



ISSN: 0067-2904

CutPoints and Separations in Alpha- Connected Topological Spaces

Balqees K. Mahmoud*, Yousif Y. Yousif

Department of Mathematics, College of Education for Pure Sciences, (Ibn Al-Haitham), University of Baghdad, Baghdad, Iraq

Received: 1/7/2020

Accepted: 20/2/2021

Abstract

This paper introduces cutpoints and separations in α -connected topological spaces, which are constructed by using the union of vertices set and edges set for a connected graph, and studies the relationships between them. Furthermore, it generalizes some new concepts.

Keywords: cutpoints, separation of space, surrounding set, connected space.

نقاط القطع والانفصالات في فضاءات الفا التوبولوجية المتصلة

بلقيس خليل*، يوسف يعقوب يوسف

قسم الرياضيات، كلية التربية للعلوم الصرفة (ابن الهيثم)، جامعة بغداد، بغداد، العراق

الخلاصة

يقدم هذا البحث نقاط القطع والانفصالات في فضاءات الفا التوبولوجية المبنية باستخدام مجموعة الاتحاد بين مجموعة الرؤوس ومجموعة الاضلاع للبيان المتصل ويدرس العلاقات بينها. اضافة الى تعميم بعض المفاهيم الجديدة.

1- Introduction

The concepts of α -open sets and α -closed sets in topological spaces were defined and studied firstly in [1] 1965 by O. Njastad. After this study, many mathematicians have generalized and derived other definitions in there researches. These studies were developed and many new relationships between these terms were found [2, 3].

In general topology, the important concepts of cutpoints and cutpoint spaces have been studied in connected topological spaces [4-9]. Also, separations are associated with the term of cutpoints. Many references have taken these terms and discussed the relationships between them [10, 11].

Throughout the last decades, the graph theory has been an essential part of combinatorial applications. The fundamental ideas appeared by Euler in 1736, when he solved a problem by introducing a graph that he constructed [12]. Thereafter, the graph theory became an important part of mathematics. As the applications have been increasingly appearing in multiple aspects in mathematics, there has been a growing interest in this concept [13, 14]. Our research topic has further relations with additional previous works [15, 16].

In this paper, we introduce the concept of cutpoints, with the topological viewpoint in α -connected topological spaces, and study separations and connectedness in the α -topological spaces. Finally, we prove some relationships and give a counter example.

*Email: balqees.khaleel1003a@ihcoedu.uobaghdad.edu.iq

2. Preliminaries and basic definitions

We need to recall some basic topological definitions and remarks with some definitions and facts for a graph.

Let (X, τ) be any topological space and A be a subset of X . If A satisfied the condition $A \subseteq \text{Int}(\text{Cl}(\text{Int}(A)))$ for all $A \subseteq X$, then A is called an α -open set. The set of all α -open sets form α -topology of X , which is denoted as τ_α . So, the pair (X, τ_α) is called an α -topological space, and the complement of A is an α -closed set or $\text{Cl}(\text{Int}(\text{Cl}(A))) \subseteq A$.

The interior of these spaces is denoted as $\alpha\text{-Int}(A)$, which means the union of all α -open sets containing A , and the closure is denoted as $\alpha\text{-Cl}(A)$, which means the intersection of all α -closed sets contained in A .

We conclude, by definition, that every open set is α -open, but the converse is not true in general.

We must refer to our topological space that it's structured under special conditions about the element of the set that built it.

If G is any graph with V_G vertices set and E_G edges set, then we take the set $X = V_G \cup E_G$ and define τ_G on it, then the pair (X, τ_G) is called topological space of the graph G with ground set $V_G \cup E_G$, and τ_G satisfied the conditions of general topology [11].

Definition 2.1 [4]: A topological space X is called connected space if there do not exist two disjoint, nonempty, open sets A and B such that $X = A \cup B$. Otherwise, X is disconnected.

Any subset B of X is connected if the subspace of B is connected.

Definition 2.2 [11]: A topological space X is called separated space if there are two unordered disjoint pairs of subsets of X such that each one is the complement of the other (unordered pair means if $B \subseteq X$, then $(B, X \setminus B) = (X \setminus B, B)$).

