alpha}&=-v_{xxx}+3u\,v_x\ \ ,\\ w_t^{\alpha}&=-w_{xxx}+3u\,w_x, \end{aligned}\right\}\qquad (5.4),$$

with the initial conditions

$$\left.\begin{aligned} u(x,0)&=\frac{1}{3}(\beta-2k^2)+2k^2tanh^2[kx]\\ v(x,0)&=-\frac{4k^2c_0(\beta+k^2)}{3{c_1}^2}+\frac{4k^2(\beta+k^2)}{3c_1}tanh[kx]\\ w(x,0)&=c_{0+}+c_1tanh[kx] \end{aligned}\right\}\qquad (5.5),$$

and the boundary conditions will be found from the analytical solution

$$\left.\begin{aligned} u(x,t)&=\frac{1}{3}(\beta-2k^2)+2k^2\tanh^2[k(x-ct)]\\ v(x,t)&=-\frac{4k^2c_0(\beta+k^2)}{3{c_1}^2}+\frac{4k^2(\beta+k^2)}{3c_1}\tanh[k(x-ct)]\\ w(x,t)&=\ c_0+c_1\tanh[k(x-ct)] \end{aligned}\right\}\qquad (5.6),$$

where $k, c_0, c_1\neq 0$ and β are arbitrary constants. By using the Laplace transform of a given system to convert it into another system of PDEs

$$\left.\begin{aligned} \frac{sU(x,s)-u(x,0)}{s^{1-\alpha}}&=\mathcal{L}\left[\frac{1}{2}u_{xxx}-3uu_x+3(vw)_x\right],\\ \frac{sV(x,s)-v(x,0)}{s^{1-\alpha}}&=\mathcal{L}[-v_{xxx}+3u\,v_x],\\ \frac{sW(x,s)-w(x,0)}{s^{1-\alpha}}&=\mathcal{L}[-w_{xxx}+3u\,w_x]. \end{aligned}\right\}\qquad (5.7),$$

Again, by using the relations (3.4) and (3.3) for nonlinear terms, we can obtain the first few Adomian's polynomials of $\bar{A}_n$, A_n, $\bar{B}_n$, B_n , $\bar{C}_n$, C_n, $\bar{D}_n$ and D_n, respectively:

and, $\bar{B}_0=(v_0w_0)_x$, $\bar{B}_1=(v_0w_1)_x+(v_1w_0)_x+(v_1w_1)_x$, $\bar{B}_2=(v_0w_2)_x+(v_2w_0)_x+(v_1w_2)_x+(v_2w_1)_x+(v_2w_2)_x$, $\bar{B}_3=(v_0w_3)_x+(v_3w_0)_x+(v_1w_3)_x+(v_3w_1)_x+(v_2w_3)_x+(v_3w_2)_x$, $\vdots$	also, $B_0=(v_0w_0)_x$, $B_1=(v_0w_1)_x+(v_1w_0)_x$, $B_2=(v_0w_2)_x+(v_2w_0)_x+(v_1w_2)_x$, $B_3=(v_0w_3)_x+(v_3w_0)_x+(v_1w_2)_x+(v_1w_2)_x$, $\vdots$

and so on to the other polynomials $\bar{C}_n, C_n, \bar{D}_n$ and D_n. By using the inverse Laplace transform and according to (3.2), the zeroth components u_0 , v_0 and w_0 are written as follows:

$$\left.\begin{aligned} u_0(x,t) &= \mathcal{L}^{-1}\left[\frac{1}{s}u(x,0)\right], \\ v_0(x,t) &= \mathcal{L}^{-1}\left[\frac{1}{s}v(x,0)\right], \\ w_0(x,t) &= \mathcal{L}^{-1}\left[\frac{1}{s}w(x,0)\right], \end{aligned}\right\}$$

and the recursive relation can be written as follows:

$$\left.\begin{aligned} u_{n+1}(x,t) &= \mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}\left[\frac{1}{2}(u_n)_{xxx} - 3A_n + 3B_n\right]\right], \\ v_{n+1}(x,t) &= \mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}[-(v_n)_{xxx} + 3C_n]\right], \\ w_{n+1}(x,t) &= \mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}[-(w_n)_{xxx} + 3D_n]\right], \end{aligned}\right\}$$

where $n \geq 0$. Hence, we obtain the following components:

$$u_0 = \frac{1}{3}\beta - \frac{2}{3}k^2 + 2k^2\tanh(kx)^2,$$

$$v_0 = -\frac{4}{3}\frac{k^2c_0(\beta+k^2)}{{c_1}^2} + \frac{4}{3}\frac{k^2(\beta+k^2)}{c_1}\tanh(kx),$$

