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Some Identities of 3-Prime Near-Rings Involving Jordan Ideals and Left Generalized Derivations

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Abstract
In the current paper, we study the structure of Jordan ideals of a 3-prime near-ring which satisfies some algebraic identities involving left generalized derivations and right centralizers. The limitations imposed in the hypothesis were justified by examples.

Keywords: 3-prime near-rings, Jordan ideals, Generalized derivations.

بعض المتساويات الدالية على الحلقات المقتربة الأولية-3 المتضمنة مثاليات جوردان والمشتقات المعممة اليسارية

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الخلاصة
في المقال الحالي ندرس بنية مثاليات جوردان في الحلقات المقتربة الاولية-3 والتي تستوفي بعض المتساويات الجبرية المتضمنة اشتقاقات معممة يسارية و متمركزات يمينية. الشروط التي وضعت في الفرضيات تم تبريرها من خلال الامثلة.

1. Introduction

A right near-ring (resp. left near-ring) is a nonempty set N equipped with two binary operations + and such that (i) (N, +) is a group (not necessarily abelian), (ii) (N, ·) is a semi group, (iii) For all x, y, z ϵ N, we have $(x + y)\cdot z = x\cdot z + y\cdot z$ (resp. $z\cdot(x + y) = z\cdot x + z\cdot y$). We will denote the product of any two elements x and y in N ,i.e.; $x\cdot y$ by xy. A right near-ring (resp. left near-ring) is called zero symmetric right near-ring (resp. zero symmetric left near-ring) if $x0 = 0$ (resp. $0x = 0$), for all $x\in N$. Recall that in a right near ring (resp. left near-ring), $0x = 0$ (resp. $x0 = 0$) for all $x\in N$. The symbol Z(N) will denote the multiplicative centre of N and usually N will be 3-prime, that is, for $x, y \in N$, $xNy =\{0\}$ implies $x = 0$ or $y = 0$. Any pair of elements $x, y \in N$, $[x, y] = xy - yx$ and $x\circ y = xy + yx$ stands for Lie product and Jordan product, respectively. Recall that N is called 2-torsion free if $2x = 0$ implies $x = 0$ for all $x \in N$. For terminologies concerning near-ring theory and its applications, we refer to Pilz [1]. An additive mapping $d : N \rightarrow N$ is a derivation if $d(xy) = xd(y) + d(x)y$ for all $x, y \in N$, or equivalently as noted earlier [2], $d(xy) = d(x)y + xd(y)$ for all $x, y \in N$. Let d be a derivation of

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N, an additive mapping $F : N \to N$ is said to be a left generalized derivation associated with d if $F(xy) = d(x)y + xF(y)$ for all $x, y \in N$. In case where $d = 0$, F will be called right centralizer (i.e., an additive mapping $F : N \to N$ satisfying $F(xy) = xF(y)$ for all $x, y \in N$).

The commutative property of 3-prime near-rings with some suitable constraints on derivations and generalized derivations was established by various authors (see [2-5] and [7-11]). Some comparable results on appropriate subsets of near-rings were also obtained. Boua, et al. (2014) [3] initiated the study of the concept of Jordan ideals on near-rings; 'An additive subgroup J of N is said to be Jordan left (resp. right) ideal of N if $n \circ j \in J$ (resp. $j \circ n \in J$) for all $j \in J$, $n \in N$, and J is said to be a Jordan ideal of N if $n \circ j \in J$ and $j \circ n \in J$ for all $j \in J$, $n \in N$'. The authors proved very interesting results, that is, if Jordan ideal satisfies suitable conditions, then the near-ring must be a commutative ring. Afterwards, studies continued in this field [4-8]. Boua et al. studied commutativity of 3-prime near-rings admitting suitably constrained additive mappings, as derivations, generalized derivations and left multipliers, satisfying certain differential identities on Jordan ideals of 3-prime near-rings. It is natural to continue this line of investigation for comparable results for 3-prime near-rings having other additive mappings with Jordan ideals. In the present paper, we shall attempt to generalize the known result and study the commutativity of Jordan ideal in 3-prime near-rings satisfying certain functional identities involving left generalized derivations and right centralizers.