If A, B are two subsets of X and if $\text{Cl}(A) \cap B = A \cap \text{Cl}(B) = \emptyset$, then they are mutually separated in X (i.e. A, B are clopen) [11].

So, the separation on the subspace of the topological space X is defined as follows.

Let Y be a subspace of a topological space X , then A, B are mutually separated in X if and only if there is a disjoint bipartition $\{A, B\}$ which is a separation of Y related to the relative topology of Y [11].

Proposition 2.3 [11]: If A, B are two subsets of X and C is a connected subset of $A \cup B$. If A, B are mutually separated subsets of a topological space X , then either $C \subseteq A$ or $C \subseteq B$.

Definition 2.4 [11]: If X is a topological space, then A, B , and C are disjoint subsets of X . If there exists a separation $\{H, K\}$ of $X \setminus A$ with $B \subseteq H$ and $C \subseteq K$, then A separates B and C .

We can use the same discussion when the subsets are singletons.

Definition 2.5 [9]: If X is a topological space and A is a subset of X , then the intersection of $\text{Cl}(A)$ and $\text{Cl}(X - A)$ is called the boundary set of A .

Definition 2.6 [9]: If X is any topological space, then the component of x is the largest connected subset of X containing x , and denoted as $K(x)$.

Proposition 2.7 [11]: Let X be a topological space and K be a component of X . If $\{U, V\}$ is a separation of X , then K is contained either in U or in V .

Definition 2.8 [4]: Let X be a connected topological space and x be any point in X . If $X \setminus \{x\}$ is not connected, then x is called a cutpoint of X . Otherwise, x is a non-cutpoint.

Definition 2.9 [11]: Let X be a connected topological space and x be any point in X . If x is a cutpoint, then it is called a cutedge when it is an edge. However, it is called an endpoint if it is a non-cutpoint.

Definition 2.10 [11]: Let X be a topological space and $B \subseteq X$, then the intersection of all open sets that contain B is called the surrounding set of B , and denoted by B° .

Definition 2.11 [11]: Let X be a topological space, then every point in X which is open but not closed is called hyperedge.

When the boundary of the hyperedge contains exactly one point, it is called a loop. However, when its boundary contains at most two points, it is called an edge. Otherwise, it is a proper edge.

We refer to the endvertex v , that means that there is only one edge that is incident on v [11].

Theorem 2.12 [11]: Let X be a connected topological space, x is a cutpoint of X , then:

1. there exist non-empty open sets (U_1, U_2) in X , such that $U_1 \cup U_2 \cup \{x\} = X$, and $\{U_1 \setminus \{x\}, U_2 \setminus \{x\}\}$ is a separation of $X \setminus \{x\}$ satisfying exactly one of the following statements:

- $\{x\} = (U_1 \cup U_2)$ and it is open.
- $\{U_1, U_2, \{x\}\}$ is a partition of X with $\{x\}$ is closed.

2. there exist nonempty closed sets (F_1, F_2) in X , such that $F_1 \cup F_2 \cup \{x\} = X$, and $\{F_1 \setminus \{x\}, F_2 \setminus \{x\}\}$ is a separation of $X \setminus \{x\}$ satisfying exactly one of the following statements:

a) $\{x\} = (F_1 \cap F_2)$ and it is closed.

b) $\{F_1, F_2, \{x\}\}$ is a partition of X with $\{x\}$ is open.

Theorem 2.13 [11]: Let X be a connected topological space and x be a cutpoint of X . If $\{A_1, A_2\}$ is a separation of $X \setminus \{x\}$, then $A_i \cup \{x\}$ is connected for $i = 1, 2$.

Theorem 2.14 [11]: Let X be a connected topological space and h be a hyperedge of X . If $\{H_1, H_2, \dots, H_n\}$ is a finite collection of non-empty closed subsets of X , when it is a partition of $X \setminus \{h\}$, then every part contains a point in the boundary of x .

3. Alpha-CutPoints And Alpha-Separations

Through this section, we introduce some concepts on α -topological space and study their properties.

Definition 3.1: An α -topological space X is called separated space if there are unordered two disjoint subsets of X , such that each one is the complement of the other.

