DOI: 10.24996/ijs.2021.62.5.26

ISSN: 0067-2904

Jordan Left Derivation and Centralizer on Skew Matrix Gamma Ring

Rajaa C. Shaheen
Mathematics Department, Education College, Al-Qadisiyah University

Received: 15/6/2020 Accepted: 21/8/2020

Abstract
We define skew matrix gamma ring and describe the constitution of Jordan left centralizers and derivations on skew matrix gamma ring on a Γ -ring. We also show the properties of these concepts.

Keywords: Skew matrix ring, Gamma ring, Jordan left centralizer, Jordan left derivation and Centralizer.

اشتقاق جوردان اليساري والمركزي على حلقة كاما للمصفوفات العكسية

رجاء جفات شاهين
قسم الرياضيات، كلية التربية ، جامعة القادسية ، القادسيه، العراق

الخلاصة
قدمنا تعريف حلقة كاما للمصفوفات العكسية ووصفنا شكل اشتقاق جوردان اليساري والمركزي على حلقة كاما للمصفوفات العكسية على حلقة كاما . وكذلك قدمنا خواص هذه المفاهيم

1-Introduction

The linear ring mapping ϑ from $\mathbb{R}$ onto $\mathbb{R}$ is called a left derivation (LD) (resp., Jordan left derivation (JLD)) if ϑ (xy)=x ϑ (y)+y ϑ (x) ∀ x,y∈ $\mathbb{R}$. (if $\vartheta(x^2) = 2x\vartheta(x)$ ∀x∈ $\mathbb{R}$)[1]. Bresar and Vukman [2] introduced the concept of left derivation .We refer the readers to several references [3,4, 5, 6] for results concerning Jordan left derivations. A linear mapping $\mathfrak{S}$ from $\mathbb{R}$ onto $\mathbb{R}$ is called Jordan left centralizers (JLC) (resp., left centralizers (LC)) if $\mathfrak{S}(a^2) = \mathfrak{S}(a)a$ (resp., $\mathfrak{S}(ab) = \mathfrak{S}(a)b$ ∀ a, b ∈ $\mathbb{R}$). A linear map $\mathfrak{S}$ from $\mathbb{R}$ onto $\mathbb{R}$ is called a Jordan centralizer (JC) if $\mathfrak{S}$ satisfies $\mathfrak{S}$ (ab+ba)= $\mathfrak{S}(a)\,b$+b $\mathfrak{S}$ (a)= $\mathfrak{S}$ (b)a+a $\mathfrak{S}$ (b) $\forall a, b \in \mathbb{R}$ [7]. In [7,8,9], some results about left centralizers were presented. In [10], Hamaguchi provided the sufficient and necessary conditions for J on a skew matrix ring being Jordan derivation. He proved that there exist many Jordan derivation maps of it, which are not derivations maps. He also studied the characterization of derivation on skew matrix ring ($M_2(\mathbb{R}; \hbar, q)$) and Jordan derivation of $M_2(\mathbb{R})$ with invariant ideal. Nobusawa [11] introduced the concept of gamma ring, which was generalized by Barnes [12], as follows.

Let $\mathbb{R}$ and Γ with +, . be abelian groups,$\mathbb{R}$ is called a gamma ring if for any $a, b, z \in \mathbb{R}$ and $\alpha, \beta \in \Gamma$, then the following conditions satisfied

(1) $a\ \alpha b \in \mathbb{R}$
(2) $(a + b)\ \alpha z = a\ \alpha z + b\alpha z$
$a\ (\alpha+\beta)\ z = a\alpha\ z + a\ \beta\ z$
$a\ \alpha(b+z) = a\ \alpha\ b + a\alpha\ z$

*Email: rajaa.chaffat@qu.edu.iq

(3) $(a\ \alpha b)\beta\ z = a\ \alpha(b\beta z)$
Hamil and Majeed [13] studied the cocomutativity preserving right centralizer on a subset of a gamma rings. In [14], Majeed and Shaheen described a form of Jordan left derivation and centralizers on $M_2(\mathbb{R};\hbar,q)$. In this article, we define the skew matrix gamma ring and describe the constitution of Jordan left centralizers and the derivation of Skew matrix gamma ring on a Γ -ring M . We also show the properties of Jordan left derivation and centralizers on this ring.
Now, we shall introduce some definitions which are preliminaries in this paper.

Definition 1.1: [15]
Let M be a ring $r, q \in M$, and the endomorphism $\hbar: M \to M$, such that $\hbar(q) = q$ and $(r)q = qr$ $\forall r \in M$.
Let $M_2(M;\hbar,q)$ be a set of square 2× 2 matrices on M with multiplication

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & i \\ f & g \end{bmatrix} = \begin{bmatrix} ae+bfq & ai+bg \\ c\hbar(e)+df & c\hbar(i)q+dg \end{bmatrix}$$

and the usual addition $M_2(M;h,q)$ is said to be a skew matrix ring over M.
In this article ,we define the skew matrix gamma ring as follows.