2. Some preliminaries

To facilitate our discussion, we begin with the following known results which will be used extensively to prove our main results. We indicate that we used Lemmas in the context of right near-rings which remain true in left near-rings.

Lemma 2.1 Let N be a 3-prime near-ring and J is a nonzero Jordan ideal of N

(i) [3, Lemma3] If $J \subseteq Z(N)$, then N is a commutative ring

(ii) [10, Lemma 1.2 (iii)] If $z \in Z(N) \setminus \{0\}$ and x is an element of N such that $xz \in Z(N)$ or $zx \in Z(N)$, then $x \in Z(N)$.

Lemma 2.2 [5, Lemma 2.2] Let N be a 3-prime near-ring. If N admits a nonzero Jordan ideal J, then $j^2 \neq 0$ for all $j \in J/\{0\}$.

Lemma 2.3 [4, Corollary 3] Let N be a 2-torsion free 3-prime near-ring and J be a nonzero Jordan ideal of N. If N admits a derivation d such that $d(J) = \{0\}$, then $d = 0$ or the element of J commute under the multiplication of N.

Lemma 2.4 [11 , Lemma 5] Let N be a near-ring. If N admits a left generalized derivation F associated with a derivation d, then

$$(d(x)y + xF(y))\, z = d(x)yz + xF(y)z \text{ for all } x, y, z \in N.$$

3. Some polynomial identities in left near-rings

The present section is motivated by two previous works [8, Theorem 3.3, and 9, Theorem 2]. Our aim in the current paper is to extend these results of Jordan ideals on 3-prime near-rings admitting a nonzero left generalized derivation.

Theorem 3.1 Let N be a 2-torsion free 3-prime near-ring, J be a nonzero Jordan ideal of N, and F be a left generalized derivation associated with a derivation d . If $F(J^2) =\{ 0\}$, then $d = 0$ or the element of J commute under the multiplication of N.

Proof : By our hypothesis, we have

$$F(ij) = 0 \text{ for all } i, j\in J. \tag{3.1}$$

By replacing i by $2i^2$ in (3.1), we get $F(2i^2j) = 0$ for all $i, j\in J$. Therefore $F(i(i + i)j) = 0$ for all $i, j\in J$. By using the definition of F and our hypothesis, we can easily arrive at $d(i)(i + i)j = 0$ for all $i, j \in J$, that is $d(i)(2i)j = 0$ for all $i, j \in J$. By replacing j by $j\circ n$ in the last equation and using it, we obtain, $d(i)(2i)Nj = \{0\}$ for all $i, j \in J$. Since $J \neq \{0\}$, then by 3-primeness of N and 2-torsion freeness of N, we get

$$d(i)i = 0 \text{ for all } i \in J. \tag{3.2}$$

Since $j(j \circ n)i = (j \circ jn)i$, then by using our hypothesis we obtain $F(j(j \circ n)i) = 0$ for all $i, j\in J$, $n\in N$. An application of the definition of F together with (3.1) in the last equation gives $d(j)(j \circ n)i = 0$ for all $i, j\in J$, $n\in N$. From (3.2), we get $d(j)Nji = \{0\}$ for all $i, j\in J$. Since N is 3-prime, we arrive at $d(j) = 0$ or $ji = 0$ for all $i, j \in J$. If there exists an element $j_0 \in J$, such that $j_0i = 0$ for all $i \in J$, and by putting $i\circ n$ for i, thereby obtaining $j_0Ni = \{0\}$ for all $i \in J$. Since N is 3-prime and $J\neq \{0\}$, we get $j_0= 0$, which forces that $d(J) = \{0\}$. Then $d = 0$ or the element of J commute under the multiplication of N by Lemma 2.3.

Corollary 3.1 [7,Theorem 3.3] Let N be a 2-torsion free 3-prime near-ring and J is a nonzero Jordan right ideal of N. If N admits a nonzero derivation d such that $d(J^2) = \{0\}$, then J is commutative.

Theorem 3.2 Let N be a 2-torsion free 3-prime near-ring and J be a nonzero Jordan ideal of N. If N admits a left generalized derivation F associated with a derivation d such that F(nj) = F(n)F(j) for all j∈ J, n∈ N, then d = 0 or the element of J commute under the multiplication of N.

