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Some Properties of D-Operator on Hilbert Space

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Abstract
In this paper, we introduce a new type of Drazin invertible operator on Hilbert spaces, which is called D-operator. Then, some properties of the class of D-operators are studied. We prove that the D-operator preserves the scalar product, the unitary equivalent property, the product and sum of two D-operators are not D-operator in general but the direct product and tenser product is also D-operator.

Keywords: Hilbert space, Drazin operator, Normal operator, D-normal operator, n-normal operator, class (Q) operators.

حول بعض خواص المؤثر على فضاء هلبرت D

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الخلاصه
في هذا البحث قدمنا نوع جديد من المؤثرات من النمط قابلة للعكس درازاين على فضاءهلبرت وتم تسميته المؤثر D-.وبعد ذلك درسنا بعض الخواص

1. Introduction

Throughout this paper, H is a Hilbert space, $B(H)$ is the space of all bounded linear operators on a complex Hilbert space$H$. The Drazin inverse for a bounded linear operator on a complex Banach space was introduced by Caradus [1]. Let$T \in B(H)$, the Drazin inverse of $T,$ if it exists, is an operator $T^D \in B(H)$ such that

$TT^D = T^D T,\ T^D T T^D = T^D,\ T^{k+1}T^D = T^k$

For some integer number$k \geq 0$, the smallest integer $k \geq 0$ is called the index of T which is denoted by ind(T). It is easy to see that ind$(T) = 0$ if and only if T is an invertible operator. Then $T^D = T^{-1}$. In the following lemma, we collect some properties of Drazin operator which appeared in previous studies [2, 3].

Lemma 1.1: Let $S, T \in B(H)$ be two Drazin invertible operators, then

(a) $(T^*)^D = (T^D)^*$.

(b) $\left(T^\ell\right)^D = (T^D)^\ell$ for $\ell = 1,2, \ldots$

(c) $(S^{-1}TS)^D = S^{-1}T^D S$.

(d) If $ST = TS$, then $(ST)^D = S^D T^D = T^D S^D, S^D T = T S^D$, and $T^D S = S T^D$.

(e) If $ST = TS = 0$, then $(S+T)^D = S^D + T^D$.

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Let $T \in B(H)$, T is called normal if $TT^* = T^*T$. The purpose of this paper is to introduce a new operator to generalize the normal operator. Many authors presented generalizations of normal operators. For examples, in an earlier work [4], the author introduced a class *(Q)* of operators acting on a Hilbert space H : for any $T \in (Q)$, $T^{*2}(T)^2 = (T^*T)^2$. Then, in another article [5], the authors introduced some new classes of operators associated with Drazin invertible operator. In this section, we give a new type of operators that are associated with Drazin invertible operator, that we call D-operator.

The paper contains two sections. In section one, we investigate some basic properties that we need. In section two, we study most of the properties of D-operators.

2. Main Results

Definition 2.1:

Let $T \in B(H)$ be Drazin invertible. T is called D-operator if
$T^{*2}(T^D)^2 = (T^*T^D)^2$.

The class of all D-operators is denoted by [D]. By lemma (1.1) (d), it is easy to prove that every normal operator is D-operator but the converse is not true in general. For example, let T be a nilpotent operator, then $T^D = 0$, hence it is clear that T is D-operator but the nilpotent is not necessary normal.

In this section, we investigate some basic properties of operators in [D].

Proposition 2.2:

Let $T \in [D]$, then the following assumptions hold:

1) $\alpha T \in [D]$ for every scalar α.

2) $S \in [D]$ for every $S \in B(H)$ that is unitary equivalent to T.

3) The restriction $T|M$ of T to any closed subspace M of H that reduces T is in [D].

4) $T^D \in [D]$.

Proof:

1) The proof is straight forward.