Definition 1.2 : Skew Matrix Gamma Ring
Let M be a gamma ring r and $q \in M$ with σ from M to M, where $\sigma(q) = q$ and $\forall r \in M$ and $\alpha \in \Gamma$ $\sigma(r)\alpha\ q = q\alpha r$. Let $M_2(M;\Gamma,\sigma,q)$ be the set of square 2× 2 matrices over M with multiplication

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}\alpha\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = \begin{bmatrix} a_1\alpha b_1 + a_2\alpha b_3\alpha q & a_1\alpha b_2 + a_2\alpha b_4 \\ a_3\alpha\sigma(b_1) + a_4\alpha b_3 & a_3\,\alpha\sigma(b_2)\alpha q + a_4\alpha b_4 \end{bmatrix}$$

and the usual addition $M_2(M;\Gamma,\sigma,q)$ is said to be a skew matrix Γ −ring over M. Note that the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is denoted by $e_{11}a + e_{12}b + e_{21}c + e_{22}d$.

2 - On Skew Matrix Gamma Ring with Jordan Left Derivation

We shall describe the constitution of JLD on skew matrix gamma ring .
Let D be a JLD of $M_2(M;\Gamma,\sigma,q)$. First, we set

$$D(e_{11}a)=\begin{bmatrix} \delta_1(a) & \delta_2(a) \\ \delta_3(a) & \delta_4(a) \end{bmatrix}, D(e_{12}b)=\begin{bmatrix} \xi_1(b) & \xi_2(b) \\ \xi_3(b) & \xi_4(b) \end{bmatrix}$$

$$D(e_{21}c)=\begin{bmatrix} l_1(c) & l_2(c) \\ l_3(c) & l_4(c) \end{bmatrix}, D(e_{22}d)=\begin{bmatrix} \Omega_1(d) & \Omega_2(d) \\ \Omega_3(d) & \Omega_4(d) \end{bmatrix}$$

where $\delta_i, \Omega_i, \xi_i, l_i: M \to M$ are linear mappings .

Lemma 2.1: The mappings δ_1, δ_2 , δ_3 and δ_3 satisfy the following conditions for every a∈M, $\alpha \in \Gamma$:
$1 - \delta_1$ and δ_2 are JLD of M .
$2 - \delta_3(a\alpha a) = 0.$
$3 - \delta_4(a\alpha a) = 0.$

Proof: Since $D(e_{11}(a\alpha a)) = D(e_{11}a\alpha e_{11}a)$
and since D is JLD, then
$D(e_{11}(a\alpha a)) = 2e_{11}a\ \alpha\ D(e_{11}a)$

$$\begin{bmatrix} \delta_1(a\alpha a) & \delta_2(a\alpha a) \\ \delta_3(a\alpha a) & \delta_4(a\alpha a) \end{bmatrix}=2\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\alpha\begin{bmatrix} \delta_1(a) & \delta_2(a) \\ \delta_3(a) & \delta_4(a) \end{bmatrix}$$

$$\begin{bmatrix} \delta_1(a\alpha a) & \delta_2(a\alpha a) \\ \delta_3(a\alpha a) & \delta_4(a\alpha a) \end{bmatrix}=\begin{bmatrix} 2a\alpha\delta_1(a) & 2a\alpha\delta_2(a) \\ 0 & 0 \end{bmatrix}$$

Then $\delta_1(a\alpha a) = 2a\alpha\delta_1(a), \delta_2(a\alpha a) = 2a\alpha\delta_2(a), \delta_3(a\alpha a) = 0$ and $\delta_4(a\alpha a) = 0$.

Lemma 2.2 : The mappings $\Omega_1, \Omega_2, \Omega_3$, and $\Omega_4$satisfy the following conditions for every d∈M and $\alpha \in \Gamma$:
1- Ω_3 and $\Omega_4$are JLD of M .
2-$\Omega_1(d\alpha d) = 0.$
3-$\Omega_2(d\alpha d) = 0.$

Proof: Since $D(e_{22}d\alpha d) = D(e_{22}d\alpha e_{22}d)$
and since D is JLD, then
$D(e_{22}d\alpha d) = 2e_{22}d\ \alpha D(e_{22}d)$

$$\begin{bmatrix} \Omega_1(d\alpha d) & \Omega_2(d\alpha d) \\ \Omega_3(d\alpha d) & \Omega_4(d\alpha d) \end{bmatrix}=2\begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix}\alpha\begin{bmatrix} \Omega_1(d) & \Omega_2(d) \\ \Omega_3(d) & \Omega_4(d) \end{bmatrix}$$

$$\begin{bmatrix}\Omega_1(d\alpha d) & \Omega_2(d\alpha d)\\ \Omega_3(d\alpha d) & \Omega_4(d\alpha d)\end{bmatrix}=\begin{bmatrix}0 & 0\\ 2d\alpha\Omega_3(d) & 2d\alpha\Omega_4(d)\end{bmatrix}$$

then $\Omega_3(d\alpha d) = 2d\alpha\Omega_3(d)$, $\Omega_4(d\alpha d) = 2d\alpha\Omega_4(d)$, $\Omega_1(d\alpha d) = 0$ and $\Omega_2(d\alpha d) = 0$.

