The Numerical Solutions of Nonlinear Time-Fractional Differential Equations by LMADM

Hameeda Oda AL-Humedi*, Faeza Lafta Hasan
Department of Mathematics, College of Education for Pure Sciences, Basrah University, Basrah, Iraq

Abstract
This paper presents a numerical scheme for solving nonlinear time-fractional differential equations in the sense of Caputo. This method relies on the Laplace transform together with the modified Adomian method (LMADM), compared with the Laplace transform combined with the standard Adomian Method (LADM). Furthermore, for the comparison purpose, we applied LMADM and LADM for solving nonlinear time-fractional differential equations to identify the differences and similarities. Finally, we provided two examples regarding the nonlinear time-fractional differential equations, which showed that the convergence of the current scheme results in high accuracy and small frequency to solve this type of equations.

Keywords: Fractional order differential equations, Caputo fractional derivative, Laplace Transform, Adomian decomposition methods.

1. Introduction
Fractional calculus is a branch of mathematics that deals with real or complex number powers of the differential and integral operators. On the basis that the idea of fractional calculus was born more than three decades ago, serious efforts have been devoted to its modern study. Fractional differential equations (FDEs) are the generalization of the differential equations of integer order, studied through the theory of fractional calculus. In recent years, studies have been extensive about FDEs, due to their applications in many areas of vital research, such as physics, medicine, and engineering.

*Email:fahas90@yahoo.co.uk
Moreover, fractional calculus studies can allow the understanding of many fractal phenomena, which cannot be studied by ordinary means. There are many applications for solving FDEs, such as the fundamental function solutions to the fractional development advection-dispersion equation [1], the implicit difference approximation [2], the introduction of stable numerical schemes by replacing fractional quadrature rules to obtain the general solution for stability problems [3], the homotopy analysis method as an approximate technique to find Solitary wave solutions [4], and the Laplace and Fourier transforms [5]. In order to better understand the phenomenon described by a given nonlinear FDEs, the solutions of differential equations for the fractional order are largely involved. Fractional derivatives provide more accurate models of real world problems than integer order derivatives. Due to many applications in scientific fields, FDEs has been found to be an effective tool to describe some physical phenomena, such as propagation processes [6], properties of electrical and rheological materials, and viscosity theories [3]. It is important to resolve time FDEs. It was found that fractional time derivatives generally originate as infinitesimal generators of the time evolution when taken along the time scale boundaries. Thus, the importance of investigating FDEs arises from the need to refine the concepts of equilibrium, stability states, and evolution of time in a long time limit. In general, no method provides an accurate solution for non-linear FDEs. Several different and robust methods have been proposed to resolve FDEs to obtain approximate solutions.

In this paper, we will apply a combination of Laplace transform with MADM [1][7] to solve the general form of non-linear FDEs. [8]

\[ D^\alpha_t u(x, t) = Lu(x, t) + Nu(x, t) + g(x, t) \]  

(1.1)

with, \( n - 1 < \alpha \leq n \), and subject to the initial condition,

\[ \frac{\partial^r}{\partial t^r} u(x, 0) = u^{(r)}(x, 0) = f_r(x), r = 0(1)(n - 1) \]  

(1.2)

where, \( D_t^\alpha u(x, t) = L_t^\alpha u(x, t) \) is the Caputo fractional derivative, \( g(x, t) \) is the source term, \( L \) is the linear operator, and \( N \) is the general nonlinear operator.

2. Definitions

We will adopt the Caputo definition for the concept of the fractional derivative, which is a modification of the Riemann–Liouville, there are many papers are recommended for more details on the geometric and physical interpretations for fractional derivatives of both Riemann-Liouville and Caputo types [9, 10]. The Caputo definition has the advantage of dealing properly with the initial value problems, in which the initial conditions are given in terms of the field variables and their integer-order, which is the case in most physical processes.

2.1 The fractional derivative of \( f(x) \) in the Caputo sense is defined as [1][9]

\[ D^\alpha f(x) = I^{n-\alpha} D^n f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} f^{(n)}(t)dt \]  

(2.1)

for \( n - 1 < \alpha \leq n \), \( n \in \mathbb{N} \), \( x > 0 \), \( f \in C^n \).

In addition, we need here two of its basic properties.

