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Some Geometric Properties for a Certain Class of Meromorphic Univalent Functions by Differential Operator

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Abstract:
The major target of this paper is to study a confirmed class of meromorphic univalent functions $\mathcal{K}(\varepsilon,\tau,\alpha,\beta,v)$. We procure several results, such as those related to coefficient estimates, distortion and growth theorem, radii of starlikeness, and convexity for this class, n additionto hadamard product, convex combination, closure theorem, integral operators, and σ neighborhoods.

Keywords: Meromorphic univalent function, distortion and growth theorem, integral operators, differential operator, convex combination, integral operator and σ neighborhoods.

بعض الخصائص الهندسية لصنف مؤكد من الدوال احادية التكافؤ الميرومورفية بواسطة المؤثر التفاضلي

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الخلاصة:
الهدف الرئيسي من هذا البحث هو دراسة صنف مؤكد من الدوال احادية التكافؤ الميرومورفية $\mathcal{K}(\varepsilon,\tau,\alpha,\beta,v)$.
نحن حصلنا على العديد من النتائج، مثال لذلك، تقديرات المعامل، مبرهنة النمو والتشوية، أنصاف أقطار النجمية والتحدبية لهذا الصنف وضرب هادمرد (ضرب الالتواء)، التركيب المحدب، مبرهنة الانغلاق، المؤثر التكاملي والجوارات من النمط $-\sigma$.

1- Introduction

We have studied the geometric function theory of the blending of geometry and analysis. Its radix has begun from the 19th century, but it uninterrupted and permanent is applicable at present.
Geometric function theory is a paramount department of complex analysis, which studies the geometric results of the analytic functions.
An univalent function is meromorphiic function f in a domain of expanded complex plane, so that $f(z_1) \neq f(z_2)$ for $z_1 \neq z_2$, where z_1 and z_2 are members of the domain.
One of the fundamental troubles in the treatise of univalent functions is whether there exists an univalent mapping of plain linked domain onto a given plain linked domain. Nevertheless, at view of Riemann Mapping theorem [1] over trouble reduces to a trouble of mapping a unit disk onto a given plain linked domain, such as like stairlike, convex, close-to-convex etc.

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Let $\mathcal{N}$ denotes the analytic and meromorphic univalent class form functions in the punctured unit disk $U^* = \{z \in \mathbb{C}: 0 < |z| < 1\} = U - \{0\}$ and let $\mathcal{A}(j)$ indicates the subclass of $\mathcal{N}$ made up of functions of the form:

$$f(z) = z^{-1} - \sum_{j=1}^{\infty} a_j z^j, \quad (a_k \geq 0, j \in \mathbb{N} = \{1, 2, \dots\}), \tag{1}$$

which are analytic functions and meromorphic univalent functions in U^*.
A function $f \in \mathcal{A}(j)$ is said to be meromorphically starlike of order γ if

$$Re\left\{-\frac{zf'(z)}{f(z)}\right\} > \gamma, (z \in U = U^* \cup \{0\}, 0 \leq \gamma < 1), \tag{2}$$

and a function $f \in \mathcal{A}(j)$is said to be meromorphically convex of order γ if

$$Re\left\{-\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > \gamma, (z \in U = U^* \cup \{0\}, 0 \leq \gamma < 1). \tag{3}$$

We indicate by $\delta^*(\gamma)$ and $\delta(\gamma)$, respectively, the classes of univalent meromorphiic starlike functions of order γ and univalent meromorphic convex functions of order γ. Such classes have been extensively calculated previously[2- 7].
The differential operator $(\mathcal{L}_{\alpha,\beta})$ [8] below will be used for this paper:

$$\begin{aligned}\mathcal{L}_{\alpha,\beta}f(z) &= z^{-1} - \sum_{j=1}^{\infty} a_j[\alpha(j+2)(\beta(j+1)+1) - \beta(j+2) + (1-\alpha+\beta)]\, z^j \\ &= z^{-1} - \sum_{j=1}^{\infty} \mathcal{H}(\alpha,\beta,j) a_j z^j, \end{aligned} \tag{4}$$

such that $\mathcal{H}(\alpha,\beta,j) = [\alpha(j+2)(\beta(j+1)+1) - \beta(j+2) + (1-\alpha+\beta)]$, $0 \leq \beta \leq \alpha, j \in \mathbb{N} = \{1, 2, \dots\}$.

Definition (1): A function $f \in \mathcal{A}(j)$ is in the class $\mathcal{K}(\varepsilon,\tau,\alpha,\beta,v)$ if the following condition is met:

$$Re\left\{1 + \frac{1}{\tau}\left(1 + \frac{z\left(\mathcal{L}_{\alpha,\beta}f(z)\right)''}{\left(\mathcal{L}_{\alpha,\beta}f(z)\right)'} - \varepsilon\right)\right\} > v\left|1 + \frac{1}{\tau}\left(\frac{z\left(\mathcal{L}_{\alpha,\beta}f(z)\right)''}{\left(\mathcal{L}_{\alpha,\beta}f(z)\right)'}\right)\right|, \qquad z \in U^*, \tag{5}$$

such that $-1 < v < \varepsilon < 1,\ \tau \in \mathbb{C} \setminus \{0\}$.

