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The Galerkin-Implicit Method for Solving Nonlinear Variable Coefficients Hyperbolic Boundary Value Problem

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Abstract
This paper has the interest of finding the approximate solution (APPS) of a nonlinear variable coefficients hyperbolic boundary value problem (NOLVCHBVP). The given boundary value problem is written in its discrete weak form (WEFM) and proved have a unique solution, which is obtained via the mixed Galerkin finite element with implicit method that reduces the problem to solve the Galerkin nonlinear algebraic system (GNAS). In this part, the predictor and the corrector techniques (PT and CT, respectively) are proved at first convergence and then are used to transform the obtained GNAS to a linear GLAS . Then the GLAS is solved using the Cholesky method (ChMe). The stability and the convergence of the method are studied. Some illustrative examples are used, where the results are given by figures that show the efficiency and accuracy for the method.

Keywords: nonlinear hyperbolic boundary value problem; Galekin finite element method; implicit method; convergence; stability.

طريقة كاليركن –الضمنية لحل مسالة القيم الحدودية الزائدية الغير خطية ذات المعاملات المتغيرة

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الخلاصة
يهتم هذا البحث بإيجاد الحل التقريبي لمسألة القيم الحدودية الزائدية الغير خطية ذات المعاملات المتغيرة حيث تمت كتابة مسالة القيم الحدودية بشكلها الضعيف ، وتم برهانه أنه يمتلك على حل وحيد ،حيث يتم الحصول عليه عبر مزج طريقة العناصر المحدودة المختلطة مع الطريقة الضمنية والتي قمات بتحويل المسالة الى مسالة ايجاد حل لنظام كاليركن الجبري الغير خطي وعند هذا الجزء تم برهان تقارب تقنية التنبأ والتصحيح اولا ومن ثم تم استخدامها لتحويل النظام كاليركن الجبري الغير خطي الى نظام كاليركن الجبري الخطي والذي تم حله باستخدام طريقة جولسكي . تمت دراسة الاستقراية والتقارب لحل هذه المسالة .اعطيت بعض الامثلة التوضحية النتائج اعطيت على شكل رسومات حيث بينت الكفاءة والدقة للطريقة.

1.Introduction

Hyperbolic partial differential equations arise in many physical problems, such as vibrating strings, and in many other fields such as fluid dynamics, optics, and others. In general, there are many researchers who are interested in the solution of boundary value problems, in

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particular the solution of the nonlinear hyperbolic boundary value problem (NLHBVP). In 2015, Feller used the Lévy Laplacian to solve a NOLVCHBVP [1]. In 2017, Mardani *et al*. used the Moving Least Squares method for the nonlinear hyperbolic telegraph equation with variable coefficients [2]. Ashyralyev and Agirseven, in 2018, solved a NOLHBVP with a time delay [3]. While in 2018, Ahmedatt *et a.l* looked at some nonlinear hyperbolic $p(x,t)$-Laplacian equations [4]. Adewole, in 2019, found the APPS of a linear hyperbolic (LHBVP) [5].

The finite element method has been studied by many researchers who are interested in this field to solve LHBVP. For example, in 2014, Quarteroni studied in his book the numerical solution for LHBVP and some especial types NOLHBVP by using GFEME [6]. In 2018, Wick studied in his book the GFEME for solving LHBVP and NOLHBVP with constant coefficients [7].

In this paper, we care about the study of the APPS of the NOLVCHBVP. The given boundary value problem is written in its WEFM, and then it is discretized using the mixed Galerkin finite element method (GFEME) for the space variable with the implicit method (IM) for the time variable (MGFEIM). It is proved that the discrete problem has a unique solution. The problem then reduces for solving the GNAS. In this point, the PT and CT are used to transform the GNAS to a GLAS, which is solved by using the ChMe. The stability and the convergence of the method are studied. A computer program is codding in Matlap to find the APPS for the problem. Some illustrative examples are given and the results are given by figures, which show the efficiency and accuracy for the considered method .

2.Description of the NOLVCHBVP

Let $\psi = \{\vec{x} = (x_1, x_2) \in \mathbb{R}^2 : 0 < \vec{x} < 1\}$, with boundary $\partial\psi$, $\varphi = \psi \times I$, $\Sigma = \partial\psi \times I$, $I = [0, \mathrm{T}]$, $0 < T < \infty$ then the NOLVCHBVP is given by:

$$w_{tt} - \sum_{r,u=1}^{2} \frac{\partial}{\partial x_u}\left[a_{ru}(\vec{x},t)\frac{\partial w}{\partial x_r}\right] + g(\vec{x},t)\, w = h(\vec{x},t,w), \qquad \forall\, (\vec{x},t) \in \varphi \tag{1}$$

$$w(\vec{x},0) = w^0(\vec{x}), \qquad \forall\, \vec{x} \in \psi \tag{2}$$

$$w_t(\vec{x},0) = w^1(\vec{x}), \qquad \forall\, \vec{x} \in \psi \tag{3}$$

$$w(\vec{x},t) = 0, \text{ on } \Sigma \tag{4}$$

where $w = w(\vec{x},t) \in H_0^2(\psi),$, $a_{ru}(\vec{x}\,,t), g(\vec{x},t) \in L^\infty(\psi)$, with $a_{ru}(\vec{x}\,,t)$ are positive functions and $h \in L^2(\psi)$ is a given function.