Lemma 2.3: If a,b∈ M and $\alpha \in \Gamma$, then

1- $\xi_1(a\alpha b) = 2a\alpha\xi_1(b) + 2b\alpha\delta_3(a)\alpha q$

2- $\xi_2(a\alpha b) = 2a\alpha\xi_2(b) + 2b\alpha\delta_4(a)$

3- $\xi_3(a\alpha b) = 0$

4- $\xi_4(a\alpha b) = 0$

Proof: Since $e_{12} = e_{11}e_{12} + e_{12}e_{11}$, then

$D(e_{12}a\alpha b) = D(e_{11}a\alpha e_{12}b + e_{12}b\alpha e_{11}a)$

$$\begin{bmatrix}\xi_1(a\alpha b) & \xi_2(a\alpha b)\\ \xi_3(a\alpha b) & \xi_4(a\alpha b)\end{bmatrix} = 2e_{11}a\alpha D(e_{12}b) + 2\, e_{12}b\,\alpha\, D(e_{11}a)$$

$$= 2\begin{bmatrix}a & 0\\ 0 & 0\end{bmatrix}\alpha\begin{bmatrix}\xi_1(b) & \xi_2(b)\\ \xi_3(b) & \xi_4(b)\end{bmatrix} + 2\begin{bmatrix}0 & b\\ 0 & 0\end{bmatrix}\alpha\begin{bmatrix}\delta_1(a) & \delta_2(a)\\ \delta_3(a) & \delta_4(a)\end{bmatrix}$$

$$=\begin{bmatrix}2a\alpha\,\xi_1(b) & 2a\,\alpha\xi_2(b)\\ 0 & 0\end{bmatrix} + \begin{bmatrix}2b\alpha\,\delta_3(a)\alpha q & 2b\,\alpha\delta_4(a)\\ 0 & 0\end{bmatrix}$$

$$=\begin{bmatrix}2a\,\alpha\xi_1(b) + 2b\,\alpha\delta_3(a)\alpha q & 2a\alpha\,\xi_2(b) + 2b\,\alpha\delta_4(a)\\ 0 & 0\end{bmatrix}$$

Lemma 2.4: The mappings $l_1, l_2, l_3,$ and l_4 satisfy the following condition for every c and d∈ M and $\alpha \in \Gamma$:

1-$l_1(d\alpha c) = 0$

2-$l_2(d\alpha c) = 0$

3-$l_3(d\alpha c) = 2d\alpha l_3(c) + 2c\alpha\sigma(\Omega_1(d))$

4-$l_4(d\alpha c) = 2d\alpha l_4(c) + 2c\alpha\sigma(\Omega_2(d))\alpha q$

Proof: Since $e_{21} = e_{22}\, e_{21} + e_{21}\, e_{22}$, then

$D(e_{21}\, d\alpha c) = D(e_{22}\, d\alpha e_{21}c + e_{21}c\alpha e_{22}\, d)$

$$\begin{bmatrix}l_1(d\alpha c) & l_2(d\alpha c)\\ l_3(d\alpha c) & l_4(d\alpha c)\end{bmatrix} = 2e_{22}\, d\alpha D(e_{21}c) + 2e_{21}c\alpha D\,(e_{22}\, d)$$

$$= \begin{bmatrix}0 & 0\\ 0 & 2d\end{bmatrix}\alpha\begin{bmatrix}l_1(c) & l_2(c)\\ l_3(c) & l_4(c)\end{bmatrix} + \begin{bmatrix}0 & 0\\ 2c & 0\end{bmatrix}\alpha\begin{bmatrix}\Omega_1(d) & \Omega_2(d)\\ \Omega_3(d) & \Omega_4(d)\end{bmatrix}$$

$$=\begin{bmatrix}0 & 0\\ 2d\alpha l_3(c) & 2d\alpha l_4(c)\end{bmatrix} + \begin{bmatrix}0 & 0\\ 2c\alpha\sigma(\Omega_1(d)) & 2c\alpha\sigma(\Omega_2(d))\alpha q\end{bmatrix}$$

$$=\begin{bmatrix}0 & 0\\ 2d\alpha l_3(c) + 2c\alpha\sigma(\Omega_1(d)) & 2d\alpha l_4(c) + 2c\alpha\sigma(\Omega_2(d))\alpha q\end{bmatrix}$$

Theorem 2.5: If M is a gamma ring and D is a JLD of $M_2(M,\Gamma;\sigma,q)$,

then $D\begin{bmatrix}a & b\\ c & d\end{bmatrix} = \begin{bmatrix}\delta_1(a) + \xi_1(b) + l_1(c) + \Omega_1(d) & \delta_2(a) + \xi_2(b) + l_2(c) + \Omega_2(d)\\ \delta_3(a) + \xi_3(b) + l_3(c) + \Omega_3(d) & \delta_4(a) + \xi_4(b) + l_4(c) + \Omega_4(d)\end{bmatrix}$

such that

1-$\delta_3(a\alpha a) = 0$,$\delta_4(a^2) = 0$,δ_1, δ_2 are Jordan left derivation of M.

2-$\Omega_1(d\alpha d) = 0, \Omega_2(d\alpha d) = 0$ Ω_3 and $\Omega_4$are Jordan left derivation of M.