Proof : Assume that

$$F(nj) = F(n)F(j) \quad \text{for all } j\in J, n\in N. \tag{3.3}$$

By replacing n by jn in (3.3) and using the definition of F with Lemma 2.4, we get d(j)nj + jF(nj) = d(j)nF(j) + jF(n)F(j) for all j ∈ J, n∈ N. From (3.3) we can simplify the last expression as follows: d(j)nj = d(j)nF(j) for all j ∈ J, n ∈ N. Equivalently, d(j)N(j - F(j)) = {0} for all j ∈ J. By 3-primeness of N we obtain

$$d(j) = 0 \text{ or } F(j) = j \text{ for all } j \in J, n \in N. \tag{3.4}$$

Suppose that there exists an element $j_0 \in J$ such that $F(j_0) = j_0$. We have $F(j_0j_0) = d(j_0)j_0 + j_0F(j_0) = d(j_0)j_0 + j_0^2$. On the other hand, $F(j_0j_0) = F(j_0)F(j_0) = j_0^2$, and by comparing the two last expressions forces, we have

$$d(j_0)j_0 = 0. \tag{3.5}$$

By replacing n by j_0 and j by $(j_0 \circ n)$ in (3.3), we get $F(j_0(j_0 \circ n)) = F(j_0)F(j_0 \circ n)$ for all n ∈ N. We use the definition of F and the fact that $F(j_0) = j_0$ to get $d(j_0)(j_0 \circ n) + j_0F(j_0 \circ n) = j_0F(j_0 \circ n)$ for all n ∈ N. Hence we conclude that $d(j_0)(j_0 \circ n) = 0$ for all n ∈ N. It is immediate from (3.5) that $d(j_0)N j_0 = \{0\}$. Since N 3-prime, we conclude that $d(j_0) = 0$. In this case (3.4) yields d(J) = {0}, then d = 0 or the element of J commute under the multiplication of N by Lemma 2.3.

Corollary 3.2 Let N be a 2-torsion free 3-prime near-ring and J a nonzero Jordan ideal of N. If N admits a nonzero derivation d such that d(nj) = d(n)d(j) for all j ∈ J , n ∈ N, then the element of J commutes under the multiplication of N.

The following example proves that the "3-primeness of N" in Theorem 3.1 and Theorem 3.2 cannot be omitted.

Example 3.1 Let S be a 2-torsion left near ring which is not abelian. We define N, J, d, F by

$$N = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix}, x, y, z, 0 \in S \right\},\ J = \left\{ \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, x, 0 \in S \right\}$$

$$F\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} = \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } d = F.$$

Then N is a left near-ring which is not 3-prime, J is a nonzero Jordan ideal of N, and F is a left generalized derivation associated with the derivation d of N. We easily can see that

(i) F(ij) = 0 for all i, j∈ J.

(ii) F(nj) = F(n)F(j) for all j∈ J, n∈ N.

But neither d = 0 nor J is commutative.

Theorem 3.3 Let N be a 2-torsion free 3-prime near-ring and J be a nonzero Jordan ideal of N. If N admits a left generalized derivation F associated with a derivation d such that F(jn) = F(n)F(j) for all j∈ J, n∈ N, then $(d(J))^2 = \{0\}$.

Proof : Assume that

$$F(jn) = F(n)F(j) \text{ for all } j\in J, n\in N. \tag{3.6}$$

By replacing n by jn in (3.6) and using the definition of F with Lemma 2.4 , we get

$$d(j)jn + jF(jn) = d(j)nF(j) + jF(n)F(j) \text{ for all } j \in J , n \in N.$$

By using (3.6) the last equation gives

$$d(j)jn = d(j)nF(j) \text{ for all } j \in J, n \in N.$$

By putting tn instead of n, where t ∈ N, we obtain d(j)N[F(j), n] = {0} for all j ∈ J, n ∈ N. By 3-primeness of N we obtain

$$d(j) = 0 \text{ or } F(j) \in Z(N) \text{ for all } j \in J. \tag{3.7}$$

Suppose that there exists an element $j_0 \in J$ such that $F(j_0) \in Z(N)$. By the assumption we get $F(j_0n) = F(n)F(j_0)$ for all n ∈ N. By replacing n by jn, where j ∈ J, in the last equation, we get

$$\begin{aligned} F(j_0jn) &= F(jn)F(j_0) \\ &= F(n)F(j)F(j_0) \\ &= F(j_0n) F(j) \end{aligned}$$