Now, let $V = H_0^1(\psi)$={ $\eta{:}\eta \in H^1(\psi)$, $\eta = 0$ on $\partial\psi$}, $w_t = p$, then the WEFM of (1- 4) is:

$$\langle\, w_{tt}\,, \eta\rangle + a(t,w,\eta) = (h(w),\eta),\ \forall\, \eta \in V \text{ are on } I, \tag{5}$$

$$(w(0),\eta) = (w^0,\eta)\ \forall\, \vec{x} \in \psi, w^0 \in V \tag{6}$$

$$(p(0),\eta) = (w^1,\eta) \quad \forall\, \vec{x} \in \psi,\ w^1 \in L^2(\varphi) \tag{7}$$

where $a(t,w,\eta) = \left(\sum_{r,u=1}^{2} a_{ru}(\vec{x},t)\frac{\partial w}{\partial x_u}\frac{\partial \eta}{\partial x_r} + g(\vec{x},t)\, w\, \eta\right)$

3. Assumptions

(i) Let κ_1 and κ_2 be two positive constants such that the following are satisfied:

a) $|a(t,w,\eta)| \le \kappa_1 \| w \|_1 \| \eta \|_1$, $\forall\, w,\eta \in V$

b) $a(t,w,w) \ge \kappa_2 \| w \|_1^2$, $\forall\, w \in V$

(ii) The function h is defined on $\varphi \times \mathbb{R}$, continuous with respect to w_j^n which satisfies the following:

a) $|h(\vec{x},t,w)| \le \beta(\vec{x},t) + \delta\, |w|$ where $\delta > 0$, $w \in \varphi$ and $\beta \in L^2(\varphi)$.

b) $|h(\vec{x},t,w_1) - h(\vec{x},t,w_2)| \le L|w_1 - w_2|$, where L is a Lipchitz constant and $w_1, w_2 \in \mathbb{R}$.

4. Discretization of The Continuity Equation (COE)

By setting $w_t = p$ in the WEFM of (5-7), then it is discretized by using the GFEME as follows: let the domain φ is divided into sub regions $\varphi_{ij} = \psi_i^n \times I_j^n$, let $\{\psi_i^n\}_{i=1}^{N(n)}$ be a triangulation of $\bar{\psi}$, and let $\{I_j^n\}_{j=0}$ be a subdivision of the interval $\bar{I}$ into Y(n) intervals, where $I_j = I_j^n \coloneqq [t_j^n, t_{j+1}^n]$ of equal length $\Delta t = \frac{T}{Y}$. Also, let $V_n \subset V = H_0^1(\psi)$ be the space of continuous piecewise affine functions in ψ. The discrete equations (DES) are written as follows:

$$\langle p_{j+1}^n - p_j^n , \eta \rangle + \Delta t\, a(w_{j+1}^n, \eta) = \Delta t\, (h(w_{j+1}^n), \eta) \ , \forall \eta \in V_n \tag{8}$$

$$w_{j+1}^n - w_j^n = \Delta t\, p_{j+1}^n \tag{9}$$

$$(w(0), \eta) = (w^0, \eta) \quad \forall \vec{x} \in \psi \tag{10}$$

$$(p(0), \eta) = (w^1, \eta) \quad \forall \vec{x} \in \psi \tag{11}$$

where $w^0 \in V$, $w^1 \in L^2(\psi)$ and $\left(w_j^n = w^n(x, t_j^n), p_j^n = p^n(\vec{x}, t_j^n)\right) \in V_n$, $\forall j = 0,1, \dots, Y - 1$.

5. The APPS of the NOLVCHBVP

To find the APPS $\bar{w}^n = (w_0^n, w_1^n, \dots, w_Y^n)$ for the DES (8)-(11), using the MGFEIM, the following steps are used:

1) Let $\{ \eta_i : i = 1,2, \dots . N, \text{with } \eta_i(\vec{x}) = 0 \text{ , on } \partial\psi \}$ be a finite basis of V_n, with using the GFEME, let

$\bar{w}^n(\vec{x}, t_j^n)$ with $\left(\bar{w}_t^n(\vec{x}, t_j^n) = \bar{p}(\vec{x}, t_j^n)\right)$ be an APPS of (8-11), then one has

$\bar{w}^n(\vec{x}, t_j^n) = \sum_{k=1}^N r_k^j \eta_i$ and $\bar{p}^n(\vec{x}, t_j^n) = \sum_{k=1}^N u_k^j \eta_i \ \forall \eta_i \in V_n$,

where $r_k^j = r_k(t_j^n)$ and $u_k^j = u_k(t_j^n)$, for each $j = 0,1, \dots, Y - 1$ are unknown constants.