3-

$\xi_1(a\alpha b) = 2a\,\alpha\xi_1(b) + 2b\alpha\,\delta_3(a)\alpha q$,$\xi_2(a\alpha b) = 2a\,\alpha\xi_2(b) + 2b\alpha\delta_4(a)$,$\xi_3(a\alpha b) = 0$ and $\xi_4(a\alpha b) = 0$

4-$l_1(d\alpha c) = 0, l_2(d\alpha c) = 0$,$l_3(d\alpha c) = 2d\alpha l_3(c) + 2c\alpha\sigma(\Omega_1(d))$and $l_4(d\alpha c) = 2d\alpha l_4(c) + 2c\alpha\sigma\big(\Omega_2(d)\big)\alpha q$.

Proof: Since $D\begin{bmatrix}a & b\\ c & d\end{bmatrix} = D(e_{11}\, a) + D(e_{12}\, b) + D(e_{21}c) + D(e_{22}\, d)$

$$=\begin{bmatrix}\delta_1(a) & \delta_2(a)\\ \delta_3(a) & \delta_4(a)\end{bmatrix} + \begin{bmatrix}\xi_1(b) & \xi_2(b)\\ \xi_3(b) & \xi_4(b)\end{bmatrix} + \begin{bmatrix}l_1(c) & l_2(c)\\ l_3(c) & l_4(c)\end{bmatrix} + \begin{bmatrix}\Omega_1(d) & \Omega_2(d)\\ \Omega_3(d) & \Omega_4(d)\end{bmatrix}$$

$$=\begin{bmatrix}\delta_1(a) + \xi_1(b) + l_1(c) + \Omega_1(d) & \delta_2(a) + \xi_2(b) + l_2(c) + \Omega_2(d)\\ \delta_3(a) + \xi_3(b) + l_3(c) + \Omega_3(d) & \delta_4(a) + \xi_4(b) + l_4(c) + \Omega_4(d)\end{bmatrix}$$

Then, by the assertion Lemmas 2.1, 2.2 , 2.3, and 2.4, we shall get the result .

Lemma 2.6 : If M is a gamma ring with 1 and D is a JLD of $M_2(M,\Gamma;\sigma,q)$, then there exist additive mappings $\delta_3, \delta_4, \xi_3, \xi_4, l_1, l_2, \Omega_1,$ and Ω_2 :M→M

and elements $\mu, \vartheta, \omega, \tau, \gamma, \aleph, \rho, \varepsilon$ in M, such that for all a,b,c,d in M, we have

$D(e_{11}\ a)=\begin{bmatrix} a\alpha\,\mu & a\alpha\,\vartheta \\ \delta_3(a) & \delta_4(a) \end{bmatrix}$,
$D(e_{12}\, b)=\begin{bmatrix} 2b\,\alpha\omega + 2\,\delta_3(b)\alpha q & 2b\,\alpha\tau + 2\delta_4(b) \\ \xi_3(b) & \xi_4(b) \end{bmatrix}$
$D(e_{21}c)=\begin{bmatrix} l_1(c) & l_2(c) \\ 2c\alpha\gamma + 2\sigma(\Omega_1(c)) & 2c\alpha\aleph + 2\sigma(\Omega_2(c))\alpha q \end{bmatrix}$, $D(e_{22}\ d)=\begin{bmatrix} \Omega_1(d) & \Omega_2(d) \\ d\alpha\rho & d\alpha\,\varepsilon \end{bmatrix}$

Proof : By Lemma 2.1.1, we have

$$\delta_1(a\alpha a) = 2a\alpha\delta_1(a)$$

then

$$\delta_1(a\alpha b + b\alpha a) = a\alpha\delta_1(b) + b\alpha\delta_1(a)$$

By putting b=1, we get

$$\delta_1(a) = a\alpha\,\delta_1(1)$$

And since $\delta_2(a\alpha a) = 2a\alpha\delta_2(a)$, then $\delta_2(a) = a\alpha\delta_2(1)$.
We choose $\mu = \delta_1(1), \vartheta = \delta_2(1)$, then we have

$$\delta_1(a) = a\alpha\,\mu$$

and

$$\delta_2(a) = a\alpha\,\vartheta$$

By the same way, we get from lemma 2.2 that $\Omega_3(d) = d\alpha\Omega_3(1),\ \Omega_4(d) = d\alpha\Omega_4(1)$.
We choose $\rho = \Omega_3(1)$ and $\varepsilon = \Omega_4(1)$, then$\Omega_3(d) = d\alpha\rho,\ \Omega_4(d) = d\alpha\,\varepsilon$.
And from lemma 2.3, we have
$\xi_1(a\alpha b) = 2a\,\alpha\xi_1(b) + 2b\alpha\,\delta_3(a)\alpha q$, and by putting a=1 ,we get
$\xi_1(a) = 2a\,\alpha\xi_1(1) + 2\,\delta_3(a)\alpha q$
and
$\xi_2(a\alpha b) = 2a\,\alpha\xi_2(b) + 2b\alpha\delta_4(a)$ also by putting b=1,we get $\xi_2(a) = 2a\,\alpha\,\xi_2(1) + 2\delta_4(a)$
By putting $\tau = \xi_2(1)$ and $\omega = \xi_1(1)$, and from lemma 2.4, we have