$= d(j_0)nF(j) + j_0F(n)F(j)$

$= d(j_0)nF(j) + j_0F(jn)$ for all $j \in J$, $n \in N$. (3.8)

Also

$F(j_0jn) = d(j_0)jn + j_0F(jn)$ for all $j \in J$, $n \in N$. (3.9)

By combining (3.8) and (3.9), we obtain $d(j_0)nF(j) = d(j_0)jn$ for all $j \in J$, $n \in N$. By putting tn, where $t \in N$, instead of n in the last equation, we conclude that $d(j_0)N[F(j), n] = \{0\}$ for all $j \in J$, $n \in N$ and 3-primeness of N, implies that $d(j_0) = 0$ or $F(J) \subseteq Z(N)$. Hence from (3.7) we conclude that

$$d(J) = \{0\} \text{ or } F(J) \subseteq Z(N) \quad (3.10)$$

Now, if $F(J) \subseteq Z(N)$, we get

$F(j(kn)) = F(kn)F(j)$

$= F(n)F(k)F(j)$

$= F(k) F(n) F(j)$

$= F(k) F(jn)$

$= F(k)d(j)n + F(k)jF(n)$

$= d(j) F(k)n + jF(kn)$ for all $j, k \in J$, $n \in N$. (3.11)

Also

$F(j(kn)) = d(j)kn + jF(kn)$ for all $j,k \in J$, $n \in N$. (3.12)

Combining (3.11) and (3.12) implies $d(j)F(k)n = d(j)kn$ for all $j, k \in J$, $n \in N$. Replacing n by F(n) in the previous equation and using the hypothesis gives $d(j)(F(kn)- kF(n)) = 0$ for all $j, k \in J$, $n \in N$. It follows that $d(j)(d(k)n - kF(n)- kF(n)) = 0$ for all $j, k \in J$, $n \in N$, hence we get $d(j)d(k)n = 0$ for all $j, k \in J$, $n \in N$, and 3- primeness of N, implies $d(j)d(k)= 0$ for all $j, k \in J$. By returning to (3.10) we conclude that $(d(J))^2= \{0\}$.

Remark 1: We tried to use (or treat) $(d(J))^2 = \{0\}$ in Theorem 3.3 to determine the structure of J, but unfortunately, we cannot find the required formulation of our theorem. This naturally allows us to ask questions that will help the reader in the future to obtain better results.

Open question 1: Let N be a 2-torsion free 3-prime near-ring and J be a nonzero Jordan right ideal of N. If N admits a nonzero derivation d such that $(d(J))^2 = \{0\}$, then J is commutative.

Open question 2: Let N be a 2-torsion free 3-prime near-ring and J be a nonzero Jordan right ideal of N. If N admits a left generalized derivation F associated with a nonzero derivation d such that $F(jn) = F(n)F(j)$ for all $j\in J$, $n\in N$, then J is commutative.

4. Some results for Jordan right ideals involving right centralizers in a left near-ring

In this section, our objective is to establish similar preciously reported results [6, Theorems 3.1 and 3.11] and explore the commutativity of a 3-prime near-ring N admitting a nonzero right centralizer T satisfying any one of the identities: (i) $T(J) \subseteq Z(N)$, (ii) $T(j \circ n) \in Z(N)$, (iii) $T([j, n]) \in Z(N)$.

Theorem 4.1. Let N be a 2-torsion free 3-prime near-ring and J a nonzero Jordan right ideal of N. If N admits a nonzero right centralizer T, then the following assertions are equivalent:

(i) $T(J) \subseteq Z(N)$;

(ii) $T(j \circ n) \in Z(N)$ for all $j \in J$, $n \in N$;

(iii) $T([j, n]) \in Z(N)$ for all $j \in J$, $n \in N$;

(iv) N is a commutative ring.