$$w_0 = c_0 + c_1\tanh(kx),$$

$$u_1 = \frac{4k^3\sinh(kx)\,\beta t^\alpha}{\Gamma(\alpha+1)\cosh(kx)^3},$$

$$v_1 = \frac{4k^3(\beta+k^2)\,\beta t^\alpha}{3\Gamma(\alpha+1)\cosh(kx)^2c_1},$$

$$w_1 = \frac{c_1k\,\beta t^\alpha}{\Gamma(\alpha+1)\cosh(kx)^2},$$

$$u_2 = \frac{4\beta^2k^4}{\Gamma(\alpha+1)^2\cosh(kx)^7}\left(\frac{(-2\cosh(kx)^2+3)\Gamma(\alpha+1)^2\cosh(kx)^3\,t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{4\left(-9k^2+\cosh(kx)^2(-\beta+5k^2)\right)\Gamma(\alpha+1)\sinh(kx)kt^{3\alpha}}{\Gamma(3\alpha+1)}\right),$$

$$v_2 = -\frac{8}{3}\frac{\beta^2(\beta+k^2)\sinh(kx)\,k^4}{c_1\Gamma(\alpha+1)^2\cosh(kx)^6}\left(\frac{\Gamma(\alpha+1)^2\cosh(kx)^3\,t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{12k^3\Gamma(2\alpha+1)\sinh(kx)\,kt^{3\alpha}}{\Gamma(3\alpha+1)}\right),$$

$$w_1 = -2c_1\beta^2\left(\frac{\sinh(kx)\,k^2t^{2\alpha}}{\Gamma(2\alpha+1)\cosh(kx)^3} + \frac{12\sinh(kx)^2\,\Gamma(2\alpha+1)k^5t^{3\alpha}}{\Gamma(3\alpha+1)\cosh(kx)^6\Gamma(\alpha+1)^2}\right),$$

$$\vdots$$

By collecting all these components, we can obtain the closed form solutions (5.6)

$$u(x,t) = \frac{1}{3}\beta - \frac{2}{3}k^2 + 2k^2\tanh(kx)^2 + \frac{4k^3\sinh(kx)\,\beta\,t^{\alpha}}{\Gamma(\alpha)\,\alpha\cosh(kx)^3} - \frac{8\beta^2k^4\left(t^{\alpha}\right)^2\sqrt{\pi}}{\Gamma(\alpha)\,\alpha\cosh(kx)^2\left(2^{\alpha}\right)^2\Gamma\left(\alpha+\frac{1}{2}\right)}$$
$$+\frac{12\beta^2k^4\left(t^{\alpha}\right)^2\sqrt{\pi}}{\Gamma(\alpha)\,\alpha\cosh(kx)^4\left(2^{\alpha}\right)^2\Gamma\left(\alpha+\frac{1}{2}\right)} - \frac{96\beta^2k^7\sinh(kx)\sqrt{\pi}\left(2^{\alpha}\right)^2\Gamma\left(\alpha+\frac{1}{2}\right)\left(t^{\alpha}\right)^3\sqrt{3}}{\Gamma(\alpha)^2\alpha^2\cosh(kx)^7\left(3^{\alpha}\right)^3\Gamma\left(\alpha+\frac{1}{3}\right)\Gamma\left(\alpha+\frac{2}{3}\right)}$$
$$-\frac{32}{3}\frac{\beta^3k^5\sinh(kx)\sqrt{\pi}\left(2^{\alpha}\right)^2\Gamma\left(\alpha+\frac{1}{2}\right)\left(t^{\alpha}\right)^3\sqrt{3}}{\Gamma(\alpha)^2\alpha^2\cosh(kx)^5\left(3^{\alpha}\right)^3\Gamma\left(\alpha+\frac{1}{3}\right)\Gamma\left(\alpha+\frac{2}{3}\right)} + \frac{160}{3}\frac{\beta^2k^7\sinh(kx)\sqrt{\pi}\left(2^{\alpha}\right)^2\Gamma\left(\alpha+\frac{1}{2}\right)\left(t^{\alpha}\right)^3\sqrt{3}}{\Gamma(\alpha)^2\alpha^2\cosh(kx)^5\left(3^{\alpha}\right)^3\Gamma\left(\alpha+\frac{1}{3}\right)\Gamma\left(\alpha+\frac{2}{3}\right)}$$