$$l_3(d\alpha c) = 2d\alpha l_3(c) + 2c\alpha\sigma(\Omega_1(d))$$

Also, we put c=1,$l_3(d) = 2d\alpha l_3(1) + 2\sigma(\Omega_1(d))$
and $l_4(d\alpha c) = 2d\alpha l_4(c) + 2c\alpha\sigma(\Omega_2(d))\alpha q$. Furthermore, we put c=1,$l_4(d) = 2d\alpha l_4(1) + 2\sigma(\Omega_2(d))\alpha q$. We then put $\gamma = l_3(1)$ and $l_4(1) = \aleph$. Then, we get the result .

Theorem 2.7: If M is a gamma ring with 1 and D :$M_2(M,\Gamma;\sigma,q) \to M_2(M,\Gamma;\sigma,q)$ is an additive mapping .Then, D is a JLD if and only if there exist additive mappings $\delta_3, l_1, \Omega_1, ,\xi_3, \delta_4, \xi_4$ and elements $\mu, \omega, \vartheta, \tau, \gamma, \rho, \aleph, \varepsilon$ satisfying the four conditions in Lemma 2.6 with the following conditions, for all a,b,c,d in M.
(i) $\delta_3(a\alpha a) = 0$.
$(ii)\delta_4(a\alpha a) = 0$.
(iii)$\Omega_1(d\alpha d) = 0$.
(iv) $\Omega_2(d\alpha d) = 0$.
(v)$\xi_3(a\alpha b) = 0$.
(vi)$\xi_4(a\alpha b) = 0$.
(vii)$l_1(d\alpha c) = 0$.
(viii) $l_2(d\alpha c) = 0$.
Particularly, a Jordan left derivation of $M_2(M,\Gamma;\sigma,q)$ is given by
$D\begin{bmatrix} a & b \\ c & d \end{bmatrix} =$
$\begin{bmatrix} a\alpha\,\mu + 2b\,\alpha\omega + 2\,\delta_3(b)\alpha q + l_1(c) + \Omega_1(d) & a\alpha\,\vartheta + 2b\,\alpha\tau + 2\delta_4(b) + l_2(c) + \Omega_2(d) \\ \delta_3(a) + \xi_3(b) + 2c\alpha\gamma + 2\sigma(\Omega_1(c)) + d\alpha\rho & \delta_4(a) + \xi_4(b) + 2c\alpha\aleph + 2\sigma(\Omega_2(c))\alpha q + d\alpha\,\varepsilon \end{bmatrix}$.
Proof: Assume that D is A Jordan left derivation of $M_2(M,\Gamma;\sigma,q)$. Then, D satisfies the above condition for some additive mappings, by Lemmas 2.6 and 2.4.
Conversely, if the mappings $\delta_3, l_1, \Omega_1, \Omega_2, \xi_3, \delta_4, \xi_4$ and elements $\mu, \omega, \vartheta, \tau, \gamma, \rho, \aleph, \varepsilon$ satisfy the above condition, then we can show that, for any A=$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $M_2(M,\Gamma;\sigma,q)$,
$D(A\alpha A) = 2D(A))\,\alpha A$,
by direct calculation.

3- On Skew Matrix Gamma Ring and Jordan Left Centralizer

Let J be a JLC of $M_2(M,\Gamma;\sigma,q)$. First, we set

$J(e_{11}a)=\begin{bmatrix}\delta_1(a) & \delta_2(a)\\ \delta_3(a) & \delta_4(a)\end{bmatrix}$, $J(e_{12}\,b)=\begin{bmatrix}l_1(b) & l_2(b)\\ l_3(b) & l_4(b)\end{bmatrix}$

$J(e_{21}c)=\begin{bmatrix}k_1(c) & k_2(c)\\ k_3(c) & k_4(c)\end{bmatrix}$, $J(e_{22}\,d)=\begin{bmatrix}\eta_1(d) & \eta_2(d)\\ \eta_3(d) & \eta_4(d)\end{bmatrix}$

where $\delta_i, k_i, \eta_i, l_i: M \to M$ are linear maps.

Lemma 3.1 : The mappings δ_1, δ_2 , δ_3 and δ_3 satisfy the following conditions for every a∈M and $\alpha \in \Gamma$.

1- $\delta_1$is JLC of M .
2- $\delta_2(a\alpha a) = 0$.
3-$\delta_3(a\alpha a) = \delta_3(a)\alpha\sigma(a)$.
4- $\delta_4(a\alpha a) = 0$.