2) Using the APPs in (8-11) to get $\forall j = 0,1, \dots, Y - 1$:

$$(M + (\Delta t)^2 Q) R^{j+1} = M R^j + (\Delta t)\, M U^j + (\Delta t)^2 \vec{L}\left(t_j^n, \vec{v}^T R^{j+1}\right) \tag{12}$$

$$U^{j+1} = \frac{1}{\Delta t} (R^{j+1} - R^j) \tag{13}$$

$$M R^0 = s^0 \tag{14}$$

$$M U^0 = s^1 \tag{15}$$

where $M = (m_{ik})_{N\times N}$, $m_{ik} = (\eta_k, \eta_i)$, $Q = (q_{ik})_{N\times N}$, $q_{ik} = a(\eta_k, \eta_i)$, $\vec{L} = (L_i)_{N\times 1}$, $L_i = (h(\vec{v}^T R^{j+1}), \eta_i)$, $R_{N\times 1}^j = (r_1^j, r_2^j, \dots, r_N^j)^T$, $U_{N\times 1}^j = (u_1^j, u_2^j, \dots, u_N^j)^T$,

$s^0 = (s_i^0)_{N\times 1}$, $s_i^0 = (w^0, \eta_i)$, $s^1 = (s_i^1)_{N\times 1}$ and $s_i^1 = (w^1, \eta_i)$, $\forall i, k = 1,2, \dots, N$.

3) System (12)-(15) is GNAS and has a unique solution. To solve it, first we solve the GLAS (14) and

(15) to obtain R^0 and U^0, then the PT and the CT are utilized to solve (12) for each j($j = 0,1, \dots, Y - 1$) as follows:

In the PT, we suppose that $R^{j+1} = R^j$ in the components of $\vec{L}$ in the R.H.S of (12), then it turns to a GLAS, solving this system to get the predictor solution R^{j+1}. Then, in the CT, we resolve (12) with setting $\bar{R}^{j+1} = R^{j+1}$ (in the components of $\vec{L}$ of the R.H.S of it) to get the corrector solution R^{j+1}. Then we substitute R^{j+1} in (13) to get U^{j+1}. We can repeat this procedure if we want more than one time; this repetition can be expressed as follows:

$$\left(w_{j+1}^{(l+1)}, \eta_i\right) + (\Delta t)^2 a\left(w_{j+1}^{(l+1)}, \eta_i\right) = \left(w_j^n, \eta_i\right) + \Delta t \left(p_j^n, \eta_i\right) + (\Delta t)^2 \left(h\left(t_j^n, w_{j+1}^{(l)}\right), \eta_i\right) \tag{16}$$

$$p_{j+1}^{(l+1)} = \frac{\left(w_{j+1}^{(l+1)} - w_j^n\right)}{\Delta t} \tag{17}$$

Equation (17) tells us the iterative method which depends only on $w_{j+1}^{(l+1)}$. Thus, equation (16) is reformulated as $w^{(l+1)} = \delta(w^{(l+1)})$, where l is the number of the iterations. And this leads us to the following theorem.

Theorem (1): The DES (8-11), for sufficiently small Δt, and for any fixed $j\ (0 \le j \le Y-1)$, has a unique solution $w^n = (w_0^n, w_1^n, \ldots\ldots, w_Y^n)$, and the sequence of the corrector solution converges on $\mathbb{R}$.

Proof: Let $w^{(l+1)} = \left(w_0^{(l+1)}, w_1^{(l+1)}, \ldots\ldots, w_Y^{(l+1)}\right)$ and
$\overline{w}^{(l+1)} = \left(\overline{w}_0^{(l+1)}, \overline{w}_1^{(l+1)}, \ldots\ldots, \overline{w}_Y^{(l+1)}\right)$
where $w^{(l+1)}$ and $\overline{w}^{(l+1)}$ are two solutions of (16), so

$$\left(w_{j+1}^{(l+1)}, \eta_i\right) + (\Delta t)^2 a\left(w_{j+1}^{(l+1)}, \eta_i\right) = \left(w_j^n, \eta_i\right) + \Delta t\left(p_j^n, \eta_i\right) + (\Delta t)^2\left(h\left(t_j^n, w_{j+1}^{(l)}\right), \eta_i\right) \quad (18)$$

and

$$\left(\overline{w}_{j+1}^{(l+1)}, \eta_i\right) + (\Delta \mathrm{t})^2 a\left(\overline{w}_{j+1}^{(l+1)}, \eta_i\right) = \left(w_j^n, \eta_i\right) + \Delta \mathrm{t}\left(p_j^n, \eta_i\right) + (\Delta t)^2\left(h\left(t_j^n, \overline{w}_{j+1}^{(l)}\right), \eta_i\right) \quad (19)$$

By subtracting (19) from (18), and putting $\eta_i = (\overline{w}_{j+1}^{(l+1)} - w_{j+1}^{(l+1)})$ in the obtained equation, we get

$$\left(\overline{w}_{j+1}^{(l+1)} - w_{j+1}^{(l+1)}, \overline{w}_{j+1}^{(l+1)} - w_{j+1}^{(l+1)}\right) + (\Delta \mathrm{t})^2 a\left(\overline{w}_{j+1}^{(l+1)} - w_{j+1}^{(l+1)}, \overline{w}_{j+1}^{(l+1)} - w_{j+1}^{(l+1)}\right)$$
$$= (\Delta \mathrm{t})^2\left(h\left(\overline{w}_{j+1}^{(l)}\right) - h\left(w_{j+1}^{(l)}\right), \overline{w}_{j+1}^{(l+1)} - w_{j+1}^{(l+1)}\right) \quad (20)$$