$$v(x,t) = -\frac{4}{3}\frac{k^2c_0\beta}{c_1^2} - \frac{4}{3}\frac{k^4c_0}{c_1^2} + \frac{4}{3}\frac{k^2\tanh(kx)\,\beta}{c_1} + \frac{4}{3}\frac{k^4\tanh(kx)}{c_1} + \frac{4}{3}\frac{k^3\beta^2t^{\alpha}}{\Gamma(\alpha)\,\alpha\cosh(kx)^2c_1}$$
$$+\frac{4}{3}\frac{k^5\beta t^{\alpha}}{\Gamma(\alpha)\,\alpha\cosh(kx)^2c_1} - \frac{8}{3}\frac{\beta^3\sinh(kx)\,k^4\left(t^{\alpha}\right)^2\sqrt{\pi}}{c_1\Gamma(\alpha)\,\alpha\cosh(kx)^3\left(2^{\alpha}\right)^2\Gamma\left(\alpha+\frac{1}{2}\right)} - \frac{64}{3}\frac{\beta^3\sinh(kx)^2k^7\sqrt{\pi}\left(2^{\alpha}\right)^2\Gamma\left(\alpha+\frac{1}{2}\right)\left(t^{\alpha}\right)^3\sqrt{3}}{c_1\Gamma(\alpha)^2\alpha^2\cosh(kx)^6\left(3^{\alpha}\right)^3\Gamma\left(\alpha+\frac{1}{3}\right)\Gamma\left(\alpha+\frac{2}{3}\right)}$$
$$-\frac{8}{3}\frac{\beta^2\sinh(kx)\,k^6\left(t^{\alpha}\right)^2\sqrt{\pi}}{c_1\Gamma(\alpha)\,\alpha\cosh(kx)^3\left(2^{\alpha}\right)^2\Gamma\left(\alpha+\frac{1}{2}\right)} - \frac{64}{3}\frac{\beta^2\sinh(kx)^2k^9\sqrt{\pi}\left(2^{\alpha}\right)^2\Gamma\left(\alpha+\frac{1}{2}\right)\left(t^{\alpha}\right)^3\sqrt{3}}{c_1\Gamma(\alpha)^2\alpha^2\cosh(kx)^6\left(3^{\alpha}\right)^3\Gamma\left(\alpha+\frac{1}{3}\right)\Gamma\left(\alpha+\frac{2}{3}\right)},$$

$$w(x,t) = c_0 + c_1\tanh(kx) + \frac{c_1k\beta t^{\alpha}}{\Gamma(\alpha)\,\alpha\cosh(kx)^2} - \frac{2c_1\beta^2\sinh(kx)\,k^2\left(t^{\alpha}\right)^2\sqrt{\pi}}{\alpha\left(2^{\alpha}\right)^2\Gamma(\alpha)\,\Gamma\left(\alpha+\frac{1}{2}\right)\cosh(kx)^3}$$
$$-\frac{16c_1\beta^2\sinh(kx)^2\sqrt{\pi}\left(2^{\alpha}\right)^2\Gamma\left(\alpha+\frac{1}{2}\right)k^5\left(t^{\alpha}\right)^3\sqrt{3}}{\alpha^2\Gamma(\alpha)^2\left(3^{\alpha}\right)^3\Gamma\left(\alpha+\frac{1}{3}\right)\Gamma\left(\alpha+\frac{2}{3}\right)\cosh(kx)^6}.$$

The numerical results are listed in Tables 2a, 2b, and 2c for $u(x,t), v(x,t)$and $w(x,t)$, respectively, where $k = 0.1, \beta = 1.5, c_1 = 1.5, c_0 = 1.5$ and $c = -1.5$.

Table 1-The numerical results in comparison with the analytical solutions for various values of α , x and t for Example 5.1

		α = 0.5		α = 0.75		α = 1.0	
t	***x***	**Abs. error1**	**Abs. error2**	**Abs. error1**	**Abs. error2**	**Abs. error1**	**Abs. error2**
0.1	0.25	0.033573465	0.035078182	0.005755138	0.005927608	0.000592224	0.000608784
	0.50	0.035736217	0.038926279	0.006480788	0.006846431	0.001175507	0.001210615
	0.75	0.040155012	0.045415638	0.007558588	0.008161559	0.001722470	0.001780365
	1.00	0.046766167	0.054738582	0.008888380	0.009802174	0.002156088	0.002243827
0.3	0.25	0.096961220	0.104779961	0.028420468	0.030463317	0.002118922	0.002566041
	0.50	0.101948887	0.118524935	0.031898155	0.036229079	0.004915917	0.005863828
	0.75	0.113738563	0.141073582	0.037993742	0.045135726	0.008399976	0.009963146