Proof: Since $J(e_{11}a\alpha a) = J(e_{11}a\alpha e_{11}a)$

and since J is JLC, then

$J(e_{11}a\alpha a) = J(e_{11}a)\alpha e_{11}a$

$$\begin{bmatrix}\delta_1(a\alpha a) & \delta_2(a\alpha a)\\ \delta_3(a\alpha a) & f_4(a\alpha a)\end{bmatrix}=\begin{bmatrix}\delta_1(a) & \delta_2(a)\\ \delta_3(a) & \delta_4(a)\end{bmatrix}\alpha\begin{bmatrix}a & 0\\ 0 & 0\end{bmatrix}$$

$$\begin{bmatrix}\delta_1(a\alpha a) & \delta_2(a\alpha a)\\ \delta_3(a\alpha a) & \delta_4(a\alpha a)\end{bmatrix}=\begin{bmatrix}\delta_1(a)\alpha a & 0\\ \delta_3(a)\alpha\sigma(a) & 0\end{bmatrix}$$

Then $\delta_1(a\alpha a) = \delta_1(a)\alpha a, \delta_2(a\alpha a) = 0, \delta_3(a\alpha a) = \delta_3(a)\alpha\sigma(a)$ and $\delta_4(a\alpha a) = 0$.

Lemma 3.2 : If d∈M and $\alpha \in \Gamma$, then the mappings η_1, η_2, η_3 and η_4 satisfy the following

1- η_2, η_4 are JLCs of M .
2- $\eta_1(d\alpha d) = 0$.
3- $\eta_3(d\alpha d) = 0$.

Proof: Since

$J(e_{22}d\alpha d) = J((e_{22}d)\alpha\,(e_{22}d))$

and since J is JLC, then

$$J(e_{22}d\alpha d) = J(e_{22}d)\alpha\, e_{22}d\ \begin{bmatrix}H_1(d\alpha d) & \eta_2(d\alpha d)\\ \eta_3(d\alpha d) & \eta_4(d\alpha d)\end{bmatrix} = \begin{bmatrix}\eta_1(d) & \eta_2(d)\\ \eta_3(d) & \eta_4(d)\end{bmatrix}\alpha\begin{bmatrix}0 & 0\\ 0 & d\end{bmatrix} = \begin{bmatrix}0 & \eta_2(d)\alpha d\\ 0 & \eta_4(d)\alpha d\end{bmatrix}.$$

Then $\eta_1(d\alpha d) = 0, \eta_3(d\alpha d) = 0$ & η_2, η_4 are JLCs of M.

Lemma 3.3 : The mappings l_1, l_2, l_3 and l_4 satisfy the following condition for every a, b∈ M and $\alpha \in \Gamma$:

For every a, b∈ M,

1- $l_1(a\alpha b) = l_1(b)\alpha a$.
2- $l_2(a\alpha b) = \delta_1(a)\alpha b$.
3- $l_3(a\alpha b) = l_3\,(b)\alpha\sigma(a)$.
4- $l_4(a\alpha b) = \delta_3(a)\alpha\sigma(b)\alpha q$.

Proof: Since $J(e_{12}a\alpha b) = J(e_{11}a\alpha e_{12}b + e_{12}b\alpha e_{11}a)$

$$\begin{bmatrix}l_1(a\alpha b) & l_2(a\alpha b)\\ l_3(a\alpha b) & l_4(a\alpha b)\end{bmatrix} = J(e_{11}a)\alpha e_{12}b + J(\,e_{12}b\,)\,\alpha e_{11}a$$

$$= \begin{bmatrix}\delta_1(a) & \delta_2(a)\\ \delta_3(a) & \delta_4(a)\end{bmatrix}\alpha\begin{bmatrix}0 & b\\ 0 & 0\end{bmatrix} + \begin{bmatrix}l_1(b) & l_2(b)\\ l_3(b) & l_4(b)\end{bmatrix}\alpha\begin{bmatrix}a & 0\\ 0 & 0\end{bmatrix}$$

$$=\begin{bmatrix}0 & \delta_1(a)\alpha b\\ 0 & \delta_3(a)\alpha\sigma(b)\alpha q\end{bmatrix} + \begin{bmatrix}l_1(b)\alpha a & 0\\ l_3\,(b)\alpha\sigma(a) & 0\end{bmatrix}$$

$$=\begin{bmatrix}l_1(b)\alpha a & \delta_1(a)\alpha b\\ l_3\,(b)\alpha\sigma(a) & \delta_3(a)\alpha\sigma(b)\alpha q\end{bmatrix},$$

then $l_1(a\alpha b) = l_1(b)\alpha a, l_2(a\alpha b) = \delta_1(a)\alpha b, l_3(a\alpha b) = l_3\,(b)\alpha\sigma(a)$ and $l_4(a\alpha b) = \delta_3(a)\alpha\sigma(b)\alpha q$.

Lemma 3.4 : The mappings k_1, k_2, k_3 and k_4 satisfy the following conditions for every c ,d∈ M, $\alpha \in \Gamma$:

1- $k_1(d\alpha c) = \eta_2(d)\,\alpha c\alpha q$.
2- $k_2(d\alpha c) = \eta_2(c)\alpha d$.
3- $k_3(d\alpha c) = \eta_4(d)\alpha c$.

4- $k_4(d\alpha c) = k_4(c)\alpha d$.