From Assumption 3 (ib), the 2^{nd} term in the L.H.S of (20) is positive. Then by applying Assumption 3 (iib) on h in R.H.S of (20), and by using the Cauchy Schwarz inequality on this side, we deduce that

$$\left\|\delta\left(\overline{w}_{j+1}^{(l)}\right) - \delta\left(w_{j+1}^{(l)}\right)\right\|_0 = \left\|\overline{w}_{j+1}^{(l+1)} - w_{j+1}^{(l+1)}\right\|_0 \le \lambda\left\|\overline{w}_{j+1}^{(l)} - w_{j+1}^{(l)}\right\|_0 \quad (21)$$

where $\lambda = (\Delta \mathrm{t})^2 L < 1$, for sufficiently small Δt.
which implies that δ is contractive. Also, since $\{w^{(l)}\} \in \mathbb{R}\ \forall\, l$, then $\delta\left(w^{(l+1)}\right) = w^{(l+1)} \in \mathbb{R}\ \forall\, l$, *i.e* $\delta(w) \in \mathbb{R}$, hence, by theorem (1) in [8], the sequence $\{w^{(l)}\}$ converges to a point in $\mathbb{R}$.

6. Stability

Lemma (2): For sufficiently small Δt, the following are satisfied:

$$\left\|w_j^n\right\|_1^2 \le \bar{d},\ \left\|p_j^n\right\|_1^2 \le \bar{d},\ \sum_{j=0}^{Y-1}\left\|w_{j+1}^n - w_j^n\right\|_1^2 \le \bar{d}, \text{ and } \sum_{j=0}^{Y-1}\left\|p_{j+1}^n - p_j^n\right\|_1^2 \le \bar{d}$$

for each $j = 0, 1, \ldots, Y$, where $\bar{d}$ represents a various constant.

Proof: By substituting $\eta = p_{j+1}^n$ in (8) and rewriting the 1^{st} term in the L.H.S of the obtained equation, it becomes

$$\left\|p_{j+1}^n\right\|_0^2 - \left\|p_j^n\right\|_0^2 + \left\|p_{j+1}^n - p_j^n\right\|_0^2 + \Delta \mathrm{t}\, a\left(w_{j+1}^n, p_{j+1}^n\right) = \Delta t\left(h(w_{j+1}^n), p_{j+1}^n\right) \quad (22)$$

Since,

$$\Delta t\, a\left(w_{j+1}^n, p_{j+1}^n\right) = \frac{1}{2}\left[a\left(w_{j+1}^n - w_j^n, w_{j+1}^n - w_j^n\right) + a\left(w_{j+1}^n, w_{j+1}^n\right) - a\left(w_j^n, w_j^n\right)\right] \quad (23)$$

and by substituting (23) in the L.H.S of (22), then by summing both sides of the obtained equation, for $j = 0$ to $j = l-1$, and setting $c = \max(1, \frac{k_2}{2})$, the result leads to the inequality

$$c\left\|p_l^n\right\|_0^2 + c\sum_{j=0}^{l-1}\left\|p_{j+1}^n - p_j^n\right\|_0^2 + c\left\|w_l^n\right\|_1^2 + c\sum_{j=0}^{l-1}\left\|w_{j+1}^n - w_j^n\right\|_1^2 \le \left\|p_0^n\right\|_0^2 +$$
$$\frac{k_2}{2}\left\|w_0^n\right\|_1^2 + \sum_{j=0}^{l-1}\Delta t\left(h(w_{j+1}^n), p_{j+1}^n\right) \quad (24)$$

Now, we use the assumptions on h, to get that

$$\left|\left(h(w_{j+1}^n), p_{j+1}^n\right)\right| \le \left\|\gamma_j\right\|_0^2 + \lambda\left\|w_{j+1}^n\right\|_1^2 + \bar{\lambda}\left\|p_{j+1}^n\right\|_0^2,\ \bar{\lambda} = \lambda + 1 \quad (25)$$

But $\left\|w_{j+1}^n\right\|_1^2 = 2\left\|w_{j+1}^n - w_j^n\right\|_1^2 + 2\left\|w_j^n\right\|_1^2$
and $\left\|p_{j+1}^n\right\|_0^2 = 2\left\|p_{j+1}^n - p_j^n\right\|_0^2 + 2\left\|p_j^n\right\|_0^2$

By substituting this equality in (25), then substituting the obtained inequality in the R.H.S. of (24) after applying the Cauchy-Schwartz inequality, assuming that $d = \max(2\delta, 2\bar{\delta})$, we conclude the inequality as

$$c\|p_l^n\|_0^2 + (c - d\Delta t)\sum_{j=0}^{l-1}\left\|p_{j+1}^n - p_j^n\right\|_0^2 + c\|w_l^n\|_1^2 + (c - d\Delta t)\sum_{j=0}^{l-1}\left\|w_{j+1}^n - w_j^n\right\|_1^2 \le \|p_0^n\|_0^2$$

$$+\frac{k_2}{2}\|w_0^n\|_1^2 + \|\gamma\|_Q^2 + d(\Delta t)\sum_{j=0}^{l-1}\left\|w_j^n\right\|_1^2 + d(\Delta t)\sum_{j=0}^{l-1}\left\|p_j^n\right\|_0^2 \quad (26)$$

Now, let $\Delta t < c/d$, and the 2^{nd} and 4^{th} terms in the L.H.S of (26) are positives, by using the discrete Gronwall's (DGs) inequality [10], we deduce

$\mathrm{c}(\| p_l^n \|_0^2 + \|w_l^n\|_1^2) \le ae^{\sum_{j=0}^{l-1} d(\Delta t)} = ae^{ld(\Delta t)} \le b,$

where $a = \|p_0^n\|_0^2 + \frac{k_2}{2}\|w_0^n\|_1^2 + \|\gamma\|_Q^2$

which implies

$\| w_l^n \|_1^2 \le d_1 = \frac{b}{c}$, and $\| p_l^n \|_0^2 \le d_1$, for any arbitrary index l.