If A, B are two subsets of X and if $\alpha\text{-Cl}(A) \cap B = A \cap \alpha\text{-Cl}(B) = \emptyset$, then they are mutually α -separated in X . (i.e. A, B are both α -clopen).

Definition 3.2: An α -connected topological space X is defined, when we do not have two disjoint, nonempty, α -open sets, such that their union is equal to X , and X is α -disconnected if the condition is not hold.

Any subset B of X is an α -connected if the subspace of B is α -connected.

Definition 3.3: Let X be an α -topological space, A be an α -connected subset of X , then an α -component of X can be defined as the maximal α -connected subspace of X containing A , denoted as $K_\alpha(A)$.

$K_{\alpha Y}(A)$ is the α -connected component of a subspace Y of X containing A . Also, we define the α -component of a point $x \in X$ without x itself by an α -adherent component, denoted as $K_\alpha(x) \setminus \{x\}$.

The following proposition is a generalization of proposition (2.3).

Proposition 3.4: Let X be an α -topological space, K is an α -component of X . If $\{A_1, A_2\}$ is an α -separation of X , then either $K \subseteq A_1$ or $K \subseteq A_2$.

The next definition is a generalization of definition (2.4).

Definition 3.5: Let X be an α -topological space and A, B, C are disjoint subsets of X ;

- If $\{H, K\}$ is an α -separation of $X \setminus A$ with $B \subseteq H$ and $C \subseteq K$, then A α -separates B and C .

- If the subsets contain only one point, we can say that one of them α -separates the other two points which are contained in H and K , respectively.

- If we have x, y, z distinct points in one α -component K of X , then y α -disconnects x and z with relation to the α -topology on X , if and only if it α -disconnects x and z with relation to the α -subspace topology on K .

The next proposition shows the relationship between being an α -separating point and being an α -disconnect point.

Proposition 3.6: Let X be an α -topological space and $x, y, z \in X$. If y α -separates x and z , then y α -disconnects x and z .

Proof: Since y α -separates x from z , then there exists an α -separation $\{U, V\}$ of $X \setminus \{y\}$ such that $x \in U, z \in V$.

When the α -component of $X \setminus \{y\}$ is containing x were the same as the one containing z , then we obtain a non-empty intersection with both U and V , which is a contradiction with proposition (3.4).

The following definition is a generalization of the cutpoint definition.

Definition 3.7: If X is an α -connected topological space and x is any point in X , then x is called an α -cutpoint of X if $X \setminus \{x\}$ is α -disconnected. If not, then x is a non α -cutpoint.

The following theorem is a generalization of theorem (2.1.8) [11].

Theorem 3.8: Let X be an α -connected topological space and x be an α -cutpoint of X , then there exist two nonempty α -closed sets F_1, F_2 in X , such that $F_1 \cup F_2 \cup \{x\} = X$, $\{F_1 \setminus \{x\}, F_2 \setminus \{x\}\}$ is an α -separation of $X \setminus \{x\}$ and exactly only one of the next statements holds:

1. $\{x\} = (F_1 \cap F_2)$ with $\{x\}$ is α -closed.

2. $\{F_1, F_2, \{x\}\}$ is a partition of X with $\{x\}$ is α -open.

Proof: Since the α -topology is a topology, the proof is similar to (2.1.8) [11].

The following theorem is a generalization of theorem (2.1.9) [11].

Theorem 3.9: Let X be an α -connected topological space and x be an α -cutpoint of X , then there exist two nonempty α -open sets U_1, U_2 in X , such that $U_1 \cup U_2 \cup \{x\} = X$, $\{U_1 \setminus \{x\}, U_2 \setminus \{x\}\}$ is an α -separation of $X \setminus \{x\}$ and exactly only one of the next statements holds:

1. $\{x\} = (U_1 \cap U_2)$ with $\{x\}$ is α -open.
2. $\{U_1, U_2, \{x\}\}$ is a partition of X with $\{x\}$ is α -closed.