2- Coefficient Estimates

The theorem below provides a sufficient condition for a function in $K(\varepsilon,\tau,\alpha,\beta,v)$.

Theorem (1): A function $f \in \mathcal{A}(j)$ is in the class $\mathcal{K}(\varepsilon,\tau,\alpha,\beta,v)$ if and only if

$$\sum_{j=1}^{\infty} j[(j+|\tau|)(1-v) + (v-\varepsilon)]\,\mathcal{H}(\alpha,\beta,j)a_j \leq (1-v)(1+|\tau|) + (\varepsilon - v), \tag{6}$$

such that $-1 < v < \varepsilon < 1,\ \tau \in \mathbb{C} \setminus \{0\}$.
The outcome is sharp for the function

$$f(z) = z^{-1} - \frac{(1-v)(1+|\tau|) + (\varepsilon - v)}{j[(j+|\tau|)(1-v) + (v-\varepsilon)]\mathcal{H}(\alpha,\beta,j)} z^j, \qquad j \in \mathbb{N}. \tag{7}$$

Proof: Suppose that (6) is true for $|z| = 1$. Then

$$Re\left\{1 + \frac{1}{\tau}\left(1 + \frac{z\left(\mathcal{L}_{\alpha,\beta}f(z)\right)''}{\left(\mathcal{L}_{\alpha,\beta}f(z)\right)'} - \varepsilon\right)\right\} - v\left|1 + \frac{1}{\tau}\left(\frac{z\left(\mathcal{L}_{\alpha,\beta}f(z)\right)''}{\left(\mathcal{L}_{\alpha,\beta}f(z)\right)'}\right)\right| > 0,$$

if

$$\begin{aligned} 1 + \frac{1}{|\tau|}\left(\frac{(1+\varepsilon) - \sum_{j=1}^{\infty} j(j-\varepsilon)\mathcal{H}(\alpha,\beta,j)a_j}{-1 - \sum_{j=1}^{\infty} j\mathcal{H}(\alpha,\beta,j)a_j}\right) \\ -v\left[1 + \frac{1}{|\tau|}\left(\frac{2 - \sum_{j=1}^{\infty} j(j-1)\mathcal{H}(\alpha,\beta,j)a_j}{-1 - \sum_{j=1}^{\infty} j\mathcal{H}(\alpha,\beta,j)a_j}\right)\right] > 0, \end{aligned}$$

that is if

$$\sum_{j=1}^{\infty} j[(j+|\tau|)(1-v)+(v-\varepsilon)]\,\mathcal{H}(\alpha,\beta,j)a_j \le (1-v)(1+|\tau|)+(\varepsilon-v).$$

Conversely, presume that $f \in \mathcal{K}(\varepsilon,b,\alpha,\beta,v)$, then

$$Re\left\{1+\frac{1}{\tau}\left(1+\frac{z\left(\mathcal{L}_{\alpha,\beta}f(z)\right)''}{\left(\mathcal{L}_{\alpha,\beta}f(z)\right)'}-\varepsilon\right)\right\} > v\left|1+\frac{1}{\tau}\left(\frac{z\left(\mathcal{L}_{\alpha,\beta}f(z)\right)''}{\left(\mathcal{L}_{\alpha,\beta}f(z)\right)'}\right)\right|,$$

$$Re\left\{1+\frac{1}{\tau}\left(\frac{(1+\varepsilon)-\sum_{j=1}^{\infty} j(j-\varepsilon)\mathcal{H}(\alpha,\beta,j)a_j}{-1-\sum_{j=1}^{\infty} j\mathcal{H}(\alpha,\beta,j)a_j}\right)\right\} > v\left|1+\frac{1}{\tau}\left(\frac{2-\sum_{j=1}^{\infty} j(j-1)\mathcal{H}(\alpha,\beta,j)a_j}{-1-\sum_{j=1}^{\infty} j\mathcal{H}(\alpha,\beta,j)a_j}\right)\right|. \tag{8}$$

By letting $z \to 1^-$ along the real axis, we get

$$1+\frac{1}{|\tau|}\left(\frac{(1+\varepsilon)-\sum_{j=1}^{\infty} j(j-\varepsilon)\mathcal{H}(\alpha,\beta,j)a_j}{-1-\sum_{j=1}^{\infty} j\mathcal{H}(\alpha,\beta,j)a_j}\right) > v\left[1+\frac{1}{|\tau|}\left(\frac{2-\sum_{j=1}^{\infty} j(j-1)\mathcal{H}(\alpha,\beta,j)a_j}{-1-\sum_{j=1}^{\infty} j\mathcal{H}(\alpha,\beta,j)a_j}\right)\right].$$

Thus, by the theorem of maximum modulus, the simple math operation drives the required variation

$$\sum_{j=1}^{\infty} j[(j+|\tau|)(1-v)+(v-\varepsilon)]\,\mathcal{H}(\alpha,\beta,j)a_j \le (1-v)(1+|\tau|)+(\varepsilon-v),$$

This completes the proof.