2) Since S is unitary equivalent to T, then $S = UTU^*$, where U is unitary operator. Thus,

$$
\begin{aligned}
S^{*2}(S^D)^2 &= (UT^*U^*)^2(UT^DU^*)^2 \\
&= (UT^*U^*)(UT^*U^*)(UT^DU^*)(UT^DU^*) \qquad (UU^* = I) \\
&= UT^*T^*T^DT^DU^* \\
&= UT^{*2}(T^D)^2U^* \\
&= U(T^*T^D)^2U^* \\
&= UT^*T^DT^*T^DU^* \\
&= (UT^*U^*)(UT^DU^*)(UT^*U^*)(UT^DU^*) \\
&= S^*S^DS^*S^D \\
&= (S^*S^D)^2.
\end{aligned}
$$

Hence, $S \in [D]$.

3)

$$
\begin{aligned}
(T|M)^{*2}((T|M)^D)^2 &= (T|M)^*(T|M)^*(T|M)^D(T|M)^D \\
&= (T^*|M)(T^*|M)(T^D|M)(T^D|M) \\
&= (T^*T^*|M)(T^DT^D|M) \\
&= \left(T^{*2}|M\right)((T^D)^2|M) \\
&= \left(T^{*2}\left(T^D\right)^2\right)|M \\
&= ((T^*T^D)^2)|M \\
&= \left((T^*T^D)(T^*T^D)\right)|M \\
&= \left(((T^*T^D)|M)((T^*T^D))|M\right) \\
&= \left((T^*|M)(T^D|M)(T^*|M)(T^D|M)\right) \\
&= \left((T^*|M)(T^D|M)\right)^2 \\
&= ((T|M)^*(T|M)^D)^2.
\end{aligned}
$$

Hence $T|M \in [D]$.

4) Since $T \in [D]$, then $T^{*2}(T^D)^2 = (T^*T^D)^2$. Thus $T^*T^*T^DT^D = T^*T^DT^*T^D$

By taking the adjoint of both sides of the above equation, we have

$$(T^*)^D(T^*)^DTT = (T^*)^DT(T^*)^DT$$

Hence, $$((T^D)^*)^2T^2 = ((T^D)^*T)^2.$$

Therefore, $((T^D)^*)^2((T^D)^D)^2 = ((T^D)^*(T^D)^D)^2$.

Thus, $T^D \in [D]$

Proposition 2.3:

The set of all D-operators on H is a closed subset of $B(H)$.

Proof:

Let $\langle T_k \rangle$ be a sequence of D-operators such that $T_k \to T$. It is enough to show that T is D-operator. Since $T_k \to T$ then $T_k{}^* \to T^*$ and $T_k{}^D \to T^D$. Hence, $T_k{}^* T_k{}^D \to T^*T^D$, then we get that

$$\left(T_k{}^* T_k{}^D\right)^2 \to (T^*T^D)^2 \quad (1)$$

On the other hand, we obtain that $T_k{}^{*2} \to T^{*2}$ and $\left(T_k{}^D\right)^2 \to (T^D)^2$. Hence, $T_k{}^{*2}\left(T_k{}^D\right)^2 \to T^{*2}(T^D)^2$ (2)

Therefore, from equations (1) and (2), we conclude that

$$\begin{aligned}\|T^{*2}(T^D)^2 - (T^*T^D)^2\| &= \left\|T^{*2}(T^D)^2 - T_k^{*2}\left(T_k^D\right)^2 + T_k^{*2}\left(T_k^D\right)^2 - (T^*T^D)^2\right\| \\ &\le \left\|T^{*2}(T^D)^2 - T_k^{*2}\left(T_k^D\right)^2\right\| + \left\|T_k^{*2}\left(T_k^D\right)^2 - (T^*T^D)^2\right\| \\ &= \left\|T^{*2}(T^D)^2 - T_k^{*2}\left(T_k^D\right)^2\right\| + \left\|\left(T_k^*T_k^D\right)^2 - (T^*T^D)^2\right\| \to 0 \text{ as } k \to \infty.\end{aligned}$$

Hence, $T^{*2}(T^D)^2 = (T^*T^D)^2$. Thus $T \in [D]$

Proposition 2.4:

Let $S, T \in [D]$. If $[T, S] = [T, S^*] = 0$, then $ST \in [D]$.