**Lemma 2.1:** If \( n - 1 < \alpha \leq n \), \( n \in \mathbb{N} \), \( f \in C^n_\mathbb{R} \) and \( \mu \geq -1 \), then [1][9]

\[ D^\alpha I^n f(x) = f(x) \]  

(2.2)

\[ I^n D^\alpha f(x) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(0) \frac{x^k}{k!} \]  

(2.3)

3. An Analysis of the ADM and Modified ADM

Now consider the general non-linear FDEs written in an operator’s form (1.1). By applying the inverse operator to both sides of (1.1), we get:

\[ u(x, t) = f_r(x) + L_t^{\alpha}[Lu(x, t) + Nu(x, t) + g(x, t)] \]  

(3.1)

Then the recursive relation according to the standard Adomian decomposition method (ADM) [7][11] is expressed as follows

\[ u_0(x, t) = f_r(x) + L_t^{\alpha}[g(x, t)] \]

\[ u_{n+1}(x, t) = L_t^{\alpha}[Lu(x, t) + A_n] \]  

(3.2)
We present MADM that will facilitate the calculations and accelerate the convergence by decomposing the nonlinear term into two parts, taking into consideration not to repeat the term for more than one time in computing the polynomials. Therefore, the polynomials for the nonlinear term \( N(u) = \bar{N}(m_1, m_2) \) can be obtained as follows: [12]

\[
\bar{A}_1(u_0, u_1) = \bar{N}(m_{10}, m_{21}) + \bar{N}(m_{11}, m_{20}) + \bar{N}(m_{11}, m_{21}),
\]

\[
\bar{A}_2(u_0, u_1, u_2) = \bar{N}(m_{10}, m_{22}) + \bar{N}(m_{12}, m_{20}) + \bar{N}(m_{12}, m_{21}) + \bar{N}(m_{12}, m_{22}) + \bar{N}(m_{12}, m_{22}):
\]

\[
\bar{A}_n = \sum_{\rho=0}^{\infty} \rho! (\bar{N}(m_1, m_2)) = \sum_{j=1}^{\infty} \bar{N}(m_{1k}, m_{2h}), \quad i, k, h = 0, 1, 2, ... \tag{3.4},
\]

where \( \rho \) is a decompose of the nonlinear term and \( m_1, m_2 \) represent the dependent variable \( u \).

4. Laplace operation

The Laplace transform (LT) is a powerful tool in applied mathematics and engineering. It will allow us to transform FDEs into algebraic equations and then, by solving these algebraic equations, we can obtain the unknown function by using the inverse Laplace transform.

4.1 Laplace transform

Given a function \( f(x) \) defined for \( 0 < x < \infty \), the LT for \( F(s) \) is defined by

\[
F(s) = L[f(x)] = \int_0^\infty f(x) e^{-sx} dx \tag{4.1},
\]

at least for those \( s \) for which the integral converges.

4.2 Laplace Transform Properties [6]

1. \( L[f(x) \pm g(x)] = F(s) \pm G(s) \),
2. \( L[x^\alpha] = \frac{\Gamma(\alpha+1)}{s^\alpha} \),
3. \( L[f^{(n)}(x)] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - ... - f^{(n-1)}(0) \),
4. \( L[x^n f(x)] = (-1)^n F^{(n)}(s) \),
5. \( L \left( \int_0^x f(t) \, dt \right) = \frac{F(s)}{s} \),
6. \( L \left( \int_0^\infty f(x-t) g(t) \, dt \right) = F(s) G(s) \).

Lemma 4.1: The Laplace transform of Caputo fractional derivative for \( n - 1 < \alpha \leq n \) \( n \in \mathbb{N} \) can be obtained in the form of [11] [6]

\[
L[D^\alpha f(x)] = \frac{s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - ... - f^{(n-1)}(0)}{s^{n-\alpha}} \tag{4.2}.
\]

5. Applications

In this section, we consider two examples that demonstrate the performance and efficiency of the present algorithms for solving nonlinear FDEs. By the comparison with the exact solution, we report the absolute error which is defined by \( \text{Abs.error1} = |\text{exact solution} - \text{LMADM}| \) and \( \text{Abs.error2} = |\text{exact solution} - \text{LADM}| \). All our calculations were achieved by using MAPLE software.