Proof. It is easy to see that (iv) ⇒ (i), (iv) ⇒ (ii) and (iv) ⇒ (iii).

(i) ⇒ (iv). Assume that

$$T(j) \in Z(N) \text{ for all } j \in J. \quad (4.1)$$

It follows that $T(j(2j)) \in Z(N)$ for all $j \in J$, which implies that $jT(2j) \in Z(N)$ for all $j \in J$, so that (4.1) together with Lemma 2.1(ii)give either $T(2j) = 0$ or $j \in Z(N)$ for all $j \in J$. By using the 2-torsion freeness of N, we get

$$T(j) = 0 \text{ or } j \in Z(N) \text{ for all } j \in J. \quad (4.2)$$

Suppose that there exists $j_0 \in J$ such that $T(j_0) = 0$. By using our hypothesis, we have $T(j_0 \circ n) \in Z(N)$ for all $n \in N$, which gives $j_0T(n) \in Z(N)$ for all $n \in N$. By putting j_0n in place of n and using Lemma 2.1(ii), we obtain $j_0T(n) = 0$ or $j_0 \in Z(N)$ for all $n \in N$. By substituting yt instead of n, where $y, t \in N$, we arrive at $j_0NT(t) = \{0\}$ or $j_0 \in Z(N)$ for all $t \in N$. Since $T \neq 0$, the last expression leads to $j_0 \in Z(N)$, and (4.2) can be reduced to $J \subseteq Z(N)$, which forces that N is a commutative ring by Lemma 2.1(i).

(ii) ⇒ (iv). Assume that

$$T(j \circ n) \in Z(N) \text{ for all } j \in J, n \in N. \quad (4.3)$$

By substituting jn instead of n, we arrive at $T(j(j \circ n)) \in Z(N)$ for all $j \in J$, $n \in N$. Then $jT(j \circ n) \in Z(N)$ for all $j \in J$, $n \in N$. From (4.3) with Lemma 2.1(ii), we obtain

$$T(j \circ n) = 0 \text{ or } j \in Z(N) \text{ for all } j \in J, n \in N. \quad (4.4)$$

Suppose that there exists $j_0 \in J$ such that

$$T(j_0 \circ n) = 0 \text{ for all } n \in N. \quad (4.5)$$

By substituting j_0 instead of n in (4.5) and by 2-torsion free of N, we arrive at:

$$j_0 T(j_0) = 0 \quad (4.6)$$

By substituting $T(j_0)n$ instead of n in (4.5) and using (4.6), we obtain $T(j_0)NT(j_0) = \{0\}$. Hence $T(j_0) = 0$ by 3-primeness of N. By returning to (4.5), we get $j_0T(n) = 0$ for all $n \in N$. By substituting nm instead of n, we get $j_0NT(m) = \{0\}$ for all $m \in N$. Since N is 3-prime and $T \neq 0$, we conclude that $j_0 = 0$. Therefore (4.4) can be reduced to $J \subseteq Z(N)$, which forces that N is a commutative ring by Lemma 2.1(i).

(iii) ⇒ (iv). Suppose that

$$T([j, n]) \in Z(N) \text{ for all } j \in J, n \in N.$$

By substituting jn instead of n, we obtain $jT([j, n]) \in Z(N)$. According to Lemma 2.1(ii), we conclude that $T([j, n])= 0$ or $j \in Z(N)$ for all $j \in J$, $n \in N$, which means that

$$T([j, n]) = 0 \text{ for all } j \in J, n \in N \quad (4.7)$$

By substituting $[i, n]$ instead of n in (4.7), we can easily arrive at

$$[i, n]T(j) = 0 \text{ for all } i, j \in J, n \in N \quad (4.8)$$

By taking $j \circ m$, where $m \in N$, instead of j in (4.8), we obtain $[i, n]T(jm) + [i, n]T(mj) = 0$ for all $i, j \in J$, $n \in N$, and using (4.7) lastly gives $2[i, n]T(mj) = 0$ for all $i, j \in J$ and $n, m \in N$. The 2-torsion freeness of N implies that $[i, n]mT(j)= 0$ for all $i, j \in J$, $n, m \in N$. Equivalently, $[i, n] N T(j) = \{0\}$ for all $i, j \in J$, $n \in N$ and by 3-primeness of N, we get

$$i \in Z(N) \text{ or } T(j) = 0 \text{ for all } i, j \in J.$$

If $T(j) = 0$ for all $j \in J$, then (4.7) produces $jT(n) = 0$ for all $j \in J$, $n \in N$. Putting mn, where $m \in N$ in place of n in the last result, implies that $jNT(n) = \{0\}$ for all $j \in J$, $n \in N$. By the 3-primeness of N, we get either $J = 0$ or $T = 0$, which contradicts our assumptions.