Proof:

Since $[T, S] = [T, S^*] = 0$, then by lemma(1.1) (d) we have

$[T, S^D] = [T^D, S] = [T^D, S^*] = [T^*, S^D] = 0$.

Moreover, since $S, T \in [D]$, then

$T^{*2}(T^D)^2 = (T^*T^D)^2$ and $S^{*2}(S^D)^2 = (S^*S^D)^2$. Therefore,

$$\begin{aligned}(ST)^{*2}((ST)^D)^2 &= (ST)^*(ST)^*(ST)^D(ST)^D \\ &= T^*S^*T^*S^*S^DT^DS^DT^D \\ &= T^*T^*T^DT^DS^*S^*S^DS^D \\ &= T^{*2}S^{*2}(T^D)^2(S^D)^2 \\ &= T^*T^*S^*S^*T^DS^DT^DS^D \\ &= T^*S^*T^*S^*T^DS^DT^DS^D \\ &= (ST)^*(ST)^*(ST)^D(ST)^D \\ &= ((ST)^*(ST)^D)^2\end{aligned}$$

Thus $ST \in [D]$

Proposition 2.5:

Let $S, T \in [D]$. If $ST = TS = 0$, then $S + T \in [D]$.

Proof:

(f) Since $S, T \in [D]$, then $T^{*2}(T^D)^2 = (T^*T^D)^2$ and $S^{*2}(S^D)^2 = (S^*S^D)^2$. Since $ST = TS = 0$, then $S^*T^* = T^*S^* = 0$, and by lemma (1.1)(e) we have $(S + T)^D = S^D + T^D$. Hence

$$\begin{aligned}(S+T)^{*2}((S+T)^D)^2 &= (S+T)^*(S+T)^*(S+T)^D(S+T)^D \\ &= (S^* + T^*)(S^* + T^*)(S^D + T^D)(S^D + T^D) \\ &= (S^{*2} + T^{*2})((S^D)^2 + (T^D)^2) \qquad (S^*T^* = T^*S^* = 0) \\ &= S^{*2}(S^D)^2 + T^{*2}(T^D)^2 \\ &= (S^*S^D)^{*2} + (T^*T^D)^2 \\ &= (S^*S^D + T^*T^D)(S^*S^D + T^*T^D) \\ &= (S^* + T^*)(S^D + T^D)(S^* + T^*)(S^D + T^D) \\ &= ((S+T)^*(S+T)^D)^2\end{aligned}$$

Thus $S + T \in [D]$

The following example shows that the propositions (2.4) and (2.5) are not necessarily true in general.

Example 2.6:

1) Let $S = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

Therefore,

$S^D = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ and $T^D = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$.

It can be easily checked that $S, T \in [D]$ and $ST \neq TS$. Note that,

$(ST)^D = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

But, it is easy to compute that $ST \notin [D]$.

2) Let $T = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Clearly $T \in [D]$, but $T + I \notin [D]$

The following corollary is a straightforward result from proportion (2.4).

Corollary 2.7:

If $T \in [D]$, then $T^n \in [D]$ for all positive integers n.

Theorem 2.8:

Let $T_1, T_2, \dots, T_n \in [D]$, then

1) $T_1 \oplus T_2 \oplus \dots \oplus T_n \in [D]$.

2) $T_1 \otimes T_2 \otimes \dots \otimes T_n \in [D]$.

Proof:

1) Since $T_i \in [D]\ \forall i = 1,2, \dots, n$, then

$T_i^{*2}\left(T_i^D\right)^2 = \left(T_i^* T_i^D\right)^2$. Hence,

$$\begin{aligned}
&(T_1 \oplus T_2 \oplus \dots \oplus T_n)^{*2}((T_1 \oplus T_2 \oplus \dots \oplus T_n)^D)^2 \\
&= (T_1^{*2} \oplus T_2^{*2} \oplus \dots \oplus T_n^{*2})((T_1^D)^2 \oplus (T_2^D)^2 \oplus \dots \oplus (T_n^D)^2) \\
&= T_1^{*2}(T_1^D)^2 \oplus T_2^{*2}(T_2^D)^2 \oplus \dots \oplus T_n^{*2}(T_n^D)^2 \\
&= (T_1^* T_1^D)^2 \oplus (T_2^* T_2^D)^2 \oplus \dots \oplus (T_n^* T_n^D)^2 \\
&= T_1^* T_1^D T_1^* T_1^D \oplus T_2^* T_2^D T_2^* T_2^D \oplus \dots \oplus T_n^* T_n^D T_n^* T_n^D \\
&= (T_1^* T_1^D \oplus T_2^* T_2^D \oplus \dots \oplus T_n^* T_n^D)(T_1^* T_1^D \oplus T_2^* T_2^D \oplus \dots \oplus T_n^* T_n^D) \\
&= \left((T_1^* \oplus T_2^* \oplus \dots \oplus T_n^*)(T_1^D \oplus T_2^D \oplus \dots \oplus T_n^D)\right)^2 \\
&= ((T_1 \oplus T_2 \oplus \dots \oplus T_n)^*(T_1 \oplus T_2 \oplus \dots \oplus T_n)^D)^2
\end{aligned}$$

2) Let $x_1, x_2, \dots, x_n \in H$, then

$$\begin{aligned}
&(T_1 \otimes T_2 \otimes \dots \otimes T_n)^{*2}((T_1 \otimes T_2 \otimes \dots \otimes T_n)^D)^2(x_1 \otimes x_2 \otimes \dots \otimes x_n) \\
&= (T_1^{*2} \otimes T_2^{*2} \otimes \dots \otimes T_n^{*2})((T_1^D)^2 \otimes (T_2^D)^2 \otimes \dots \otimes (T_n^D)^2)(x_1 \otimes x_2 \otimes \dots \otimes x_n) \\
&= T_1^{*2}(T_1^D)^2(x_1) \otimes T_2^{*2}(T_2^D)^2(x_2) \otimes \dots \otimes T_n^{*2}(T_n^D)^2(x_n) \\
&= (T_1^* T_1^D)^2(x_1) \otimes (T_2^* T_2^D)^2(x_2) \otimes \dots \otimes (T_n^* T_n^D)^2(x_n) \\
&= \left(T_1^* T_1^D(x_1) \otimes T_2^* T_2^D(x_2) \otimes \dots \otimes T_n^* T_n^D(x_n)\right)^2 \\
&= \left((T_1^* \otimes T_2^* \otimes \dots \otimes T_n^*)(T_1^D \otimes T_2^D \otimes \dots \otimes T_n^D)\right)^2 (x_1 \otimes x_2 \otimes \dots \otimes x_n) \\
&= ((T_1 \otimes T_2 \otimes \dots \otimes T_n)^*(T_1 \otimes T_2 \otimes \dots \otimes T_n)^D)^2(x_1 \otimes x_2 \otimes \dots \otimes x_n)
\end{aligned}$$

In the following theorem, we compute the Drazin invertible operator for some special matrix.

Theorem 2.9:

Let $T = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}$, where a, b, c are non-zero complex numbers such that $a \neq b$, then one of the following forms of Drazin invertible can be satisfied:

1) $T^D = 0$.

2) $T^D = \begin{pmatrix} 0 & -c/((a-b)b) \\ 0 & 1/b \end{pmatrix}$.

3) $T^D = \begin{pmatrix} 1/a & c/((a-b)a) \\ 0 & 0 \end{pmatrix}$.

4) $T^D = \begin{pmatrix} 1/a & -c/(ab) \\ 0 & 1/b \end{pmatrix}$.