Example 1: Consider the following time-fractional nonlinear dispersive KdV equation:

\[
\frac{\partial^\alpha u}{\partial t^\alpha} + a(u^2)u + (u u_{xx})_x + (u_{xx}) u_x = 0 \tag{5.1},
\]

with the initial condition

\[
u(x, 0) = \frac{1}{2} (1 + \cosh(Bx)), \tag{5.2}.
\]

and the exact solution is

\[
u(x, t) = \frac{1}{2} (1 + \cosh(B(x - \lambda t^\alpha))), \tag{5.3},
\]
where \( a \) is a real valued constant and \((u(u)_{xx})_x\) and \((u(u)_{x})_{xx}\) are dispersive terms. \( \frac{\partial^6 u}{\partial x^6} = D^6_t u \) is applied to study the behaviour of weakly nonlinear ion acoustic waves in a plasma comprised of cold ions and hot isothermal electrons in the presence of a uniform magnetic field. The LT of (5.1) is converted to

\[
\frac{sU(x, s) - u(x, 0)}{s^{1+\alpha}} = -\mathcal{L}[a(u^2)_x + (u(u)_{xx})_x + (u(u)_{x})_{xx}].
\]

We shall apply (3.2) and then substitute the initial conditions (5.2).

\[
U(x, s) = \frac{1}{s}u(x, 0) - \frac{1}{s^{\alpha}}\mathcal{L}[2aA_n + 2B_n + 3C_n],
\]

|\( N(u) = uu_x = N(m_1, m_2) \) | While with the standard ADM, we get the following components:
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tilde{A}<em>0 = u_0u</em>{0x} )</td>
<td>( A_0 = u_0u_{0x} )</td>
</tr>
<tr>
<td>( \tilde{A}<em>1 = u_0u</em>{1x} + u_1u_{0x} + u_1u_{1x} )</td>
<td>( A_1 = u_0u_{1x} + u_1u_{0x} )</td>
</tr>
<tr>
<td>( \tilde{A}<em>2 = u_0u</em>{2x} + u_2u_{0x} + u_2u_{2x} + u_2u_{1x} + u_2u_{2x} )</td>
<td>( A_2 = u_0u_{2x} + u_2u_{0x} + u_1u_{2x} + u_2u_{1x} )</td>
</tr>
<tr>
<td>( \tilde{A}<em>3 = u_0u</em>{3x} + u_3u_{0x} + u_1u_{3x} + u_3u_{1x} + u_2u_{3x} + u_3u_{2x} + u_3u_{3x} )</td>
<td>( A_3 = u_0u_{3x} + u_3u_{0x} + u_1u_{2x} + u_2u_{1x} )</td>
</tr>
</tbody>
</table>

\[
\text{and,} \quad \tilde{B}_1 = u_0u_{1xxx} + u_1u_{0xxx} + u_1u_{1xxx}, \quad \tilde{B}_2 = u_0u_{2xxx} + u_2u_{0xxx} + u_1u_{2xxx} + u_2u_{1xxx}, \quad \tilde{B}_3 = u_0u_{3xxx} + u_3u_{0xxx} + u_1u_{3xxx} + u_3u_{1xxx} + u_2u_{3xxx} + u_3u_{2xxx} + u_3u_{3xxx},
\]

while \( \tilde{B}_1 = u_0u_{1xxx} + u_1u_{0xxx} + u_1u_{1xxx} \), \( \tilde{B}_2 = u_0u_{2xxx} + u_2u_{0xxx} + u_1u_{2xxx} + u_2u_{1xxx} \), \( \tilde{B}_3 = u_0u_{3xxx} + u_3u_{0xxx} + u_1u_{3xxx} + u_3u_{1xxx} + u_2u_{3xxx} + u_3u_{2xxx} + u_3u_{3xxx} \), respectively:

Thus, in the same way, we calculate the other polynomials of \( \tilde{C}_n \) and \( C_n \), using the inverse LT and, according to (3.2), the zeroth component \( u_0 \) is written as follows:

\[
u_0(x, t) = \mathcal{L}^{-1}\left[\frac{1}{s}u(x, 0)\right],
\]

and the recursive relation can be written as follows:

\[
u_0 = \frac{1}{2}(1 + \cosh(Bx)), \quad u_{n+1}(x, t) = -\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}[2aA_n + 2B_n + 3C_n]\right].
\]

Both zeroth and the first components are similar in ADM and MADM, but the other components are different, which is why the approximate solution has higher accuracy and faster approximation to the exact solution than LADM, where the other components can be written as follows:

\[
u_1 = -\frac{1}{4}\frac{\sinh(Bx)(2aB + 2B^3 + \cosh(Bx)B(5B^2 + 2a))t^\alpha}{\Gamma(1 + \alpha)}
\]

\[
u_2 = \frac{1}{8}\frac{1}{\Gamma(1 + 3\alpha)\Gamma(1 + \alpha)^2}\left((2a^2 - 6\cosh(Bx)^2)(a + 6B^2)(a + B^2)
- 2\cosh(Bx)^3(a + 10B^2)(5B^2 + 2a) + B^4\left(11 + 48\cosh(Bx)\right)
+ aB^2\left(13 + 21\cosh(Bx)\right)\right)\frac{1}{\Gamma(2\alpha + 1)}(2a + 1)\sinh(Bx)(5B^2 + 2a)B^3 + 2a
+ \frac{1}{8}\frac{1}{\Gamma(2\alpha + 1)}(B^2(-50B^4 - 40aB^2 - 8a^2))
\]