Proof: Since $\{x\}$ is an α -cutpoint of X , then there are two nonempty α -clopen sets \tilde{U}_1, \tilde{U}_2 in $X \setminus \{x\}$, such that $\{\tilde{U}_1, \tilde{U}_2\}$ is an α -separation of X . We can express these sets by two nonempty α -open sets U_1, U_2 in X obtained from relative α -topology, such that $\tilde{U}_1 = U_1 \cap (X \setminus \{x\}) = U_1 \setminus \{x\}$ and $\tilde{U}_2 = U_2 \cap (X \setminus \{x\}) = U_2 \setminus \{x\}$.

We have

$$X = (X \setminus \{x\}) \cup \{x\} = \tilde{U}_1 \cup \tilde{U}_2 \cup \{x\} = (U_1 \setminus \{x\}) \cup (U_2 \setminus \{x\}) \cup \{x\} \subseteq U_1 \cup U_2 \cup \{x\}.$$

On the other hand, we have

$$U_1 \cap U_2 = (U_1 \cap U_2) \cap X = (U_1 \cap U_2 \cap (X \setminus \{x\})) \cup (U_1 \cap U_2 \cap \{x\}) = ((U_1 \cap (X \setminus \{x\})) \cap (U_2 \cap (X \setminus \{x\}))) \cup (U_1 \cap U_2) \cap \{x\} = (\tilde{U}_1 \cap \tilde{U}_2) \cup (U_1 \cap U_2) \cap \{x\} = (U_1 \cap U_2) \cap \{x\} \subseteq \{x\}.$$

Now we discuss the probabilities of $\{x\}$. If $\{x\} = U_1 \cap U_2$, then x lies in both U_1 and U_2 , so the intersection of two α -open sets is α -open by topology conditions. This satisfies case (1) of the theorem.

Another case to discuss is if $x \notin U_1 \cap U_2$, i.e. $U_1 \cap U_2 = \emptyset$, then either x lies in U_1 or in U_2 , or not in any one of them. In the previous discussion, we had that x lies in one of them, i.e. $x \in (U_1 \cup U_2)$ means that $U_1 \cup \{x\} \cup U_2 = X$, so $\{U_1, U_2\}$ is an α -separation of X , which is a contradiction with the connectedness of X . Now we must have $x \notin (U_1 \cup U_2)$. But since $(U_1 \cup U_2) \cup \{x\} = X$, then $\{U_1, U_2, \{x\}\}$ is a partition of X , and $\{x\}$ is the α -complement of $(U_1 \cup U_2)$ in X (which is α -open) by topology conditions, i.e. $\{x\}$ is α -closed. This satisfies case (2) of the theorem.

We deduce the next corollary from the previous theorems (3.8, 3.9).

Corollary 3.10: If X is an α -connected topological space, x is an α -cutpoint of X , and there is an α -separation $\{A, B\}$ of $X \setminus \{x\}$, then $\{x\}$ is α -closed, where both A, B are α -open, or $\{x\}$ is α -open where both A, B are α -closed.

Proof: The proof is similar to the proof of theorem (3.2) [5].

The next simple example shows the application of the above theorems (3.8) and (3.9) by choosing an α -cutpoint (α -open or α -closed) with its separation (α -closed or α -open) sets.

Example 3.11: Let G be a graph with $V_G = \{v_1, v_2\}$ vertices set and $E_G = \{e\}$ edges set consists of only one edge between the two vertices, and let $X = V_G \cup E_G = \{v_1, v_2, e\}$ be a set with $\tau_X = \{\emptyset, X, \{e\}\}$. Then (X, τ) is a topological space such that the elements of τ are open sets and their complements are closed sets; $F_X = \{X, \emptyset, \{v_1, v_2\}\}$.

Then the α -topology of X is such that $\tau_{\alpha X} = \{\emptyset, X, \{e\}, \{v_1, e\}, \{v_2, e\}\}$, hence $(X, \tau_{\alpha X})$ is an α -topological space such that all elements of $\tau_{\alpha X}$ are α -open sets and their complements are α -closed sets; $F_{\alpha X} = \{X, \emptyset, \{v_1, v_2\}, \{v_2\}, \{v_1\}\}$.

The following definition is a generalization of the surrounding set.