Theorem (1) immediately gives the next result.

Corollary (1): Let $f \in \mathcal{K}(\varepsilon,\tau,\alpha,\beta,v)$. Then

$$a_j \le \frac{(1-v)(1+|\tau|)+(\varepsilon-v)}{j[(j+|\tau|)(1-v)+(v-\varepsilon)]\mathcal{H}(\alpha,\beta,j)}, \qquad j=1,2,\ldots . \tag{9}$$

The equality in (9) is obtained for the function f given by (7).

3- Distortion and Growth Theorem

We now mention the following distortion and growth of class $K(\varepsilon,\tau,\alpha,\beta,v)$.

Theorem (2): Let the function $f \in \mathcal{K}(\varepsilon,\tau,\alpha,\beta,v)$. Then

$$\frac{1}{|z|}-\frac{(1-v)(1+|\tau|)+(\varepsilon-v)}{[(1+|\tau|)(1-v)+(v-\varepsilon)]\mathcal{H}(\alpha,\beta,1)}|z| \le |f(z)| \le \frac{1}{|z|}+\frac{(1-v)(1+|\tau|)+(\varepsilon-v)}{[(1+|\tau|)(1-v)+(v-\varepsilon)]\mathcal{H}(\alpha,\beta,1)}|z|. \tag{10}$$

The outcome is sharp for the function

$$f(z)=z^{-1}-\frac{(1-v)(1+|\tau|)+(\varepsilon-v)}{[(1+|\tau|)(1-v)+(v-\varepsilon)]\mathcal{H}(\alpha,\beta,1)}z\,.$$

Proof: Of the function

$$f(z)=z^{-1}-\sum_{j=1}^{\infty} a_j z^j$$

$$|f(z)| \le \frac{1}{|z|}+\sum_{j=1}^{\infty} a_j|z|^j \le \frac{1}{|z|}+\frac{(1-v)(1+|\tau|)+(\varepsilon-v)}{[(1+|\tau|)(1-v)+(v-\varepsilon)]\mathcal{H}(\alpha,\beta,1)}|z|. \tag{11}$$

Likewise

$$|f(z)| \ge \frac{1}{|z|}-\sum_{j=1}^{\infty} a_j|z|^j \ge \frac{1}{|z|}-\frac{(1-v)(1+|\tau|)+(\varepsilon-v)}{[(1+|\tau|)(1-v)+(v-\varepsilon)]\mathcal{H}(\alpha,\beta,1)}|z|. \tag{12}$$

Plural (2.11) and (2.12), we get (2.10).

Theorem (3): Let the function $f \in \mathcal{K}(\varepsilon,\tau,\alpha,\beta,v)$. Then

$$\frac{1}{|z|^2}-\frac{(1-v)(1+|\tau|)+(\varepsilon-v)}{[(1+|\tau|)(1-v)+(v-\varepsilon)]\mathcal{H}(\alpha,\beta,1)} \leq |f'(z)|$$
$$\leq \frac{1}{|z|^2}+\frac{(1-v)(1+|\tau|)+(\varepsilon-v)}{[(1+|\tau|)(1-v)+(v-\varepsilon)]\mathcal{H}(\alpha,\beta,1)}. \quad (13)$$

The outcome is sharp for the function

$$f(z)=z^{-1}-\frac{(1-v)(1+|\tau|)+(\varepsilon-v)}{[(1+|\tau|)(1-v)+(v-\varepsilon)]\mathcal{H}(\alpha,\beta,1)}z.$$

Proof: Of the function

$$f(z)=z^{-1}-\sum_{j=1}^{\infty}a_j z^j$$

$$|f'(z)| \leq \frac{1}{|z|^2}+\sum_{j=1}^{\infty} j a_j |z|^j \leq \frac{1}{|z|^2}+\frac{(1-v)(1+|\tau|)+(\varepsilon-v)}{[(1+|\tau|)(1-v)+(v-\varepsilon)]\mathcal{H}(\alpha,\beta,1)}. \quad (14)$$

Likewise

$$|f(z)| \geq \frac{1}{|z|^2}-\sum_{j=1}^{\infty} j a_j |z|^j \geq \frac{1}{|z|^2}-\frac{(1-v)(1+|\tau|)+(\varepsilon-v)}{[(1+|\tau|)(1-v)+(v-\varepsilon)]\mathcal{H}(\alpha,\beta,1)}. \quad (15)$$

Plural (14) and (15), we get (13).