By collecting these components, we obtain the following numerical solutions:
\[ u(x, t) = \frac{1}{2} + \frac{1}{2} \cosh(Bx) - \frac{1}{4} \frac{1}{\Gamma(1 + \alpha)} \left\{ \frac{1}{\Gamma(1 + 3\alpha)} \right\} \left( 2a^2 - 6 \cosh(Bx)^2 (a + B^2) \right) + 2 \cosh(Bx)^3 (a + 10B^2)(5B^2 + 2a) + B^4 \left( 11 + 48 \cosh(Bx) \right) + aB^2 (13 + 21 \cosh(Bx)) \right) t^{3\alpha} \Gamma(2\alpha + 1) \sinh(Bx)(5B^2 + 2a)B^3 \right) \]
\[ + \frac{2k^2 a_1}{b_0 \Gamma(1+\alpha)} \right), \quad k = -0.5, \quad a_1 = 1, \quad a_1 = 2.5, \quad b_0 = 0.5. \]

**Example 5.2:** Consider the solution of generalized Hirota–Satsuma coupled KdV of time-fractional order[4]

\[
\begin{align*}
\frac{1}{2} u_{xxx} - 3uu_x + 3(vw)_x, \\
\frac{1}{2} v_{xxx} + 3u v_x, \\
\frac{1}{2} w_{xxx} + 3u w_x,
\end{align*}
\]

with the initial conditions

\[
\begin{align*}
u(x, 0) &= \frac{1}{3} (\beta - 2k^2) + 2k^2 \tanh^2[kx] \\
v(x, 0) &= -\frac{4k^2 c_0 (\beta + k^2) + 4k^2 (\beta + k^2)}{3c_1^2} \tanh[kx] \\
w(x, 0) &= c_{0x} + c_1 \tanh[kx]
\end{align*}
\]

and the boundary conditions will be found from the analytical solution

\[
\begin{align*}
u(x, t) &= \frac{1}{3} (\beta - 2k^2) + 2k^2 \tanh^2[k(x - ct)] \\
v(x, t) &= -\frac{4k^2 c_0 (\beta + k^2) + 4k^2 (\beta + k^2)}{3c_1} \tanh[k(x - ct)] \\
w(x, t) &= c_{0x} + c_1 \tanh[k(x - ct)]
\end{align*}
\]

where \( k, c_0, c_1 \neq 0 \) and \( \beta \) are arbitrary constants. By using the Laplace transform of a given system to convert it into another system of PDEs

\[
\begin{align*}
\frac{sU(x, s) - u(x, 0)}{s^{1-\alpha}} &= L \left[ \frac{1}{2} u_{xxx} - 3uu_x + 3(vw)_x \right], \\
\frac{sV(x, s) - v(x, 0)}{s^{1-\alpha}} &= L \left[ -u_{xxx} + 3u v_x \right], \\
\frac{sW(x, s) - w(x, 0)}{s^{1-\alpha}} &= L \left[ -w_{xxx} + 3u w_x \right].
\end{align*}
\]

Again, by using the relations (3.4) and (3.3) for nonlinear terms, we can obtain the first few Adomian’s polynomials of \( A_n, B_n, C_n, D_n, \) and \( D_n, \) respectively:

\[
\begin{align*}
A_1 &= (v_0 w_0), \\
A_2 &= (v_0 w_1 + v_1 w_0)_x + (v_0 w_2)_x + (v_1 w_2)_x + (v_2 w_1)_x, \\
A_3 &= (v_0 w_3 + v_2 w_0)_x + (v_1 w_3 + v_3 w_1)_x \}
\]