Therefore, $\| w_j^n \|_1^2 \le d_1$ and $\| p_j^n \|_0^2 \le d_1$, for each $j = 0,1,\dots,Y-1$.

Thus,

$(\Delta t)d\sum_{j=0}^{Y-1}\|w_j^n\|_1^2 + (\Delta t)d\sum_{j=0}^{Y-1}\|p_j^n\|_1^2 \le 2d_1\, d\, \Delta t\, Y = 2\mathrm{c}T = \bar{d}$.

We return to (26) with substituting $l = Y$ The 1st and the 3rd terms in the L.H.S are positives. Then by using the above results in the R.H.S. of it , keeping in mind that the first three terms in this side are bounded, we obtain

$$\sum_{j=0}^{Y-1}\left\|w_{j+1}^n - w_j^n\right\|_1^2 \le \bar{d} \quad (27)$$

$$\sum_{j=0}^{Y-1}\left\|p_{j+1}^n - p_j^n\right\|_0^2 \le \bar{d} \quad (28)$$

7. Convergence

The following definitions for the functions "almost everywhere on $\bar{I}$ " are useful in the proof of the next theorem, so let

$w_-^n(t) := w_j^n,\ t \in I_j^n,\ \forall j = 0,1,\dots,Y,$

$w_+^n(t) := w_{j+1}^n,\ t \in I_j^n, \forall j = 0,1,\dots,Y-1,$

$p_+^n(t) := p_{j+1}^n,\ t \in I_j^n, \forall j = 0,1,\dots,Y-1,$

$p_-^n(t_j^n) := p_j^n, t_j^n \in I_j^n,\ \forall j = 0,1,\dots,Y,$

Let $w_\wedge^n(t)$ be an affine function on each I_j^n, such that $w_\wedge^n(t) := w_j^n\ \forall, j = 0,1,\dots,Y,$ and $p_\wedge^n(t)$ be an affine function on each I_j^n, such that $p_\wedge^n(t) := p_j^n\ \ \forall, j = 0,1,\dots,Y.$

Theorem (3): The discrete solutions $w_-^n(t), w_+^n(t)$, and $w_\wedge^n(t)$ converge strongly in $L^2(\varphi)$, as $n \longrightarrow \infty$,

proof: From Lemma (2), we have for any $j = 0,1,\dots,Y$, that

$\left\|w_j^n\right\|_1^2 \le \bar{d}$ and $\left\|p_j^n\right\|_0^2 \le \bar{d},$

which makes

$\|w_-^n\|_{L^2(I,V)}^2$, $\| w_+^n \|_{L^2(I,V)}^2$, $\|w_\wedge^n\|_{L^2(I,V)}^2$, $\|p_-^n\|_{L^2(\varphi)}^2$, $\|p_+^n\|_{L^2(\varphi)}^2$, and $\|p_\wedge^n\|_{L^2(\varphi)}^2$ are bounded.

From the inequality (27), we get

$\Delta t\sum_{j=0}^{Y-1}\left\|w_{j+1}^n - w_j^n\right\|_0^2 \le \Delta t\bar{d} \longrightarrow 0$, as $\Delta t \longrightarrow 0$,

$$\bar{w}_+^n \longrightarrow \bar{w}_-^n \text{ is strongly (ST) in } L^2(I,V) \text{ and in } L^2(\varphi). \quad (29)$$

Also, by using the same way into the inequality (28), we get

$$p_+^n \longrightarrow p_-^n \text{ is ST in } L^2(\varphi) \quad (30)$$

By using theorem 3.2 in [9], there are subsequences of ($\{w_-^n\}$, $\{w_+^n\}$, $\{w_\wedge^n\}$) and of ($\{p_-^n\}$, $\{p_+^n\}$, $\{p_\wedge^n\}$,). Using the same notations again, they converge weakly to some w in $L^2(I,V)$, to some p in $L^2(\varphi)$, which means that

$w_-^n \longrightarrow w, w_+^n \longrightarrow w, w_\wedge^n \longrightarrow w$ is weakly in $L^2(I,V)$ and