4- Radius of starlikeness and convexity

Next, we get the region of univalency; in particular, starlikeness and convexity forthe class $\mathcal{K}(\varepsilon,\tau,\alpha,\beta,v)$.

Theorem (4): Let $f \in K(\varepsilon,\tau,\alpha,\beta,v)$. Then f is starlike of order $\gamma,(0 \leq \gamma < 1)$ in $|z| < r = r_1(\varepsilon,\tau,\alpha,\beta,v,\gamma)$, where

$$r_1(\varepsilon,\tau,\alpha,\beta,v,\gamma)=\inf_j\left\{\frac{(1-\gamma)j[(j+|\tau|)(1-v)+(v-\varepsilon)]\mathcal{H}(\alpha,\beta,j)}{(j+2-\gamma)(1-v)(1+|\tau|)+(\varepsilon-v)}\right\}^{\frac{1}{j+1}},$$

$j=1,2,\dots$. (16)

is a must for every $|z|$ is sharp for each j, with the extreme function of form (7).

Proof: Let $f \in \mathcal{K}(\varepsilon,\tau,\alpha,\beta,v)$, then by Theorem (1)

$$\sum_{j=1}^{\infty}\frac{j[(j+|\tau|)(1-v)+(v-\varepsilon)]}{(1-v)(1+|\tau|)+(\varepsilon-v)}\mathcal{H}(\alpha,\beta,j)\,a_j \leq 1. \quad (17)$$

For $0 \leq \gamma < 1$, we need to show that

$$\left|\frac{zf'(z)}{f(z)}+1\right| \leq 1-\gamma,$$

we have to show that

$$\left|\frac{zf'(z)+f(z)}{f(z)}\right| \leq \left|\frac{-\sum_{j=1}^{\infty}(j+1)a_j z^{j+1}}{1-\sum_{j=1}^{\infty}a_j z^{j+1}}\right| \leq \frac{\sum_{j=1}^{\infty}(j+1)a_j|z|^{j+1}}{1-\sum_{j=1}^{\infty}a_j z^{j+1}} \leq 1-\gamma.$$

Subsequently

$$\sum_{j=1}^{\infty}\left(\frac{j+2-\gamma}{1-\gamma}\right)a_j|z|^{j+1} \leq 1.$$

This inequality is enough to consider

$$|z|^{j+1} \leq \frac{(1-\gamma)j[(j+|\tau|)(1-v)+(v-\varepsilon)]\mathcal{H}(\alpha,\beta,j)}{(j+2-\gamma)(1-v)(1+|\tau|)+(\varepsilon-v)}.$$

Therefore

$$|z| \leq \left\{\frac{(1-\gamma)j[(j+|\tau|)(1-v)+(v-\varepsilon)]\mathcal{H}(\alpha,\beta,j)}{(j+2-\gamma)(1-v)(1+|\tau|)+(\varepsilon-v)}\right\}^{\frac{1}{j+1}}. \quad (18)$$

Let $|z| = r_1(\varepsilon,\tau,\alpha,\beta,v,\gamma)$ in (18), We obtain the radius of the starlikeness, and from it the proof of Theorem (4) is complete. ■

Theorem (5): Let $f \in \mathcal{K}(\varepsilon,\tau,\alpha,\beta,v)$. Then, f is convex of order $\gamma, (0 \le \gamma < 1)$ in $|z| < r = r_2(\varepsilon,\tau,\alpha,\beta,v,\gamma)$, where

$$r_2(\varepsilon,\tau,\alpha,\beta,v,\gamma) = \inf_{j}\left\{\frac{(1-\gamma)[(j+|\tau|)(1-v)+(v-\varepsilon)]\mathcal{H}(\alpha,\beta,j)}{(j+2-\gamma)(1-v)(1+|\tau|)+(\varepsilon-v)}\right\}^{\frac{1}{j+1}},$$

$j = 1,2,\dots$. (19)

is obligatory for each $|z|$ sharp per j, with the extreme function of the from (7).

Proof: Let $f \in \mathcal{K}(\varepsilon,\tau,\alpha,\beta,v)$, then by Theorem (1)

$$\sum_{j=1}^{\infty}\frac{j[(j+|\tau|)(1-v)+(v-\varepsilon)]}{(1-v)(1+|\tau|)+(\varepsilon-v)}\mathcal{H}(\alpha,\beta,j)\,a_j \le 1. \tag{20}$$

For $0 \le \gamma < 1$, we have to show that

$$\left|\frac{zf''(z)}{f'(z)}+2\right| \le 1-\gamma,$$

We need to show that

$$\left|\frac{zf''(z)+2f'(z)}{f'(z)}\right| \le \left|\frac{-\sum_{k=1}^{\infty} j(j+1)a_j z^{j+1}}{1-\sum_{j=1}^{\infty} j a_j z^{j+1}}\right| \le \frac{\sum_{j=1}^{\infty} j(j+1)a_j z^{j+1}}{1-\sum_{j=1}^{\infty} j a_j z^{j+1}} \le 1-\gamma.$$