Therefore we conclude that $J \subseteq Z(N)$, which forces that N is a commutative ring by Lemma 2.1 (i).

The following example proves that the 3-primeness hypothesis in Theorem 4.1 is not superfluous.

Example 4.1: Let S be a 2-torsion free left near-ring. We define N, J and T by :

$$N = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} / x, y \in S \right\}, J = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix} / k \in S \right\} \text{and } T\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$$

It is easy to see that *N* is a 2-torsion free left near-ring which is not 3-prime, *J* is a nonzero Jordan right ideal of *N*, and *T* is a nonzero right centralizer such that:

(i) $T(J) \subseteq Z(N)$;

(ii) $T(j \circ n) \in Z(N)$ for all $j \in J$, $n \in N$;

(iii) $T([j, n]) \in Z(N)$ for all $j \in J$, $n \in N$.

But *N* is not a commutative ring.

5. Results in right near-ring involving right Jordan ideals and right centralizes

Theorem 5.1. Let N be a 2-torsion free 3-prime near-ring and J be a nonzero Jordan right ideal of N. Then there is no nonzero right centralizer T satisfying $T(i \circ j) = 0$ for all $i, j \in J$.

Proof. Assume that

$$T(i \circ j) = 0 \text{ for all } i, j \in J.$$

For $i = j$, we have $jT(j) = 0$ for all $j \in J$ by 2-torsion freeness of N. By substituting $i \circ ni$ instead of j and using $i \circ ni = (i \circ n)i$ in the last equation, we can easily arrive at $(i \circ ni)iNT(i) = \{0\}$ for all $i \in J$, $n \in N$. By 3-primeness of N, we get

$$(i \circ ni)i = 0 \text{ or } T(i) = 0 \text{ for all } i \in J, n \in N \quad (5.1)$$

Suppose that there exists an element $i_0 \in J\backslash\{0\}$, such that $(i_0 \circ ni_0)\, i_0 = 0$. Then $i_0\, ni_0^2 = -ni_0^3$ for all $n \in N$. By substituting nm instead of n in last equation, we get

$$\begin{aligned} i_0\, nmi_0^2 &= -nmi_0^3 \\ &= (-n)\,(-i_0)mi_0^2 \text{ for all } n, m \in N. \end{aligned}$$

So we get $[-i_0, n]Ni_0^2 = \{0\}$ for all $n \in N$. By the 3-primeness of N and Lemma 2.2, we obtain $[-i_0, n] = 0$ for all $n \in N$, which means that $-i_0 \in Z(N)$. By returning to the hypothesis and substituting $-i_0$ instead of i, we conclude that $2jT(-i_0) = 0$ for all $j \in J$. The 2-torsion freeness of N leads to $jT(-i_0) = 0$ for all $j \in J$. By replacing $j \circ n$ instead of j in the last equation, we get $jNT(-i_0) = \{0\}$ for all $j \in J$. Since $J \neq \{0\}$, then 3-primeness of N implies that $T(-i_0) = 0$ and hence $T(i_0) = 0$. Therefore, from (5.1) we obtain

$$T(j) = 0 \quad \text{for all} \quad j \in J \tag{5.2}$$

By substituting $j \circ n$ instead of j in (5.2) and using it, we get $jT(n) = 0$ for all $j \in J$, $n \in N$, and by putting nm instead of n, we obtain $jNT(m) = \{0\}$ for all $j \in J$, $m \in N$. Since $T \neq 0$, by 3-primeness of N, we conclude that $J = \{0\}$; which is a contradiction.