\[
\begin{align*}
B_1 &= (v_0 w_1)_x + (v_1 w_0)_x + (v_2 w_0)_x, \\
B_2 &= (v_0 w_2)_x + (v_1 w_1)_x + (v_2 w_2)_x + (v_3 w_1)_x + (v_2 w_3)_x + (v_3 w_2)_x, \\
B_3 &= (v_0 w_3)_x + (v_3 w_0)_x + (v_4 w_1)_x + (v_3 w_3)_x + (v_4 w_2)_x + (v_5 w_2)_x, \\
B_4 &= (v_0 w_4)_x + (v_5 w_0)_x + (v_6 w_1)_x + (v_5 w_2)_x + (v_6 w_2)_x + (v_7 w_1)_x + (v_6 w_3)_x + (v_7 w_2)_x + (v_8 w_2)_x, \\
\end{align*}
\]

\[
\begin{align*}
C_1 &= (v_0 w_1)_x + (v_1 w_0)_x + (v_2 w_0)_x, \\
C_2 &= (v_0 w_2)_x + (v_1 w_1)_x + (v_2 w_2)_x + (v_3 w_1)_x + (v_2 w_3)_x + (v_3 w_2)_x, \\
C_3 &= (v_0 w_3)_x + (v_2 w_0)_x + (v_3 w_1)_x + (v_3 w_3)_x + (v_4 w_1)_x + (v_3 w_2)_x + (v_4 w_2)_x + (v_5 w_1)_x + (v_4 w_3)_x + (v_5 w_2)_x + (v_6 w_2)_x, \\
C_4 &= (v_0 w_4)_x + (v_3 w_0)_x + (v_4 w_1)_x + (v_4 w_2)_x + (v_5 w_1)_x + (v_5 w_2)_x + (v_6 w_1)_x + (v_5 w_3)_x + (v_6 w_2)_x + (v_7 w_1)_x + (v_6 w_3)_x + (v_7 w_2)_x + (v_8 w_1)_x + (v_7 w_3)_x + (v_8 w_2)_x + (v_9 w_2)_x, \\
\end{align*}
\]

\[
\begin{align*}
D_1 &= (v_0 w_1)_x + (v_1 w_0)_x + (v_2 w_0)_x, \\
D_2 &= (v_0 w_2)_x + (v_1 w_1)_x + (v_2 w_2)_x + (v_3 w_1)_x + (v_2 w_3)_x + (v_3 w_2)_x, \\
D_3 &= (v_0 w_3)_x + (v_2 w_0)_x + (v_3 w_1)_x + (v_3 w_3)_x + (v_4 w_1)_x + (v_3 w_2)_x + (v_4 w_2)_x + (v_5 w_1)_x + (v_4 w_3)_x + (v_5 w_2)_x + (v_6 w_1)_x + (v_5 w_3)_x + (v_6 w_2)_x + (v_7 w_1)_x + (v_6 w_3)_x + (v_7 w_2)_x + (v_8 w_1)_x + (v_7 w_3)_x + (v_8 w_2)_x + (v_9 w_1)_x + (v_8 w_3)_x + (v_9 w_2)_x + (v_{10} w_2)_x, \\
\end{align*}
\]

\[
\begin{align*}
D_4 &= (v_0 w_4)_x + (v_3 w_0)_x + (v_4 w_1)_x + (v_4 w_2)_x + (v_5 w_1)_x + (v_5 w_2)_x + (v_6 w_1)_x + (v_5 w_3)_x + (v_6 w_2)_x + (v_7 w_1)_x + (v_6 w_3)_x + (v_7 w_2)_x + (v_8 w_1)_x + (v_7 w_3)_x + (v_8 w_2)_x + (v_9 w_1)_x + (v_8 w_3)_x + (v_9 w_2)_x + (v_{10} w_1)_x + (v_9 w_3)_x + (v_{10} w_2)_x + (v_{11} w_2)_x, \\
\end{align*}
\]
and so on to the other polynomials $\tilde{C}_n, C_n, \tilde{D}_n$ and $D_n$. By using the inverse Laplace transform and according to (3.2), the zeroth components $u_0, v_0$ and $w_0$ are written as follows:

$$
\begin{align*}
   u_0(x, t) &= \mathcal{L}^{-1}\left[\frac{1}{s} u(x, 0)\right], \\
   v_0(x, t) &= \mathcal{L}^{-1}\left[\frac{1}{s} v(x, 0)\right], \\
   w_0(x, t) &= \mathcal{L}^{-1}\left[\frac{1}{s} w(x, 0)\right],
\end{align*}
$$

and the recursive relation can be written as follows:

$$
\begin{align*}
   u_{n+1}(x, t) &= \mathcal{L}^{-1}\left[\frac{1}{s^2} L\left(\frac{1}{2} (u_n)_{xxx} - 3A_n + 3B_n\right)\right], \\
   v_{n+1}(x, t) &= \mathcal{L}^{-1}\left[\frac{1}{s^2} L\left(-(v_n)_{xxx} + 3C_n\right)\right], \\
   w_{n+1}(x, t) &= \mathcal{L}^{-1}\left[\frac{1}{s^2} L\left(-(w_n)_{xxx} + 3D_n\right)\right],
\end{align*}
$$

where $n \geq 0$. Hence, we obtain the following components:

$$
\begin{align*}
   u_0 &= \frac{1}{3} \beta - \frac{2}{3} k^2 + 2k^2 \tanh(kx)^2, \\
   v_0 &= -\frac{4k^2 c_0 (\beta + k^2)}{3 c_1^2} + \frac{4k^2 (\beta + k^2)}{c_1} \tanh(kx), \\
   w_0 &= c_0 + c_1 \tanh(kx),
\end{align*}
$$

$$
\begin{align*}
   u_1 &= \frac{4k^3 \sinh(kx) \beta t^\alpha}{\Gamma(\alpha + 1) \cosh(kx)^3}, \\
   v_1 &= \frac{4k^3 (\beta + k^2) \beta t^\alpha}{3\Gamma(\alpha + 1) \cosh(kx)^2 c_1}, \\
   w_1 &= \frac{c_1 k \beta t^\alpha}{\Gamma(\alpha + 1) \cosh(kx)^2},
\end{align*}
$$

$$
\begin{align*}
   u_2 &= \frac{4\beta^2 k^4}{\Gamma(\alpha + 1) \cosh(kx)^7}\left(\frac{(-2 \cosh(kx)^2 + 3)\Gamma(\alpha + 1)^2 \cosh(kx)^3 t^{2\alpha}}{\Gamma(2\alpha + 1)} + \frac{4(-9k^2 + \cosh(kx)^2 (-\beta + 5k^2))\Gamma(\alpha + 1) \sinh(kx) k t^{3\alpha}}{\Gamma(3\alpha + 1)}\right),
\end{align*}
$$

$$
\begin{align*}
   v_2 &= -\frac{8}{3} \frac{\beta^2 (\beta + k^2) \sinh(kx) k^4}{c_1 \Gamma(\alpha + 1)^2 \cosh(kx)^6}\left(\frac{\Gamma(\alpha + 1)^2 \cosh(kx)^3 t^{2\alpha}}{\Gamma(2\alpha + 1)} + \frac{12k^3 \Gamma(2\alpha + 1) \sinh(kx) k t^{3\alpha}}{\Gamma(3\alpha + 1)}\right),
\end{align*}
$$

$$
\begin{align*}
   w_1 &= -2c_1 \beta^2 \left(\frac{\sinh(kx) k^2 t^{2\alpha}}{\Gamma(2\alpha + 1) \cosh(kx)^3} + \frac{12 \sinh(kx)^2 \Gamma(2\alpha + 1) k^5 t^{3\alpha}}{\Gamma(3\alpha + 1) \cosh(kx) \Gamma(\alpha + 1)^2}\right).
\end{align*}
$$

By collecting all these components, we can obtain the closed form solutions (5.6)
\[ u(x, t) = \frac{1}{3} \beta - \frac{2}{3} k^2 + 2 k^2 \tanh(kx) + \frac{4 k^3 \sinh(kx) \beta \Gamma(\alpha)}{\Gamma(\alpha) \alpha \cosh(kx)^3} - \frac{8 \beta^2 k^4 \rho^2 \sqrt{\pi}}{\Gamma(\alpha) \alpha \cosh(kx)^3 (2\alpha^2)^3 \Gamma(\alpha + \frac{1}{2})} \]

\[ + \frac{12 \beta^2 k^4 \rho^2 \sqrt{\pi}}{\Gamma(\alpha) \alpha \cosh(kx)^4 (2\alpha^2)^2 \Gamma(\alpha + \frac{1}{2})} - \frac{96 \beta^2 k^7 \sinh(kx) \sqrt{\pi} (2\alpha^2)^2 \Gamma(\alpha + \frac{1}{2}) \rho^3 \sqrt{3}}{\Gamma(\alpha)^2 \alpha^2 \cosh(kx)^7 (3\alpha^2)^3 \Gamma(\alpha + \frac{1}{3}) \Gamma(\alpha + \frac{2}{3})} \]