Hence

$$\sum_{j=1}^{\infty}\frac{j(j+2-\gamma)}{1-\gamma}a_j|z|^{j+1} \le 1.$$

This is enough to consider

$$|z|^{j+1} \le \frac{(1-\gamma)[(j+|\tau|)(1-v)+(v-\varepsilon)]\mathcal{H}(\alpha,\beta,j)}{(j+2-\gamma)(1-v)(1+|\tau|)+(\varepsilon-v)}.$$

Therefore

$$|z| \le \left\{\frac{(1-\gamma)[(j+|\tau|)(1-v)+(v-\varepsilon)]\mathcal{H}(\alpha,\beta,j)}{(j+2-\gamma)(1-v)(1+|\tau|)+(\varepsilon-v)}\right\}^{\frac{1}{j+1}}. \tag{21}$$

By setting $|z| = r_2(\varepsilon,\tau,\alpha,\beta,v,\gamma)$ in (2.18), we obtain the radius of convexity, which will finish proving the Theorem (5).

5- Hadamard product

In the subsidiary Theorem, we get the Hadamard product of the functions f and $g \in \mathcal{K}(\varepsilon,\tau,\alpha,\beta,v)$.

Theorem (6): If

$$f(z) = z^{-1} - \sum_{j=1}^{\infty} a_j z^j \quad \text{and} \quad g(z) = z^{-1} - \sum_{j=1}^{\infty} b_j z^j$$

be in the class $\mathcal{K}(\varepsilon,\tau,\alpha,\beta,v)$, then the Hadamard product of f and g, given by

$$(f*g)(z) = z^{-1} - \sum_{j=1}^{\infty} a_j b_j z^j,$$

is in the class $\mathcal{K}(\mu,\tau,\alpha,\beta,v)$, where

$$\mu \le \frac{[(1-v)(1+|\tau|)+(\varepsilon-v)]^2[(j+|\tau|)(1-v)+v] - j[(j+|\tau|)(1-v)+(v-\varepsilon)]^2\mathcal{H}(\alpha,\beta,j)[(1-v)(1+|\tau|)-v]}{[(1-v)(1+|\tau|)+(\varepsilon-v)]^2 + j[(j+|\tau|)(1-v)+(v-\varepsilon)]^2\mathcal{H}(\alpha,\beta,j)}.$$

Proof: Let $f, g \in \mathcal{K}(\varepsilon,\tau,\alpha,\beta,v)$. By Theorem (1), we get

$$\sum_{j=1}^{\infty}\frac{j[(j+|\tau|)(1-v)+(v-\varepsilon)]}{(1-v)(1+|\tau|)+(\varepsilon-v)}\mathcal{H}(\alpha,\beta,j)\,a_j \le 1 \tag{22}$$

and

$$\sum_{j=1}^{\infty}\frac{j[(j+|\tau|)(1-v)+(v-\varepsilon)]}{(1-v)(1+|\tau|)+(\varepsilon-v)}\mathcal{H}(\alpha,\beta,j)\,b_j \le 1. \tag{23}$$

We must find the greatest value μ, such that

$$\sum_{j=1}^{\infty} \frac{j\big[(j+|\tau|)(1-v)+\big(v-\mu\big)\big]}{(1-v)(1+|\tau|)+\big(\mu-v\big)} \mathcal{H}(\alpha,\beta,j)\, a_j b_j \le 1\,. \tag{24}$$

By Cauchy--Schwarzs inequality, we get

$$\sum_{j=1}^{\infty} \frac{j[(j+|\tau|)(1-v)+(v-\varepsilon)]}{(1-v)(1+|\tau|)+(\varepsilon-v)} \mathcal{H}(\alpha,\beta,j) \sqrt{a_j b_j} \le 1\,. \tag{25}$$

So, it suffices to show that

$$\frac{j\big[(j+|\tau|)(1-v)+\big(v-\mu\big)\big]}{(1-v)(1+|\tau|)+\big(\mu-v\big)} \mathcal{H}(\alpha,\beta,j) a_j b_j \le \frac{j[(j+|\tau|)(1-v)+(v-\varepsilon)]}{(1-v)(1+|\tau|)+(\varepsilon-v)} \mathcal{H}(\alpha,\beta,j) \sqrt{a_j b_j}\,,$$

that is

$$\sqrt{a_j b_j} \le \frac{[(j+|\tau|)(1-v)+(v-\varepsilon)]\big[(1-v)(1+|\tau|)+\big(\mu-v\big)\big]}{\big[(j+|\tau|)(1-v)+\big(v-\mu\big)\big][(1-v)(1+|\tau|)+(\varepsilon-v)]}\,. \tag{26}$$