$p_-^n \longrightarrow p$, $p_+^n \longrightarrow p, p_\wedge^n \longrightarrow p$ is weakly in $L^2(\varphi)$,
By using the first compactness theorem[9], we get , $w_\wedge^n \longrightarrow w$ ST in $L^2(\varphi)$. Also, $w_+^n \longrightarrow w$ and $w_-^n \longrightarrow w$ ST in $L^2(\varphi)$.
Now, let $\{V_n\}_{n=1}^{\infty}$ be a sequence of subspaces of V, where V_n is as defined above. Then by using the Galerkin approach, for each $\eta \in V$, there exists a sequence $\{\eta_n\}$, with $\eta_n \in V_n$ for each n, such that $\eta_n \longrightarrow \eta$ ST in $L^2(\varphi)$.
Consider that $\xi(t) \in C^2[0,T]$, for which $\xi(T) = \xi'(T) = 0$ and $\xi(0) = \xi'(0) \neq 0$. Let $\xi^n(t)$ be a piecewise continuous (CP) interpolation of $\xi(t)$ with respect to I_j^n, and let $\zeta = \eta\, \xi(t)$, with $\zeta^n = \eta_n \xi^n(t)$, with

$\zeta_-^n := \eta_n \xi_-^n(t)$, $t \in I_j^n$, $j = 0,1,\dots,Y-1$, $\eta_n \in V_n$,

$\zeta_+^n := \eta_n \xi_+^n(t)$, $t \in I_j^n$, $j = 0,1,\dots,Y-1$, $\eta_n \in V_n$,

$\zeta_\wedge^n := \eta_n \xi^n(t)$, $t \in I$, $\eta_n \in V_n$,

By substituting $\eta = \zeta_{j+1}^n$ in eq.(8), then summing both sides of the obtained equation for $j = 0$, to $j = Y-1$, and by using the discrete integrating by parts (DIBP) for the 1^{st} term in the L.H.S., eq.(8) one can get that

$$-\int_0^T (p_-^n, (\zeta_\wedge^n)')\, dt + \int_0^T a(w_+^n, \zeta_+^n)\, dt = \int_0^T (h(t_-^n, w_+^n), \zeta_+^n)\, dt + (p_0^n, \eta_n)\, \xi(0) \quad (31)$$

On the other hand, from (9), one has

$$((w_\wedge^n)', \eta)(\xi^n)' = (p_+^n, \eta)(\xi^n)'$$

By integrating both sides on $[0,T]$, and by applying the DIBP for the 1^{st} term in the L.H.S of the obtained equation, we have

$$-\int_0^T (w_+^n, \eta)\, (\xi^n(t))'' dt = \int_0^T (p_+^n, \eta)(\xi^n)' dt + (w_0^n, \eta)(\xi^n(0))' \quad (32)$$

Also, since
$\xi^n(t) \longrightarrow \xi(t)$ in $C(I) \subset L^2(I)$, $\eta_n \longrightarrow \eta$ ST in $L^2(I,V)$ and in $L^2(\psi)$, then we get that
$\zeta_+^n = \eta_n \xi_+^n \longrightarrow \eta\, \xi = \zeta$ ST in $L^2(I,V)$ and in $L^2(\varphi)$, $\eta_n \xi^n(0) \longrightarrow \eta\, \xi(0)$ ST in $L^2(\varphi)$,
$(\zeta_\wedge^n)' = \eta_n \xi'^n \longrightarrow \eta\, \xi' = \eta\zeta'$ ST in $L^2(I,V)$.
And since $t_-^n \longrightarrow t$ ST in $L^\infty(I)$, w_+^n , w_-^n, $w_\wedge^n \longrightarrow w$ ST in $L^2(\varphi)$, $w_0^n \longrightarrow w^0$ ST in V and $p_0^n \longrightarrow w^1$ ST in $L^2(\psi)$.
Then from these convergences and the assumptions on h, one can passage to the limit in (31) and in (32), then we get

$$-\int_0^T (p, \eta)\, \xi' dt + \int_0^T a(w, \eta)\, \xi\, dt = \int_0^T (h(t,w), \eta)\, \xi\, dt + (w^1, \eta)\, \xi(0) \quad (33)$$

and

$$-\int_0^T (w, \eta)\xi''(t)\, dt = \int_0^T (p, \eta)\xi'(t) dt + (w^0, \eta)\xi'(0) \quad (34)$$

The following cases appear:
Case (I): Choose $\xi(t) \in C^2[0,T]$, with $\xi(T) = \xi'(T) = \xi(0) = \xi'(0) = 0$, and put $\xi'(0) = 0$ in eq.(33) and $\xi(0) = 0$ in eq.(34), then using IBP for the 1^{st} term for each resulted equation, we have

$$\int_0^T (w_t, \eta)\xi'(t)\, dt = \int_0^T (p, \eta)\xi'(t) dt \implies w_t = p\,,$$

$$\int_0^T (w_{tt}, \eta)\, \xi\, dt + \int_0^T a(w, \eta)\, \xi\, dt = \int_0^T (h(t,w), \eta)\xi\, dt, \quad (35)$$

Then

$(w_{tt}, \eta) + a(w, \eta) = (h(t,w), \eta)$, $\eta \in V$ a.e. on I.

Case (II) : Choose the $\xi(t) \in D[0,T]$, with $\xi(0) \neq 0$, $\xi(T) = 0$, and use IBP for the 1^{st} term in the L.H.S of (35), we obtain

$$-\int_0^T (w_t, \eta)\, \xi' dt + \int_0^T a(w, \eta)\, \xi\, dt = \int_0^T (h(t,w), \eta)\xi\, dt + w_t(0),\ \eta)\xi(0) \quad (36)$$

Let $p = w_t$ in eq.(33), we subtract the resulting equation from (36) to yield

$(w_t(0), \eta)\xi(0) = (w^1, \eta)\xi(0) \implies (w_t(0), \eta) = (w^1, \eta)$, $\forall \eta$

then $w_t(0) = w^1(0)$.