\[ v(x, t) = \frac{3}{4} \beta \frac{2}{3} k^2 c_0 \frac{\beta k^2}{c_1} - \frac{4}{3} \frac{k^4 c_0}{c_1} + \frac{4}{3} \frac{k^2 \tanh(kx) \beta}{c_1} + \frac{4}{3} \frac{k^4 \tanh(kx)}{c_1} + \frac{4}{3} \frac{k^2 \beta^2 \rho}{\Gamma(\alpha) \alpha \cosh(kx)^2 c_1} + \frac{4}{3} \frac{k^3 \beta^2 \rho}{\Gamma(\alpha) \alpha \cosh(kx)^3 c_1} \]

\[ w(x, t) = \frac{c_1 k \rho^2}{\Gamma(\alpha) \alpha \cosh(kx)^2} - \frac{2 c_1 \beta^2 \sinh(kx) k^2 \rho^2 \sqrt{\pi}}{\alpha (2\alpha^2)^2 \Gamma(\alpha) \Gamma(\alpha + \frac{1}{2}) \cosh(kx)^3} \]

\[ - \frac{16 c_1 \beta^2 \sinh(kx)^2 (3\alpha^2)^3 \Gamma(\alpha + \frac{1}{3}) \Gamma(\alpha + \frac{2}{3}) \cosh(kx)^6}{\alpha^2 \Gamma(\alpha)^2 (3\alpha^2)^3 \Gamma(\alpha + \frac{1}{3}) \Gamma(\alpha + \frac{2}{3}) \cosh(kx)^6} \]

The numerical results are listed in Tables 2a, 2b, and 2c for \( u(x, t) \), \( v(x, t) \) and \( w(x, t) \), respectively, where \( k = 0.1, \beta = 1.5, c_1 = 1.5, c_0 = 1.5 \) and \( c = -1.5 \).

**Table 1:** The numerical results in comparison with the analytical solutions for various values of \( \alpha, x \) and \( t \) for Example 5.1

<table>
<thead>
<tr>
<th>t</th>
<th>x</th>
<th>Abs. error1</th>
<th>Abs. error2</th>
<th>Abs. error1</th>
<th>Abs. error2</th>
<th>Abs. error1</th>
<th>Abs. error2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.25</td>
<td>0.033573465</td>
<td>0.035078182</td>
<td>0.005755138</td>
<td>0.005927608</td>
<td>0.000592224</td>
<td>0.000608784</td>
</tr>
<tr>
<td></td>
<td>0.50</td>
<td>0.035736217</td>
<td>0.038926279</td>
<td>0.006480788</td>
<td>0.006846431</td>
<td>0.001175507</td>
<td>0.001210615</td>
</tr>
<tr>
<td></td>
<td>0.75</td>
<td>0.040155012</td>
<td>0.045415638</td>
<td>0.007558588</td>
<td>0.008161559</td>
<td>0.001722470</td>
<td>0.001780365</td>
</tr>
<tr>
<td>1.0</td>
<td>0.25</td>
<td>0.046766167</td>
<td>0.054738582</td>
<td>0.01175507</td>
<td>0.001210615</td>
<td>0.001722470</td>
<td>0.001780365</td>
</tr>
<tr>
<td></td>
<td>0.50</td>
<td>0.101948887</td>
<td>0.118524935</td>
<td>0.03189817</td>
<td>0.036229079</td>
<td>0.004915917</td>
<td>0.005863828</td>
</tr>
<tr>
<td></td>
<td>0.75</td>
<td>0.113738563</td>
<td>0.141073582</td>
<td>0.037993742</td>
<td>0.045135726</td>
<td>0.008399976</td>
<td>0.00963146</td>
</tr>
</tbody>
</table>
Table 2-Comparison of the numerical results of the analytical solutions with LMADM and LADM for various values of $\alpha$, $x$ and $t$ for Example 5.2.