And from (25), we will get

$$\sqrt{a_j b_j} \le \frac{(1-v)(1+|\tau|)+(\varepsilon-v)}{j[(j+|\tau|)(1-v)+(v-\varepsilon)]\mathcal{H}(\alpha,\beta,j)}\,. \tag{27}$$

Through (26) and (27), it is sufficient to prove that

$$\frac{(1-v)(1+|\tau|)+(\varepsilon-v)}{j[(j+|\tau|)(1-v)+(v-\varepsilon)]\mathcal{H}(\alpha,\beta,j)} \le \frac{[(j+|\tau|)(1-v)+(v-\varepsilon)]\big[(1-v)(1+|\tau|)+\big(\mu-v\big)\big]}{\big[(j+|\tau|)(1-v)+\big(v-\mu\big)\big][(1-v)(1+|\tau|)+(\varepsilon-v)]},$$

To simplify that, we have

$$\mu$$
$$\le \frac{[(1-v)(1+|\tau|)+(\varepsilon-v)]^2[(j+|\tau|)(1-v)+v] - j[(j+|\tau|)(1-v)+(v-\varepsilon)]^2\mathcal{H}(\alpha,\beta,j)[(1-v)(1+|\tau|)-v]}{[(1-v)(1+|\tau|)+(\varepsilon-v)]^2 + j[(j+|\tau|)(1-v)+(v-\varepsilon)]^2\mathcal{H}(\alpha,\beta,j)}\,.$$

6- Convex combination

In the theorems below, we will show that the class $K\,(\varepsilon,\tau,\alpha,\beta,v)$ is closed under a convex linear combination.

Theorem (7): Let $f_i \in \mathcal{K}(\varepsilon,\tau,\alpha,\beta,v)$, such that

$$f_i(z) = z^{-1} - \sum_{j=1}^{\infty} a_{j,i} z^{j}\ ,\big(a_{j,i} \ge 0, i = 1{,}2\,\big).$$

Then

$$\varpi(z) = \zeta f_1(z) + (1-\zeta) f_2(z), \qquad (0 \le \zeta \le 1)$$

Also in class $\mathcal{K}(\varepsilon,\tau,\alpha,\beta,v)$.

Proof: From $0 \le \zeta \le 1$, we get

$$\varpi(z) = z^{-1} - \sum_{j=1}^{\infty} (\zeta\, a_{j,1} + (1-\zeta)\, a_{j,2}) z^{j}\,,$$

We note that

$$\sum_{j=1}^{\infty} j[(j+|b|)(1-v)+(v-\varepsilon)]\mathcal{H}(\alpha,\beta,j)\,(\zeta\, a_{j,1} + (1-\zeta)\, a_{j,2})$$

$$= \zeta \sum_{j=1}^{\infty} j[(j+|b|)(1-v)+(v-\varepsilon)]\mathcal{H}(\alpha,\beta,j) a_{j,1}$$

$$+(1-\zeta) \sum_{j=1}^{\infty} j[(j+|b|)(1-v)+(v-\varepsilon)]\mathcal{H}(\alpha,\beta,j) a_{j,2} \le (1-v)(1+|b|)+(\varepsilon-v).$$

Using Theorem (1), $\varpi \in \mathcal{K}(\varepsilon,\tau,\alpha,\beta,v)$.

7- Closure theorem

In the following, we prove the closure theorem.

Theorem (8): Let

$$f_i(z) = z^{-1} - \sum_{j=1}^{\infty} a_{j,i} z^{j} \in \mathcal{K}(\varepsilon, \tau, \alpha, \beta, v)\,, \quad i \in \{1, 2, \dots, m\} \ \textit{and} \ \ 0 < \vartheta_i < 1$$

So that

$$\sum_{i=1}^{m} \vartheta_i = 1\,.$$

Then, we select function $\mathcal{G}$ which is defined as

$$\mathcal{G}(z) = \sum_{i=1}^{m} \vartheta_i f_i(z)$$

also in class $\mathcal{K}(\varepsilon, \tau, \alpha, \beta, v)$.