Definition 3.12: Let X be an α -topological space and $B \subseteq X$, the intersection of all α -open sets containing B is called α -surrounding set of B , denoted by $B^{\alpha\circ}$.

We can express the last two theorems in another way by using the above definition, as we see in the next proposition.

Proposition 3.13: Let X be an α -connected topological space and x is an α -cutpoint of X , then one of the next statements is holding for any α -separation $\{B_1, B_2\}$ of $X \setminus \{x\}$:

1. $\{x\}$ is α -closed, B_i is α -open in X , and $\alpha\text{-Cl}(B_i) = B_i \cup \{x\}$ is α -closed, for $i = 1, 2$
2. $\{x\}$ is α -open, B_i is α -closed in X , and $B_i^{\alpha\circ} = B_i \cup \{x\}$ is α -open, for $i = 1, 2$.

Proof: Suppose that x is an α -cutpoint of X and $\{B_1, B_2\}$ is an α -separation of $X \setminus \{x\}$. Since any α -cutpoint is either α -open or α -closed by corollary (3.10), so if $\{x\}$ is α -closed, then there exist α -open sets B_1, B_2 such that $\{B_1 \setminus \{x\}, B_2 \setminus \{x\}, \{x\}\}$ is an α -separation of $X \setminus \{x\}$. Therefore, $\{B_1, B_2, \{x\}\}$ is a partition of X by theorem (3.9). Also, $\alpha\text{-Cl}(B)$ is α -closed from its definition which means that $\alpha\text{-Cl}(B_1) = B_1 \cup \{x\}$ and $\alpha\text{-Cl}(B_2) = B_2 \cup \{x\}$ are α -closed. Hence, $\alpha\text{-Cl}(B_i) = B_i \cup \{x\}$, for $i = 1, 2$. In the same way, we can prove the second part, that is, $B_i^{\alpha\circ} = B_i \cup \{x\}$ is α -open, for $i = 1, 2$.

In example (3.11), we can see the application of the last proposition. If $\{x\} = \{e\}$ is the α -cutpoint which is α -open and $B_1, B_2 = \{v_1\}, \{v_2\}$ are α -closed sets, then $B_i^{\alpha\circ} = \{v_1, e\}, \{v_2, e\}$ for $i = 1, 2$.

Theorem 3.14: If X is an α -connected topological space and x is an α -cutpoint of X . If $\{A, B\}$ is an α -separation of $X \setminus \{x\}$, then $A \cup \{x\}$ is an α -connected (so is $B \cup \{x\}$).

Proof: The proof is in the same way of (2.1.13) in [11], since α -topology is a topology.

Corollary 3.15: Let X be an α -connected topological space and x and y are two α -cutpoints of X . If there are two α -separations $\{A, B\}$ of $X \setminus \{x\}, \{C, D\}$ of $X \setminus \{y\}$ such that $x \in C$ and $y \in A$, then $D \subseteq A$ and $B \subseteq C$.

Proof: Since x is an α -cutpoint of X , and we have an α -separation of $X \setminus \{x\}$, so by using the previous theorem (3.14), we get that $D \cup \{y\}$ is α -connected. This leads to $D \cup \{y\} \subseteq X \setminus \{x\}$. Hence, by the α -separation of the space, either $D \cup \{y\} \subseteq A$, and by the hypothesis $y \in A$ we obtain that $D \subseteq A$, or $D \cup \{y\} \subseteq B$.

By the same way and using the hypothesis with α -separation property, we get the second result.

Definition 3.16: If X is an α -topological space and $A \subseteq X$, then the α -boundary point of A is the set of intersections α -Cl(A) and α -Cl($X - A$).

Definition 3.17: If X is an α -topological space, then every point in X which is α -open but not α -closed is called α -hyperedge.

The next corollary gives the relation between α -open cutpoints and its α -boundary.

Proposition 3.18: If X is an α -connected topological space, then h is an α -hyperedge of X . Let $\{H_1, H_2, \dots, H_n\}$ be a finite non-empty α -closed subsets of X , when it is a partition of $X \setminus \{h\}$. Then every part contains a point in the α -boundary of x .

Proof: The proof is similar to (2.1.14) in [11].