Theorem 5.2 Let N be a 2-torsion free 3-prime near-ring and J is a nonzero Jordan right ideal of N. If N admits a nonzero right centralizer T, then the following assertions are equivalent:

(i) $T([j, n]) = 0$ for all $j \in J$, $n \in N$;

(ii) N is a commutative ring.

Proof. It is easy to see that (ii) ⇒ (i)

(i) ⇒ (ii). Assume that $T([j, n]) = 0$ for all $j \in J$, $n \in N$. Then

$$T(jn) = T(NJ) \quad \text{for all} \quad j \in J, n \in N \tag{5.3}$$

By substituting nm instead of n in (5.3) and using it, we get $jT(nm) = nT(mj) = nT(jm)$ for all $j \in J$, $n \in N$.

It follows that

$$jnT(m) = njT(m) \quad \text{for all} \quad j \in J, n, m \in N \tag{5.4}$$

By substituting mt instead of m in (5.4), we get $[j, n]NT(t) = \{0\}$ for all $j \in J$, $n, t \in N$. Since $T \neq 0$, the 3-primeness of N implies that $J \subseteq Z(N)$. Therefore N is a commutative ring by Lemma 2.1(i).

The following example proves that the 3-primeness hypothesis in Theorem 5.2 is not superfluous.

Example 5.1: Let S be a 2-torsion free right near-ring. We define N, J and T by

$$N = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} / x, y \in S \right\}, \ J = \left\{ \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix} / k \in S \right\} \text{and } T\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix}$$

It is easy to see that N is a 2-torsion free right near-ring which is not 3-prime, J is a nonzero Jordan right ideal of N, and T is a nonzero right centralizer such that $T([j, n]) = 0$ for all $j \in J$, $n \in N$. But N is not a commutative ring.

Theorem 5.3 Let N be a 2-torsion free 3-prime near-ring and J be a nonzero Jordan right ideal of N. T_1 and T_2 are nonzero right centralizers of N. If $T_1(i)T_2(j) \in Z(N)$ for all $i, j \in J$, such that $T_1(N)J \neq \{0\}$ and $T_2 \circ T_1 \neq 0$, then N is a commutative ring.

Proof. Suppose that

$$T_1(i)T_2(j) \in Z(N) \quad \text{for all} \quad i, j \in J \tag{5.5}$$

By replacing i by $i \circ ni$, where $n \in N$, in (5.5) and using the fact that $i \circ ni = (i \circ n)i$, we find that $(i \circ n)T_1(i)T_2(j) \in Z(N)$ for all $i, j \in J$, $n \in N$. By Lemma 2.1(ii), we obtain

$$T_1(i)T_2(j) = 0 \ \text{ or } \ (i \circ n) \in Z(N) \ \text{ for all } \ i, j \in J, n \in N. \tag{5.6}$$

If there exists a nonzero element $i_0 \in J$ such that $i_0 \circ n \in Z(N)$ for all $n \in N$, we substitute ni_0 instead of n and use Lemma 2.1 (ii), we obtain

$$i_0 \circ n = 0 \ \text{ or } \ i_0 \in Z(N) \ \text{ for all } \ n \in N \tag{5.7}$$

For $n = i_0$, in (5.7) and using 2-torsion freeness of N, it can be reduced to $i_0^2 = 0$ or $i_0 \in Z(N)$ for all $n \in N$. Since $(i_0 \circ n)i_0 = 0$ for all $n \in N$, we obtain $i_0 Ni_0 = \{0\}$, which cannot hold because $i_0 \neq 0$. Therefore, $i_0 \in Z(N)$. Since $i_0 \circ n \in Z(N)$ for all $n \in N$, we conclude that $(2n)i_0 \in Z(N)$ for all $n \in N$. By Lemma 2.1(ii) we obtain $2n \in Z(N)$ for all $n \in N$, then $2n^2 \in Z(N)$ for all $n \in N$. Since $2n^2 = (2n)n \in Z(N)$, then Lemma 2.1(ii) and the 2-torsion freeness of N force that $N \subseteq Z(N)$, which assures that N is a commutative ring by Lemma 2.1(i).