	1.00	0.132084780	0.173510663	0.046606630	0.057430218	0.012447274	0.014816238
0.5	0.25	0.155554096	0.172377348	0.057480538	0.063928107	0.002632510	0.004702507
	0.50	0.161352982	0.197018958	0.063819825	0.077488933	0.008822848	0.013211326
	0.75	0.177954506	0.236770101	0.075837002	0.098378281	0.017312487	0.024549383
	1.00	0.204571142	0.293705450	0.093302035	0.127463065	0.027966547	0.038933974

Table 2-Comparison of the numerical results of the analytical solutions with LMADM and LADM for various values of α , x and t for Example 5.2,

Table 2a							
		α = 0.5		α = 0.75		α = 1.0	
t	**x**	**Abs. error1**	**Abs. error2**	**Abs. error1**	**Abs. error2**	**Abs.error1**	**Abs. error2**
0.1	0.25	1.23364010^{-4}	1.23783210^{-4}	3.0836010^{-5}	3.0884010^{-5}	$5.\ 10^{-10}$	$5.2\ 10^{-9}$
	0.50	1.60614210^{-4}	1.61448810^{-4}	4.4608110^{-5}	4.4703810^{-5}	$4.\ 10^{-10}$	$9.6\ 10^{-9}$
	0.75	1.97071410^{-4}	1.98313910^{-4}	5.8159410^{-5}	5.8301810^{-5}	$3.\ 10^{-10}$	$1.40\ 10^{-8}$
	1.00	2.32564210^{-4}	2.34203810^{-4}	7.1424310^{-5}	7.1612210^{-5}	$2.\ 10^{-10}$	$1.82\ 10^{-8}$
0.3	0.25	2.74590110^{-4}	2.767682110^{-4}	9.1317610^{-5}	9.1886710^{-5}	5.0810^{-8}	$1.723\ 10^{-7}$
	0.50	3.18265110^{-4}	3.22601710^{-4}	1.11380010^{-4}	1.12513010^{-4}	$4.63\ 10^{-8}$	$2.943\ 10^{-7}$
	0.75	3.60380010^{-4}	3.66836510^{-4}	1.30893410^{-4}	1.32580410^{-4}	$4.09\ 10^{-8}$	$4.102\ 10^{-7}$
	1.00	4.00751510^{-4}	4.09270910^{-4}	1.49768510^{-4}	1.51994410^{-4}	$3.45\ 10^{-8}$	$5.217\ 10^{-7}$
0.5	0.25	3.77593210^{-4}	3.82279610^{-4}	1.47756310^{-4}	1.49552410^{-4}	4.00510^{-7}	$9.771\ 10^{-7}$
	0.50	4.15397510^{-4}	4.24728510^{-4}	1.67492010^{-4}	1.71068210^{-4}	$3.754\ 10^{-7}$	$1.5235\ 10^{-7}$
	0.75	4.51175310^{-4}	4.65067610^{-4}	1.86404610^{-4}	1.91728910^{-4}	$3.443\ 10^{-7}$	$2.0537\ 10^{-7}$
	1.00	4.84788910^{-4}	5.03119810^{-4}	2.04413910^{-4}	2.11439310^{-4}	$3.072\ 10^{-7}$	$2.5627\ 10^{-7}$

Table 2b							
		α = 0.5		α = 0.75		α = 1.0	
t	x	**Abs. error1**	**Abs. error2**	**Abs. error1**	**Abs. error2**	**Abs. error1**	**Abs. error2**
0.1	0.25	$5.1533255\ 10^{-4}$	$5.1533269\ 10^{-4}$	$1.8783730\ 10^{-4}$	$1.8783732\ 10^{-4}$	$1.505\ 10^{-8}$	$1.505\ 10^{-8}$
	0.50	$5.1293837\ 10^{-4}$	$5.1293892\ 10^{-4}$	$1.8720289\ 10^{-4}$	$1.8720295\ 10^{-4}$	$1.493\ 10^{-8}$	$1.493\ 10^{-8}$
	0.75	$5.0991844\ 10^{-4}$	$5.0991965\ 10^{-4}$	$1.8633835\ 10^{-4}$	$1.8633849\ 10^{-4}$	$1.472\ 10^{-8}$	$1.473\ 10^{-8}$
	1.00	$5.0628770\ 10^{-4}$	$5.0628983\ 10^{-4}$	$1.8524797\ 10^{-4}$	$1.8524821\ 10^{-4}$	$1.444\ 10^{-8}$	$1.446\ 10^{-8}$
0.3	0.25	$6.3647537\ 10^{-4}$	$6.3647608\ 10^{-4}$	$2.8303898\ 10^{-4}$	$2.8303917\ 10^{-4}$	$4.0541\ 10^{-7}$	$4.0545\ 10^{-7}$
	0.50	$6.3144311\ 10^{-4}$	$6.3144595\ 10^{-4}$	$2.8132302\ 10^{-4}$	$2.8132376\ 10^{-4}$	$4.0133\ 10^{-7}$	$4.0149\ 10^{-7}$
	0.75	$6.2565362\ 10^{-4}$	$6.2565994\ 10^{-4}$	$2.7926519\ 10^{-4}$	$2.7926684\ 10^{-4}$	$3.9521\ 10^{-7}$	$3.9558\ 10^{-7}$