Proof: For each $i \in \{1, 2, \dots, m\}$, we have

$$\sum_{j=1}^{\infty} \frac{j[(j + |b|)(1 - v) + (v - \varepsilon)]}{(1 - v)(1 + |b|) + (\varepsilon - v)} \mathcal{H}(\alpha, \beta, j)\, a_{j,i} \le 1,$$

since

$$\mathcal{G}(z) = \sum_{i=1}^{m} \vartheta_i\, f_i(z) = \sum_{i=1}^{m} \vartheta_i \left(z^{-1} - \sum_{j=1}^{\infty} a_{j,i} z^{j} \right) = z^{-1} - \sum_{j=1}^{\infty} \left(\sum_{i=1}^{m} \vartheta_i\, a_{j,i} \right) z^{j}.$$

And therefore,

$$\sum_{j=1}^{\infty} \frac{j[(j + |b|)(1 - v) + (v - \varepsilon)]}{(1 - v)(1 + |b|) + (\varepsilon - v)} \mathcal{H}(\alpha, \beta, j) \left(\sum_{i=1}^{m} \vartheta_i\, a_{j,i} \right)$$
$$= \sum_{i=1}^{m} \vartheta_i \left(\sum_{j=1}^{\infty} \frac{j[(j + |b|)(1 - v) + (v - \varepsilon)]}{(1 - v)(1 + |b|) + (\varepsilon - v)} \mathcal{H}(\alpha, \beta, j) a_{j,i} \right)$$
$$\le \sum_{i=1}^{m} \vartheta_i = 1.$$

Then, $\mathcal{G} \in \mathcal{K}(\varepsilon, \tau, \alpha, \beta, v)$ and the proof is complete.

8- Integral Operators

Theorem (2.2.8): Let $f \in \mathcal{K}(\varepsilon, \tau, \alpha, \beta, v)$. Then the integral operator [2] defined by

$$\mathcal{F}(z) = \text{ƀ} \int_0^1 \text{ꞑ}^{\text{ƀ}} f(\text{ꞑ} z)\, d\text{ꞑ}\,, (0 < \text{ꞑ} \le 1, 0 < \text{ƀ} < \infty),$$

is also in the class $\mathcal{K}(\varphi, \tau, \alpha, \beta, v)$, such that

$$\le \frac{\text{ƀ}[(j + |\tau|)(1 - v) + v][(1 - v)(1 + |\tau|) + (\varepsilon - v)] - [(j + |\tau|)(1 - v) + (v - \varepsilon)][(j + \text{ƀ} + 1)(1 - v)(1 + |\tau|) - v]}{\text{ƀ}[(1 - v)(1 + |\tau|) + (\varepsilon - v)] + [(j + |\tau|)(1 - v) + (v - \varepsilon)]}^{\varphi}. \quad (28)$$

Proof: Let $f \in \mathcal{K}(\varepsilon, \tau, \alpha, \beta, v)$, and therefore we have

$$\mathcal{F}(z) = \text{ƀ} \int_0^1 \text{ꞑ}^{\text{ƀ}} f(\text{ꞑ} z)\, d\text{ꞑ} = \text{ƀ} \int_0^1 \left(\text{ꞑ}^{\text{ƀ}-1} z^{-1} - \sum_{j=1}^{\infty} \text{ꞑ}^{j+\text{ƀ}} a_j z^{j} \right) d\text{ꞑ}$$
$$= z^{-1} - \sum_{j=1}^{\infty} \frac{\text{ƀ}}{j + \text{ƀ} + 1} a_j z^{j}.$$

We suffice it to prove that

$$\sum_{j=1}^{\infty} \frac{\text{ƀ} j[(j + |\tau|)(1 - v) + (v - \varphi)]}{(j + \text{ƀ} + 1)(1 - v)(1 + |\tau|) + (\varphi - v)} \mathcal{H}(\alpha, \beta, j)\, a_j \le 1, \qquad (29)$$

since the function $f \in \mathcal{K}(\varepsilon, \tau, \alpha, \beta, v)$, then we will get

$$\sum_{j=1}^{\infty} \frac{j[(j + |\tau|)(1 - v) + (v - \varepsilon)]}{(1 - v)(1 + |\tau|) + (\varepsilon - v)} \mathcal{H}(\alpha, \beta, j)\, a_j \le 1\,.$$

Note that (29) is satisfied if the following inequality is satisfied

$$\frac{\mathfrak{t}\jmath[(\jmath+|\tau|)(1-v)+(v-\varphi)]}{(\jmath+\mathfrak{t}+1)(1-v)(1+|\tau|)+(\varphi-v)}\mathcal{H}(\alpha,\beta,\jmath) \leq \frac{\jmath[(\jmath+|\tau|)(1-v)+(v-\varepsilon)]}{(1-v)(1+|\tau|)+(\varepsilon-v)}\mathcal{H}(\alpha,\beta,\jmath).$$

And by rewriting the inequality, we will get

$$\mathfrak{t}[(\jmath+|\tau|)(1-v)+(v-\varphi)][(1-v)(1+|\tau|)+(\varepsilon-v)] \leq [(\jmath+\mathfrak{t}+1)(1-v)(1+|\tau|)+(\varphi-v)][(\jmath+|\tau|)(1-v)+(v-\varepsilon)]$$

As a solution to φ, we will get

$$\varphi \leq \frac{\mathfrak{t}[(\jmath+|\tau|)(1-v)+v][(1-v)(1+|\tau|)+(\varepsilon-v)]-[(\jmath+|\tau|)(1-v)+(v-\varepsilon)][(\jmath+\mathfrak{t}+1)(1-v)(1+|\tau|)-v]}{\mathfrak{t}[(1-v)(1+|\tau|)+(\varepsilon-v)]+[(\jmath+|\tau|)(1-v)+(v-\varepsilon)]}. \quad (30)$$

Thus, the right side of (30) is an increasing function of $\jmath$.