Corollary 3.19: Let X be an α -connected topological space and x be an α -open cutpoint of X , then x α -separates at most two α -boundary points of x .

Proof: It is similar to the proof of (2.1.15) in [11].

Theorem 3.20: If X is an α -connected topological space and h is an α -hyperedge of X with finitely many of α -boundary points, then each α -component of $X \setminus \{h\}$ includes an α -boundary point of h . Specifically, $X \setminus \{h\}$ contains at most 2 α -connected components, when h is an edge.

Proof: The proof is similar to that in lemma (2.1.16) in [11].

Corollary 3.21: Let X be an α -connected topological space. x is a cutedge (i. e. it is a proper edge), then an α -adherent component $K_\alpha(x) \setminus \{x\}$ contains exactly two α -connected components.

Proof: Suppose that x is a cutedge of X , so x is an α -cutpoint of X . If x is α -open (hyperedge), then it α -separates at most two α -boundary points of x by corollary (3.19), and by (3.20), $X \setminus \{x\}$ contains two α -connected component points of x that is an α -adherent component. If not, then there is an α -separation $\{A_1, A_2\}$ of $X \setminus \{x\}$ and A_1, A_2 are α -open sets, such that α -Cl(A_i) = $A_i \cup \{x\}$ for $i = 1, 2$ by proposition (3.13), and α -connected by theorem (3.14). Hence, it is an α -adherent component which contains two α -connected components of x .

4. Conclusions

Alpha-cutpoints and alpha-separations have been formulated and their relationships have been discussed, which opens the way for future work to new concepts associated with them, in addition to studying some concepts of graph in alpha topological spaces, like prepaths, paths, and cycles.

References

1. Njastad, O. **1965**. On Some Classes of Nearly Open Sets, *Pacific J. Math.*, **15**: 961-970.
2. Andrijevic, D. **1984**. Some Properties of the Topology of α -Sets, *Math. Vesnik*, **36**: 1-10.
3. Levine. N. **1970**. Generalized Closed Sets in Topology, *Rend. Circ. Math. Palermo*, **19**: 89 – 96.
4. Gemignani, M. C. **1972**. *Elementary Topology*. Buffalo. State University of New York.
5. Honari, B. and Bahrapour, Y. **1999**. Cut-Point Spaces, *Pro. Amer. Math. Soc. U.S.A.*, **127**(9): 2797-2803.
6. Kamboj, D. K. and Kumar V. **2010**. Cut points in some connected topological spaces, *Topology Appl.*, **157**: 629–634.
7. Kamboj, D. K. and Kumar, V. **2012**. A study of Cut points in connected topological spaces, *Topology Appl.*, **159**: 40–48.

8. Whyburn, G. T. **1968**. Cut points in general topological spaces, *Proc. Nat. Acad. Sci. U.S.A.*, **61**: 380–387.
9. Willard, S. **1970**. *General topology*. London-Don Mills. Addison-Wesley Publishing Co. Reading Mass. Ontaio.
10. Kelley, J. L. **1955**. *General Topology*. New York. D. Van Nostrand company.
11. Vella, A. **2005**. *A Fundamentally Topological Perspective on Graph Theory*. Ph. D. Thesis, Waterloo, Ontaio, Canada.
12. Harary, F. **1969**. *Graph Theory*. Addison – Wesley, Reading, MA.
13. Bondy, J. A. and Murty, U. S. **1976**. *Graph Theory with Applications*. London. Macmillan Press.
14. Brouwer, A. E. **1971**. *On connected spaces in which each connected subset has at most one endpoint*. Amsterdam. Technical Report 22. Wisk. Sem. Vrije Univ.
15. Sarmad, S. and Yousif, Y. Y. **2020**. Supra Rough Membership Relations and Supra Fuzzy Digrahs on Related Topologies, *Iraqi Journal of Science*, **Special Issue**: 28-34. doi: 10.24996/ij.2020.SI.1.5.
16. Ashaea G. S. and Yousif, Y. Y. **2020**. Weak and Strong Forms of ω -Perfect Mappings, *Iraqi Journal of Science*, **Special Issue**: 45-55. doi: 10.24996/ij.2020.SI.1.7.