Proof:
Let $T^D = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}$, where $m_1, m_2, m_3, m_4 \in \mathbb{C}$, then

$$TT^D = T^D T \quad (1)$$

$$T^D T T^D = T^D \quad (2)$$

Thus, from eq (1), it is easy to conclude that

$$\begin{pmatrix} am_1 + cm_3 & am_2 + cm_4 \\ bm_3 & bm_4 \end{pmatrix} = \begin{pmatrix} am_1 & cm_1 + bm_2 \\ am_3 & cm_3 + bm_4 \end{pmatrix}.$$

Therefore we get that

$$am_1 + cm_3 = am_1 \quad (3)$$

$$am_2 + cm_4 = cm_1 + bm_2 \quad (4)$$

From (3), we get m_3 =0.
This implies that, from eq (2), the following matrix equation:

$$\begin{pmatrix} am_1{}^2 & am_1 m_2 + m_4(cm_1 + bm_2) \\ 0 & bm_4{}^2 \end{pmatrix} = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}.$$

Thus we get the following equations:

$$m_1(am_1 - 1) = 0 \quad (5)$$

$$m_4(bm_4 - 1) = 0 \quad (6)$$

$$am_1 m_2 + m_4(cm_1 + bm_2) = m_2 \quad (7)$$

Thus, from equations (5), (6) and (7), we obtain the following cases:
Case 1: If $m_1 = m_4 = 0$, then we have from eq (7) that $m_2 = 0$. Thus T^D is the zero matrix.
Case 2: If $m_1 = 0$, $m_4 = \frac{1}{b}$, then we have from eq (4) that $m_2 = -c/((a-b)b)$. Thus $T^D = \begin{pmatrix} 0 & -c/((a-b)b) \\ 0 & 1/b \end{pmatrix}$.
Case 3: If $m_1 = 1/a$, $m_4 = 0$, then we have from eq (4) that $m_2 = c/((a-b)a)$. Thus $T^D = \begin{pmatrix} 1/a & c/((a-b)a) \\ 0 & 0 \end{pmatrix}$.
Case 4: If $m_1 = 1/a$, $m_4 = 1/b$, then we have from eq (4) that $m_2 = -c/(ab)$. Thus $T^D = \begin{pmatrix} 1/a & -c/(ab) \\ 0 & 1/b \end{pmatrix}$
Remark 2.10:
Note that, from theorem (2.9), the case (1) satisfies when T is nilpotent matrix ($T^D = 0$) and case (4) satisfies when T is invertible ($T^D = T^{-1}$).
Corollary 2.11:
Let $T = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}$, where a, b, c are non-zero complex numbers such that $a \neq b$. If $T^D \neq 0$, then T is not D-operator.
Proof:
We discuss case (2) in theorem (2.8) and the other cases can be proved similarly. Note that $T = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}$, $T^* = \begin{pmatrix} a & 0 \\ c & b \end{pmatrix}$ and $T^D = \begin{pmatrix} 0 & -c/((a-b)b) \\ 0 & 1/b \end{pmatrix}$.
Hence
$T^{*2} = \begin{pmatrix} a^2 & 0 \\ c(a+b) & b^2 \end{pmatrix}$ and $(T^D)^2 = \begin{pmatrix} 0 & -c/((a-b)b^2) \\ 0 & 1/b^2 \end{pmatrix}$.
Thus

$$T^{*2}(T^D)^2 = \begin{pmatrix} 0 & -ca^2/((a-b)b^2) \\ 0 & (-c^2(a+b)/((a-b)b^2)) + 1 \end{pmatrix} \quad (8)$$

On the other hand,

$$(T^* T^D)^2 = \begin{pmatrix} -c^2 a/((a-b)b^2) & -ca/((a-b)b) \\ c/b & (-c^2 a/((a-b)b^2)) + 1 \end{pmatrix} \quad (9)$$

Assume that T is D-operator, then $T^{*2}(T^D)^2 = (T^* T^D)^2$. Therefore, from eqs (8) and (9), we obtain that c/b=0. Since $b \neq 0$, then $c = 0$, which is a contradiction. Hence M cannot be D-operator

Conclusions

The present paper discusses some elementary properties of a new class of operators, namely the D-operators. The D-operators is some generalization of normal operators. Some properties of normal operator may not be satisfied in D-operators, such as the property of the sum and the product of two D-operators, which we proved that it is not necessarily true.

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