<table>
<thead>
<tr>
<th>$\alpha = 0.5$</th>
<th>$\alpha = 0.75$</th>
<th>$\alpha = 1.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t$</td>
<td>$x$</td>
<td>Abs. error1</td>
</tr>
<tr>
<td>0.1</td>
<td>0.25</td>
<td>1.23364010$^{-4}$</td>
</tr>
<tr>
<td></td>
<td>0.50</td>
<td>1.60614210$^{-4}$</td>
</tr>
<tr>
<td></td>
<td>0.75</td>
<td>1.97071410$^{-4}$</td>
</tr>
<tr>
<td></td>
<td>1.00</td>
<td>2.32564210$^{-4}$</td>
</tr>
<tr>
<td>0.3</td>
<td>0.25</td>
<td>2.74590110$^{-4}$</td>
</tr>
<tr>
<td></td>
<td>0.50</td>
<td>3.18265110$^{-4}$</td>
</tr>
<tr>
<td></td>
<td>0.75</td>
<td>3.60380010$^{-4}$</td>
</tr>
<tr>
<td></td>
<td>1.00</td>
<td>4.00751510$^{-4}$</td>
</tr>
<tr>
<td>0.5</td>
<td>0.25</td>
<td>3.77593210$^{-4}$</td>
</tr>
<tr>
<td></td>
<td>0.50</td>
<td>4.15397510$^{-4}$</td>
</tr>
<tr>
<td></td>
<td>0.75</td>
<td>4.51175310$^{-4}$</td>
</tr>
<tr>
<td></td>
<td>1.00</td>
<td>4.84788910$^{-4}$</td>
</tr>
</tbody>
</table>

Table 2b

<table>
<thead>
<tr>
<th>$\alpha = 0.5$</th>
<th>$\alpha = 0.75$</th>
<th>$\alpha = 1.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t$</td>
<td>$x$</td>
<td>Abs. error1</td>
</tr>
<tr>
<td>0.1</td>
<td>0.25</td>
<td>5.153325510$^{-4}$</td>
</tr>
<tr>
<td></td>
<td>0.50</td>
<td>5.129383710$^{-4}$</td>
</tr>
<tr>
<td></td>
<td>0.75</td>
<td>5.099184410$^{-4}$</td>
</tr>
<tr>
<td></td>
<td>1.00</td>
<td>5.062877010$^{-4}$</td>
</tr>
<tr>
<td>0.3</td>
<td>0.25</td>
<td>6.364753710$^{-4}$</td>
</tr>
<tr>
<td></td>
<td>0.50</td>
<td>6.314431110$^{-4}$</td>
</tr>
<tr>
<td></td>
<td>0.75</td>
<td>6.256536210$^{-4}$</td>
</tr>
</tbody>
</table>
Table 2c

<table>
<thead>
<tr>
<th>$t$</th>
<th>$\alpha = 0.5$</th>
<th>$\alpha = 0.75$</th>
<th>$\alpha = 1.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\alpha = 0.5$</td>
<td>$\alpha = 0.75$</td>
<td>$\alpha = 1.0$</td>
</tr>
<tr>
<td>0.1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.25</td>
<td>5.7590972</td>
<td>5.7590988</td>
<td>5.1991751</td>
</tr>
<tr>
<td>0.50</td>
<td>5.7333410</td>
<td>5.7333471</td>
<td>2.0920852</td>
</tr>
<tr>
<td>0.75</td>
<td>5.6985919</td>
<td>5.6986055</td>
<td>2.0824236</td>
</tr>
<tr>
<td>1.00</td>
<td>5.6580165</td>
<td>5.6580404</td>
<td>2.0702380</td>
</tr>
<tr>
<td>0.3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.25</td>
<td>7.1129282</td>
<td>7.1129362</td>
<td>3.1631011</td>
</tr>
<tr>
<td>0.50</td>
<td>7.0566904</td>
<td>7.0567221</td>
<td>3.1439246</td>
</tr>
<tr>
<td>0.75</td>
<td>6.9919899</td>
<td>6.9920605</td>
<td>3.1209273</td>
</tr>
<tr>
<td>1.00</td>
<td>6.9191468</td>
<td>6.9192709</td>
<td>3.0942235</td>
</tr>
<tr>
<td>0.5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.25</td>
<td>6.6558850</td>
<td>6.6559021</td>
<td>3.3018434</td>
</tr>
<tr>
<td>0.50</td>
<td>6.5801454</td>
<td>6.5802135</td>
<td>3.2717531</td>
</tr>
<tr>
<td>0.75</td>
<td>6.4966088</td>
<td>6.4967608</td>
<td>3.2376946</td>
</tr>
<tr>
<td>1.00</td>
<td>6.4056891</td>
<td>6.4059624</td>
<td>3.1998380</td>
</tr>
</tbody>
</table>

6. Conclusions

In this article, we found the solutions of nonlinear time-fractional differential equations by combining Laplace transform with the modified Adomian decomposition method (LMADM). We conclude that the results obtained by using this method are effective; they require a small number of iterations and high accuracy to solve different nonlinear fractional differential equations and their good convergence, compared with the Laplace transform combined with the standard Adomian analysis method (LADM).

References