Neighborhoods

Following earlier investigations on the familiar meaning of neighborhoods of meromorphic functions, which was first introduced by [9], a later work [10] studies this concept with elements of many famous subclasses of analytic functions. An earlier work [11] considered a specific family of meromorphic functions with negative coefficients. Also, other studies [2, 12, 13, 14] extended this topic for a specific subclass of meromorphically multivalent functions or meromorphically univalent functions.

The $(j,\sigma)-$ neighborhood of a function $\mathfrak{f} \in A(j)$ is defined by the following relation:

$$N_{\jmath,\sigma}(\mathfrak{f}) = \left\{ \mathrm{g} \in \mathcal{A}(\jmath) : \mathrm{g}(z) = z^{-1} - \sum_{\jmath=1}^{\infty} b_\jmath z^\jmath \ and \ \sum_{\jmath=1}^{\infty} \jmath\left|a_\jmath - b_\jmath\right| \leq \sigma, 0 \leq \sigma < 1 \right\}. \quad (31)$$

For the identity function $e(z) = z$, we will get

$$N_{\jmath,\sigma}(e) = \left\{ \mathrm{g} \in \mathcal{A}(\jmath) : \mathrm{g}(z) = z^{-1} - \sum_{\jmath=1}^{\infty} b_\jmath z^\jmath \ and \ \sum_{\jmath=1}^{\infty} \jmath\left|b_\jmath\right| \leq \sigma \right\}.$$

Definition (2): A function $\mathfrak{f} \in \mathcal{A}(\jmath)$ defined by definition (1) is said to be in the class $\mathcal{K}^\rho(\varepsilon,\tau,\alpha,\beta,v)$ if there is a function $\mathrm{g} \in \mathcal{K}(\varepsilon,\tau,\alpha,\beta,v)$ such that

$$\left|\frac{\mathfrak{f}(z)}{\mathrm{g}(z)} - 1\right| < 1-\rho, \quad (z \in U, 0 \leq \rho < 1). \quad (32)$$

Theorem (10): Let $\mathrm{g} \in \mathcal{K}^\rho(\varepsilon,\tau,\alpha,\beta,v)$ and

$$\rho = 1 - \frac{\sigma[(1+|\tau|)(1-v)+(v-\varepsilon)]\mathcal{H}(\alpha,\beta,1)}{[(1+|\tau|)(1-v)+(v-\varepsilon)]\mathcal{H}(\alpha,\beta,1)-[(1-v)(1+|\tau|)+(\varepsilon-v)]}. \quad (33)$$

Then, $N_{\jmath,\sigma}(\mathrm{g}) \subset \mathcal{K}^\rho(\varepsilon,\tau,\alpha,\beta,v)$.

Proof: Let $\mathfrak{f} \in N_{\jmath,\sigma}(\mathrm{g})$. Then, we will get from (31) that

$$\sum_{\jmath=1}^{\infty} \jmath\left|a_\jmath - b_\jmath\right| \leq \sigma,$$

which denotes the coefficient inequality

$$\sum_{\jmath=1}^{\infty} \left|a_\jmath - b_\jmath\right| \leq \sigma, (\jmath \in \mathbb{N}). \quad (34)$$

Since the function $\mathrm{g} \in \mathcal{K}(\varepsilon,\tau,\alpha,\beta,v)$, then we get from Theorem (1) that

$$\sum_{\jmath=1}^{\infty} b_\jmath \leq \frac{(1-v)(1+|\tau|)+(\varepsilon-v)}{[(1+|\tau|)(1-v)+(v-\varepsilon)]\mathcal{H}(\alpha,\beta,1)},$$

and, hence, it is

$$\left|\frac{\mathfrak{f}(z)}{\mathrm{g}(z)} - 1\right| \leq \frac{\sum_{\jmath=1}^{\infty}\left|a_\jmath - b_\jmath\right|}{1-\sum_{\jmath=1}^{\infty} b_\jmath}$$

$$\leq \frac{\sigma[(1+|\tau|)(1-v)+(v-\varepsilon)]\mathcal{H}(\alpha,\beta,1)}{[(1+|\tau|)(1-v)+(v-\varepsilon)]\mathcal{H}(\alpha,\beta,1)-[(1-v)(1+|\tau|)+(\varepsilon-v)]} = 1-\rho.$$

Therefore, by using definition (2), $\mathfrak{f} \in \mathcal{K}^\rho(\varepsilon,\tau,\alpha,\beta,v)$ for σ given by equation (33).

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