	1.00	6.1913550 10^{-4}	6.1914661 10^{-4}	2.7687571 10^{-4}	2.7687861 10^{-4}	3.8712 10^{-7}	3.8775 10^{-7}
0.5	0.25	5.9557846 10^{-4}	5.9557999 10^{-4}	2.9545385 10^{-4}	2.9545444 10^{-4}	1.87134 10^{-6}	1.87153 10^{-6}
	0.50	5.8880117 10^{-4}	5.8880727 10^{-4}	2.9276132 10^{-4}	2.9276366 10^{-4}	1.84979 10^{-6}	1.85054 10^{-6}
	0.75	5.8132618 10^{-4}	5.8133978 10^{-4}	2.8971370 10^{-4}	2.8971891 10^{-4}	1.81881 10^{-6}	1.82049 10^{-6}
	1.00	5.7319055 10^{-4}	5.7321445 10^{-4}	2.8632623 10^{-4}	2.8633539 10^{-4}	1.77879 10^{-6}	1.78173 10^{-6}

Table 2c

		α = 0.5		**α = 0.75**		**α = 1.0**	
t	x	**Abs. error1**	**Abs. error2**	**Abs. error1**	**Abs. error2**	**Abs. error1**	**Abs. error2**
0.1	0.25	5.7590972 10^{-2}	5.7590988 10^{-2}	2.0991751 10^{-2}	2.0991753 10^{-2}	1.68110^{-6}	1.68210^{-6}
	0.50	5.7323410 10^{-2}	5.7323471 10^{-2}	2.0920852 10^{-2}	2.0920859 10^{-2}	1.66710^{-6}	1.66810^{-6}
	0.75	5.6985919 10^{-2}	5.6986055 10^{-2}	2.0824236 10^{-2}	2.0824252 10^{-2}	1.64510^{-6}	1.64610^{-6}
	1.00	5.6580165 10^{-2}	5.6580404 10^{-2}	2.0702380 10^{-2}	2.0702408 10^{-2}	1.61410^{-6}	1.61610^{-6}
0.3	0.25	7.1129282 10^{-2}	7.1129362 10^{-2}	3.1631011 10^{-2}	3.1631032 10^{-2}	4.530410^{-5}	4.530910^{-5}
	0.50	7.0566904 10^{-2}	7.0567221 10^{-2}	3.1439246 10^{-2}	1.439328 10^{-2}	4.485110^{-5}	4.486910^{-5}
	0.75	6.9919899 10^{-2}	6.9920605 10^{-2}	3.1209273 10^{-2}	3.1209458 10^{-2}	4.416810^{-5}	4.420810^{-5}
	1.00	6.9191468 10^{-2}	6.9192709 10^{-2}	3.0942235 10^{-2}	3.0942560 10^{-2}	4.326310^{-5}	4.333410^{-5}
0.5	0.25	6.6558850 10^{-2}	6.6559021 10^{-2}	3.3018434 10^{-2}	3.3018500 10^{-2}	2.09131 10^{-4}	2.09152 10^{-4}
	0.50	6.5801454 10^{-2}	6.5802135 10^{-2}	3.2717531 10^{-2}	3.2717792 10^{-2}	2.06721 10^{-4}	2.06805 10^{-4}
	0.75	6.4966088 10^{-2}	6.4967608 10^{-2}	3.2376946 10^{-2}	3.2377529 10^{-2}	2.03263 10^{-4}	2.03450 10^{-4}
	1.00	6.4056891 10^{-2}	6.40595624 10^{-2}	3.1998380 10^{-2}	3.1999403 10^{-2}	1.98789 10^{-4}	1.99118 10^{-4}

6. Conclusions

In this article, we found the solutions of nonlinear time-fractional differential equations by combining Laplace transform with the modified Adomian decomposition method (LMADM). We conclude that the results obtained by using this method are effective; they require a small number of iterations and high accuracy to solve different nonlinear fractional differential equations and their good convergence, compared with the Laplace transform combined with the standard Adomian analysis method (LADM).

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