Now, suppose that

$$T_1(i)T_2(j) = 0 \text{ for all } i, j \in J, n \in N. \tag{5.8}$$

By taking $i \circ n$ instead of i, where $n \in N$, in (5.8) we get $iT_1(n)T_2(j) = 0$ for all $i, j \in J$, $n \in N$. By replacing n by mn in the last equation we arrive at $iNT_1(n)T_2(j) = \{0\}$ for all $i, j \in J$, $n \in N$. Since $J \neq 0$, then by the 3-primeness of N, we get

$$T_1(n)T_2(j) = 0 \text{ for all } j \in J, n \in N. \tag{5.9}$$

By putting $j \circ t$, where $t \in N$, instead of j , we find that $T_1(n)(jT_2(t) + tT_2(j)) = 0$ for all $j \in J$, $n, t \in N$. By replacing t by $tT_1(m)$, where $m \in N$, and using (5.9), we get $T_1(n)jtT_2(T_1(m)) = 0$ for all $i \in J$, m, t, $n \in N$. That is, $T_1(n)jNT_2(T_1(m)) = \{0\}$ for all $i \in J$, m, $n \in N$. By the 3-primeness of N, we conclude that either $T_1(N)J = \{0\}$ or $T_2 \circ T_1 = 0$, which contradicts our hypothesis.

The following example proves that the 3-primeness hypothesis in Theorem 5.3 is necessary.

Example 5.2: Let S be a 2-torsion free right near-ring. We define N, J , T_1 and T_2 by:

$$N = \left\{ \begin{pmatrix} r & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & t \end{pmatrix}, r, s, t, 0 \in S \right\} \text{ and } J = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & 0 \end{pmatrix}, p, 0 \in S \right\}$$

$$T_1 \begin{pmatrix} r & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & t \end{pmatrix} = \begin{pmatrix} r & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 0 \end{pmatrix}, T_2 \begin{pmatrix} r & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & t \end{pmatrix} = \begin{pmatrix} r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to see that N is a 2- torsion free right near-ring which is not 3-prime, J is a nonzero Jordan right ideal of N, and T_1 , T_2 are nonzero right centralizers of N. Moreover, $T_1(i)T_2(j) \in Z(N)$ for all i, $j \in J$, $T_1(N)J \neq \{0\}$ and $T_2 \circ T_1 \neq 0$. However, N is not a commutative ring.

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References

1. G. Pilz, **1983**. *Near-Rings*. (*Second Edition*), North Holland /American Elsevier, Amsterdam.
2. Wang, X. **1994**. Derivations in prime near-rings, *Proc. Am. Math. Soc.* **121**: 361–366.
3. Boua, A., Oukhtite, L. and Raji, A. **2014**. Jordan ideals and derivations in prime near-rings. *Comment. Math. Univ. Carolin*. **55**(2):131–139
4. Boua, A. and Kamal, A. **2015**. Lie ideals and Jordan ideals in 3-prime near-rings with derivations. *JP. .Journal of Alg*, **37**: 167–184.
5. Boua, A. and Bell, H. **2018**. Jordan Ideals and Derivations Satisfying Algebraic Identities. *Iranian Mathematical Society* **, 44**: 1543–1554.
6. Boua. A., Ali, A. and Ul. Huque. **2018**. Several Algebriac Identities in 3-Prime Near Ring. *Kragujevac Journal of Mathematics*. **42**(2): 249–258.
7. Boua, A. **2018**. Commutativity of Near-rings With Certain Constrains on Jordan Ideals. *Bol. Soc. Paran Mat*. **36**(4): 159–170.
8. Boua, A. **2018**. Commutivity of Jordan Ideals in 3-Prime Near-Rings with Derivations. *Commun. Korean Math*. **33**(1): 37- 44.
9. Bell, H., and G. Mason.**1987**. On derivations in near-rings. *North-Holland Math. Stud*. **137**: 31-35.
10. Bell, H. **1997** On derivations in near-rings. II, Nearrings, nearfields and K-loops. *Kluwer Acad. Publ., Dordrecht*, 1910-197, Math. Appl., 426.
11. Hvala, B. **1998**. Generalized derivations on prime near-rings. *Commun. Algebra.* **26**(4): 1147–1166.