Case (III): Choose the $\xi(t) \in D[0,T]$, such that $\xi'(0) \neq 0$, $\xi(0) = 0$, and $\xi(T) = \xi'(T) = 0$. Using twice the IBP for the 1st term in the L.H.S. of (35), we get

$$\int_0^T (w,\eta)\xi'' dt + \int_0^T a(w,\eta)\,\xi\, dt = \int_0^T (h(t,w),\eta)\xi\, dt - (w(0),\eta)\xi'(0) \quad (37)$$

We rewrite (34) in the following form

$$-\int_0^T (p,\eta)\xi'(t)\, dt = \int_0^T (w,\eta)\xi''(t) dt + (w^0,\eta)\xi'(0) \quad (38)$$

By substituting (38) in (33), with $\xi(0) = 0$, then subtracting the resulting equation from (37), we get

$(w(0),\eta)\xi'(0) = (w^0, \eta)\xi'(0) \implies (w(0),\eta) = (w^0, \eta)$ for each η, then $w(0) = w^0(0)$

That is, the limit point w is a solution to the WEFM in the COE.

8. Cholesky Factorization

The Cholesky decomposition is used to solve the GLAS with two conditions, in which the coefficient matrix B must be a symmetric and positive definite. Then the matrix B can be factorized into the product of an Upper triangular matrix $U$and Lower triangular matrix U^T [8], and U can be determined as shown in the following steps:

Step 1: $u_{ii} = (b_{ii} - \sum_{k=1}^{i-1} u_{ki}^2)^{\frac{1}{2}}$ for $i = 1,2,\dots,n$

Step 2: $u_{ij} = \left(b_{ij} - \sum_{k=1}^{i-1} u_{ki}\, u_{kj}\right)/\, u_{ii}$ for $j = i+1,\dots,n$.

9. Numerical Examples

The problems in the following examples are coded by Matlap software:

Example 1: Consider the following NOLVCHBVP:

$$w_{tt} - \sum_{r,u=1}^{2} \frac{\partial}{\partial x_u}\left(a_{ru}(\vec{x},t)\frac{\partial w}{\partial x_r}\right) + g(\vec{x},t)\, w = h(\vec{x},t,w),$$

with the variables coefficients are:

$a_{11}(x_1,x_2) = 1 - x_1 + e^{(x_1/2)}$, $a_{22}(x_1,x_2) = 1 - x_2 + e^{(x_2/2)}$, $g(x_1,x_2) = 1 + e^{(0.7x_1x_2)}$

where $\varphi = \psi \times I$, $\psi = (0,1)\times(0,1)$, $I = [0,1]$

$w_t(\vec{x},0) = w^1(\vec{x})$, in ψ

$w(\vec{x},t) = 0$, on $\sum = \partial\psi \times I$

$$h(\vec{x},t,w) = \sqrt{\cos(e^{-t})}\Big[x_1(x_1-1)\Big((x_2-1)\Big(x_2\big(e^{(7x_1x_2/10)}+1\big) - \big(2x_2e^{x_2^2}-1\big)\Big) - \Big(x_2\big(x_2e^{x_2^2}-1\big) + 2\big(e^{x_2^2} - x_2 + 1\big)\Big)\Big) - x_2(x_2-1)\Big(2\big(e^{(x_1/2)} - x_1 + 1\big) + \Big(\big((x_1-1) + x_1\big)\big(e^{(x_1/2)}/2 - 1\big)\Big)\Big) - x_1x_2\sin\Big(x_1x_2\sqrt{\cos(e^{-t})}(x_1-1)(x_2-1)\Big)\Big] + w\sin(w)$$

The exact solution (EXS) of this problem is $w(\vec{x},t) = x_1x_2(1-x_1)(1-x_2)\sqrt{\cos(e^{-t})}$. Using the MGFEIM to solve this problem for $K = 9$, $Y = 20$ and $T = 1$, then the results are shown in Figure 1. (a) which shows the APPS, and Figure 1.(b) which shows the EXS at $\hat{t} = 0.5$

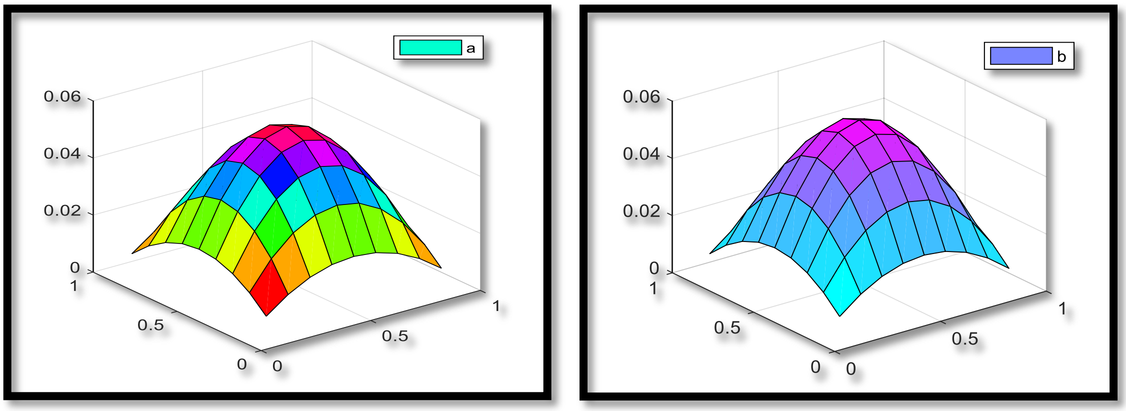

Figure 1-(a) The APPS for the NOLVCHBVP at $\hat{t} = 0.5$ and (b) the EXS for the equation at $\hat{t} = 0.5$.