Proof: Since

$$J(e_{21}\, d\alpha c) = J(e_{22}\, d\alpha e_{21} c + e_{21} c\alpha e_{22}\, d)$$

$$\begin{bmatrix} k_1(d\alpha c) & k_2(d\alpha c) \\ k_3(d\alpha c) & k_4(d\alpha c) \end{bmatrix} = J(e_{22}\, d)\alpha e_{21} c + J(e_{21} c)\alpha e_{22}\, d$$

$$= \begin{bmatrix} \eta_1(d) & \eta_2(d) \\ \eta_3(d) & \eta_4(d) \end{bmatrix} \alpha \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} + \begin{bmatrix} k_1(c) & k_2(c) \\ k_3(c) & k_4(c) \end{bmatrix} \alpha \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix}$$

$$= \begin{bmatrix} \eta_2(d)\, \alpha c\alpha q & 0 \\ \eta_4(d)\alpha\, c & 0 \end{bmatrix} + \begin{bmatrix} 0 & k_2(c)\alpha d \\ 0 & k_4(c)\alpha d \end{bmatrix}$$

$$= \begin{bmatrix} \eta_2(d)\, \alpha c\alpha q & k_2(c)\alpha d \\ \eta_4(d)\, \alpha c & k_4(c)\alpha d \end{bmatrix}$$

$k_1(d\alpha c) = \eta_2(d)\alpha c\alpha q$, $k_2(d\alpha c) = k_2(c)\alpha d$

$k_3(d\alpha c) = \eta_4(d)\, \alpha c$, $k_4(d\alpha c) = k_4(c)\alpha d$

Theorem 3.5: If M is a gamma ring and J is a Jordan left centralizer of $M_2(M, \Gamma; \sigma, q)$, then

$$J\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \delta_1(a) + l_1(b) + k_1(c) + \eta_1(d) & \delta_2(a) + l_2(b) + k_2(c) + \eta_2(d) \\ \delta_3(a) + l_3(b) + k_3(c) + \eta_3(d) & \delta_4(a) + l_4(b) + k_4(c) + \eta_4(d) \end{bmatrix}$$

such that

1- δ_1 is JLC of M , $\delta_2(a\alpha a) = 0, \delta_3(a\alpha a) = \delta_3(a)\alpha\sigma(a)$, and $\delta_4(a\alpha a) = 0$.

2- η_2, η_4 are JLCs of R , $\eta_1(d\alpha d) = 0$, and $\eta_3(d\alpha d) = 0$.

3- $l_1(a\alpha b) = l_1(b)\alpha a$, $l_2(a\alpha b) = \delta_1(a)\alpha b$, $l_3(a\alpha b) = l_3(b)\alpha\sigma(a)$, and $l_4(a\alpha b) = \delta_3(a)\alpha\sigma(b)\alpha q$.

4- $k_1(d\alpha c) = \eta_2(d)\alpha c\alpha q$ and $k_2(d\alpha c) = k_2(c)\alpha d$.

$k_3(d\alpha c) = \eta_4(d)\, \alpha c$ and $k_4(d\alpha c) = k_4(c)\alpha d$.

Proof: Since $J\begin{bmatrix} a & b \\ c & d \end{bmatrix} = J(e_{11}\, a) + J(e_{12}\, b) + J(e_{21} c) + J(e_{22}\, d)$

$$= \begin{bmatrix} \delta_1(a) & \delta_2(a) \\ \delta_3(a) & \delta_4(a) \end{bmatrix} + \begin{bmatrix} l_1(b) & l_2(b) \\ l_3(b) & l_4(b) \end{bmatrix} + \begin{bmatrix} k_1(c) & k_2(c) \\ k_3(c) & k_4(c) \end{bmatrix} + \begin{bmatrix} \eta_1(d) & \eta_2(d) \\ \eta_3(d) & \eta_4(d) \end{bmatrix}$$

$$= \begin{bmatrix} \delta_1(a) + l_1(b) + k_1(c) + \eta_1(d) & \delta_2(a) + l_2(b) + k_2(c) + \eta_2(d) \\ \delta_3(a) + l_3(b) + k_3(c) + \eta_3(d) & \delta_4(a) + l_4(b) + k_4(c) + \eta_4(d) \end{bmatrix},$$

then by lemmas 3.1, 3.2, 3.3 and 3.4 , we have the result.

Lemma 3.6 : If M is a gamma ring with 1 and J is a JLC of $M_2(M, \Gamma; \sigma, q)$, then there exist additive mappings δ_2, δ_4 , η_1 , η_3:M→M and elements $\mu, \vartheta, \tau, \aleph, \rho, \varepsilon, \theta$ in M, such that for all a, b ,c ,d in M ,

$J(e_{11}a) = \begin{bmatrix} \varepsilon\alpha a & \delta_2(a) \\ \rho\alpha\sigma(a) & \delta_4(a) \end{bmatrix}$, $J(e_{12}\, b) = \begin{bmatrix} \mu\alpha b & \varepsilon\alpha b \\ \theta\alpha\sigma(b) & \rho\, \alpha\sigma(b)\alpha q \end{bmatrix}$

$J(e_{21}c) = \begin{bmatrix} \tau\alpha c\alpha q & \varepsilon\alpha c \\ \vartheta\alpha c & \aleph\alpha c \end{bmatrix}$, $J(e_{22}\, d) = \begin{bmatrix} \eta_1(d) & \tau\alpha d \\ \eta_3(d) & \vartheta\alpha d. \end{bmatrix}$

Proof : From lemma 3.1, we have $\delta_1(a\alpha a) = \delta_1(a)\alpha a$ for all a in M.

Then, $\delta_1(a\alpha b + b\alpha a) = \delta_1(a)\alpha b + \delta_1(b)\alpha a$ for all a,b in M . By putting b=1 , we get $\delta_1(a) = \delta_1(1)\alpha a$.

And since $\delta_3(a\alpha a) = \delta_3(a)\alpha\sigma(a)$, then

$\delta_3(a\alpha b + b\alpha a) = \delta_3(a)\alpha\sigma(b) + \delta_3(b)\alpha\sigma(a)$. By Putting b=1,

then $\delta_3(a) = \delta_3(1)\alpha\sigma(a)$.

We put $\varepsilon = \delta_1(1)$ and $\rho = \delta_3(1)$, then $\delta_1(a) = \varepsilon\alpha a$ and $\delta_3(a) = \rho\alpha\sigma(a)$.

From Lemma 3.2, $\eta_2(d\alpha d) = \eta_2(d)\alpha d$, $\eta_4(d\alpha d) = \eta_4(d)\alpha d$

and $\eta_2(d) = \eta_2(1)\alpha d$, $\eta_4(d) = \eta_4(1)\alpha d$. By putting $\tau = \eta_2(1)$ and $\vartheta = \eta_4(1)$, we get $\eta_2(d) = \tau\alpha d$ and $\eta_4(d) = \vartheta\alpha d$.

By the same way, from lemma 3.3, $l_1(a\alpha b) = l_1(b)\alpha a \rightarrow l_1(a) = l_1(1)\alpha a$. We put $l_1(1) = \mu$, then $l_1(a) = \mu\alpha a$,

$l_2(a\alpha b) = \delta_1(a)\alpha b \rightarrow l_2(a) = \delta_1(a)$,

$l_3(a\alpha b) = l_3(b)\alpha\sigma(a) \rightarrow l_3(a) = l_3(1)\alpha\sigma(a)$,put $l_3(1) = \theta$ then $l_3(a) = \theta\alpha\sigma(a)$,

and $l_4(a\alpha b) = \delta_3(a)\alpha\sigma(b)\alpha q \rightarrow l_4(b) = \delta_3(1)\alpha\sigma(b)\alpha q = \rho\, \alpha\sigma(b)\alpha q$.

Also, from lemma 3.4, $k_1(d\alpha c) = \eta_2(d)\alpha c\alpha q$, then $k_1(d\alpha c) = \tau\alpha d\alpha c\alpha q$, and so $k_1(c) = \tau\alpha c\alpha q$, $k_2(d\alpha c) = k_2(c)\alpha d$, implies that $k_2(d) = k_2(1)\alpha d$. We put $\varepsilon = k_2(1)$, then we get $k_2(d) =$

$\varepsilon\alpha \mathrm{d}k_3(\mathrm{d}\alpha\mathrm{c}) = \eta_4(\mathrm{d})\,\alpha\mathrm{c} = \vartheta\alpha\mathrm{d}$ and $k_4(\mathrm{d}\alpha\mathrm{c}) = k_4(\mathrm{c})\alpha\mathrm{d}$, implies that $k_4(\mathrm{d}) = k_4(1)\alpha\mathrm{d}$. We put $\aleph = k_4(1)$, then we get $k_4(\mathrm{d}) = \aleph\alpha\mathrm{d}$.

Theorem 3.7: If M is a gamma ring with 1, then J is a JLC of $M_2(M,\Gamma;\sigma,q)$, if and only if there exist additive mappings $\delta_2, \delta_4, \eta_1, \eta_3$:M→M and elements $\mu, \vartheta, \tau, \aleph, \rho, \varepsilon, \theta$ in M, such that for all a, b, c, d in M, such that

1- $\delta_2(\mathrm{a}\alpha a) = 0$ and $\delta_4(\mathrm{a}\alpha a) = 0$.

2- $\eta_1(\mathrm{d}\alpha d) = 0$ and $\eta_3(\mathrm{d}\alpha d) = 0$.

Particularly,

$$J\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \varepsilon\alpha a + \mu\alpha b + \tau\alpha c\alpha q + \eta_1(d) & \delta_2(a) + \varepsilon\alpha b + \varepsilon\alpha c + \tau\alpha d \\ \rho\alpha\sigma(a) + \theta\alpha\sigma(b) + \vartheta\alpha c + \eta_3(d) & \delta_4(a) + \rho\,\alpha\sigma(b)\alpha q + \aleph\alpha c + \vartheta\alpha d \end{bmatrix}$$

Proof : Suppose that J is a JLC of $M_2(M,\Gamma;\sigma,q)$, then by lemma 2.6, these conditions are satisfied . Conversely, suppose that J satisfies the conditions above , then we can show that, for any A=$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $M_2(M,\Gamma;\sigma,q)$,

$$J(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\alpha\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = J(\begin{bmatrix} a & b \\ c & d \end{bmatrix})\,\alpha\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

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