Example 2: Consider the following NOLVCHBVP:

$$w_{tt} - \sum_{r,u=1}^{2} \frac{\partial}{\partial x_u}\left(a_{ru}(\vec{x},t)\frac{\partial w}{\partial x_r}\right) + g(\vec{x},t)\, w = h(\vec{x},t,w),$$

with the variables coefficients are:

$a_{11}(x_1,x_2) = e^{(1+x_1^2)}\,, a_{22}(x_1,x_2) = e^{(1+x_2^2)}\,, g(x_1,x_2) = e^{(1+x_1+x_2)}$

where $\varphi = \psi \times I\,, \psi = (0,1)\times(0,1),\ I = [0,1]$ and

$w_t(\vec{x},0) = w^1(\vec{x})$, in ψ

$w(\vec{x},t) = 0$, on $\sum = \partial\psi \times I$

$h(\vec{x},t,w) = -2x_2^2(x_2-1)^2 e^{(-3t/10)+(x_1^2+1)}[x_1^2 - (x_1-1)^2 - 4x_1(x_1-1) + 2x_1^3(x_1-1) + 2x_1^2(x_1-1)^2] - 2(x_1-1)^2 e^{(-3t/10)+(x_2^2+1)}[2x_1^2x_2^3(x_2-1) + (2x_1^2x_2^2)(x_2-1)^2 + x_1^2(x_2-1)^2 + x_1^2x_2^2 + 4x_1^2x_2(x_2-1)] + x_1^2x_2^2(x_1-1)^2(x_2-1)^2e^{(-3t/10)}\left[e^{(x_1+x_2+1)} - \sin\left(x_1^2x_2^2(x_1-1)^2(x_2-1)^2e^{(-3t/10)}\right)\right] + w\sin(w).$

The exact solution (EXS) of this problem is $w(\vec{x},t) = x_1^2x_2^2(1 - x_1 - x_2 + x_1x_2)^2e^{(-0.3t)}$. Using the MGFEIM to solve this problem for $= 9$, $Y = 20$ and $T = 1$, then the results are shown in Figure 2. (a) which shows the APPS , and Figure 2.(b) which shows the EXS at $\hat{t} = 0.5$.

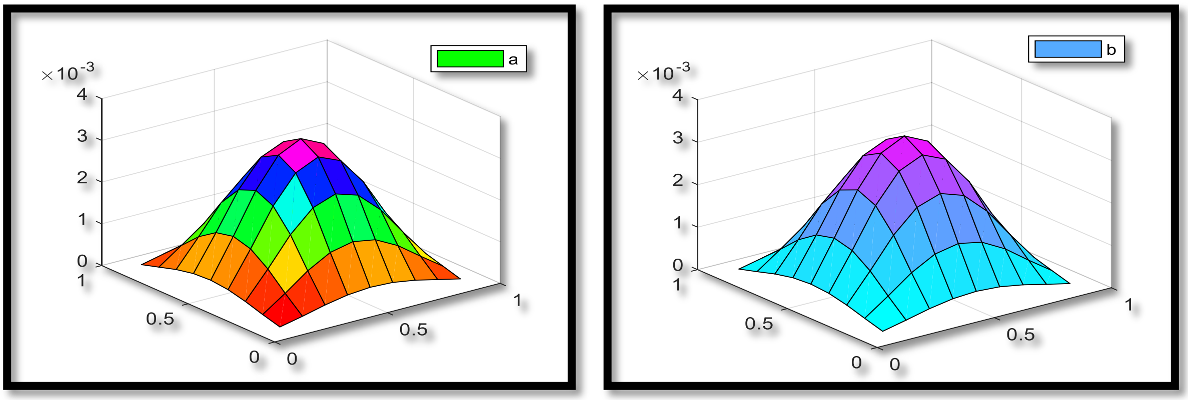

Figure 2- (a) The APPS for the NOLVCHBVP at $\hat{t} = 0.5$ and (b) the EXS for the equation at $\hat{t} = 0.5$.

10. Numerical Discussion and Conclusions

The MGFEIM is used successfully to solve the discrete of the WEFM of a certain type of NOLVCHBVP. The existence theorem of a unique convergent APPs is proved. The convergence of the PT and CT, which are used to solve the GNAS that is obtained from applying the MGFEIM, is proved and the ChMe, which is used inside these technique, is highly efficient for solving large GAS. The discrete of the WEFM proved that it is stable and convergent. The results of the considered examples showed the efficiency and accuracy of the method.

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