Some Properties of D-Operator on Hilbert Space

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Abstract

In this paper, we introduce a new type of Drazin invertible operator on Hilbert spaces, which is called D-operator. Then, some properties of the class of D-operators are studied. We prove that the D-operator preserves the scalar product, the unitary equivalent property, the product and sum of two D-operators are not D-operator in general but the direct product and tensor product is also D-operator.

Keywords: Hilbert space, Drazin operator, Normal operator, D-normal operator, n-normal operator, class (Q) operators.

1. Introduction

Throughout this paper, $H$ is a Hilbert space, $B(H)$ is the space of all bounded linear operators on a complex Hilbert space $H$. The Drazin inverse for a bounded linear operator on a complex Banach space was introduced by Caradus [1]. Let $T \in B(H)$, the Drazin inverse of $T$, if it exists, is an operator $T^D \in B(H)$ such that $TT^D = T^{D}T = T^{D}$ and $k + 1 T^{D} = T^{k}$ for some integer number $k \geq 0$, the smallest integer $k \geq 0$ is called the index of $T$, which is denoted by $\text{ind}(T)$. It is easy to see that $\text{ind}(T) = 0$ if and only if $T$ is an invertible operator. Then $T^{D} = T^{-1}$.

In the following lemma, we collect some properties of Drazin operator which appeared in previous studies [2, 3].

Lemma 1.1: Let $S, T \in B(H)$ be two Drazin invertible operators, then

(a) \((T^{\ast})^{D} = (T^{D})^{\ast}\).
(b) \((T^{\ell})^{D} = (T^{D})^{\ell}\) for $\ell = 1, 2, \ldots$
(c) \((S^{-1}T)^{D} = S^{-1}T^{D}S\).
(d) If $ST = TS$, then $S^{D}T^{D} = T^{D}S^{D}$, $S^{D}T = T^{D}S$, and $T^{D}S = ST^{D}$.
(e) If $ST = TS = 0$, then $(S + T)^{D} = S^{D} + T^{D}$.

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Let \( T \in B(H) \), \( T \) is called normal if \( TT^* = T^*T \). The purpose of this paper is to introduce a new operator to generalize the normal operator. Many authors presented generalizations of normal operators. For examples, in an earlier work [4], the author introduced a class \((Q)\) of operators acting on a Hilbert space \( H \): for any \( T \in (Q) \), \( T^{\bigotimes 2}(T)^2 = (T^*T)^2 \). Then, in another article [5], the authors introduced some new classes of operators associated with Drazin invertible operator. In this section, we give a new type of operators that are associated with Drazin invertible operator, that we call D-operator.

The paper contains two sections. In section one, we investigate some basic properties that we need. In section two, we study most of the properties of D-operators.

2. Main Results

Definition 2.1:
Let \( T \in B(H) \) be Drazin invertible. \( T \) is called D-operator if 
\[ T^{\bigotimes 2}(T)^2 = (T^*T^D)^2. \]

The class of all D-operators is denoted by \([D]\). By lemma (1.1) (d), it is easy to prove that every normal operator is D-operator but the converse is not true in general. For example, let \( T \) be a nilpotent operator, then \( T^D = 0 \), hence it is clear that \( T \) is D-operator but the nilpotent is not necessary normal.

In this section, we investigate some basic properties of operators in \([D]\).

Proposition 2.2:
Let \( T \in [D] \), then the following assumptions hold:
1) \( aT \in [D] \) for every scalar \( a \).
2) \( S \in [D] \) for every \( S \in B(H) \) that is unitary equivalent to \( T \).
3) The restriction \( T|M \) of \( T \) to any closed subspace \( M \) of \( H \) that reduces \( T \) in \([D]\).
4) \( T^D \in [D] \).

Proof:
1) The proof is straightforward.
2) Since \( S \) is unitary equivalent to \( T \), then \( S = UTU^* \), where \( U \) is unitary operator. Thus,
\[
S^{\bigotimes 2}(S)^2 = (UT^U)^2(UT^DU)^2 = (UT^U)(UT^DU^*)(UT^DU^*) = (UU^* = I)
\]
\[ U^TU^DT^DU^* = U^TU^DT^DU^* \]
\[ U^TU^DT^DU^* = (UT^U)(UT^DU^*)(UT^DU^*) = S^3S^2S^2 \]
\[ = (S^3S^2)^2. \]

Hence, \( S \in [D] \).
3) \( (T|M)^{(T|M)D} = (T|M)^{(T|M)^D} = (T|M)(T^D|M)(T^D|M) \)
\[ = (T^*T)(T^D|M)(T^D|M) \]
\[ = (T^{\bigotimes 2}|M)(T^D)^2|M \]
\[ = ((T^*T)^D)^2|M \]
\[ = ((T^*T)(T^*T)^D)|M \]
\[ = ((T^*T)(M)((T^*T)^D)|M \]
\[ = ((T^*T)(M)(T^D|M)(T^D|M)) \]
\[ = ((T^{\bigotimes 2}|M)(T^D|M)^2 = ((T|M)^{T|M)^D}. \]

Hence \( T|M \in [D] \).
4) Since \( T \in [D] \), then 
\[ T^{\bigotimes 2}(T^D)^2 = (T^*T^D)^2. \]
Thus \( T^*T^D = T^D(T^D) \)

By taking the adjoint of both sides of the above equation, we have
\[ (T^D)^2(T^D)^2 = ((T^D)^2)^2. \]

Hence,
\[ (T^D)^2(T^D)^2 = ((T^D)^2)^2. \]
Therefore, \((T^D)^2((T^D)^2 = ((T^D)^2(T^D)^2)\).

Thus, \(T^D \in [D]\)

**Proposition 2.3:**

The set of all D-operators on \(H\) is a closed subset of \(B(H)\).

**Proof:**

Let \(\{T_k\}\) be a sequence of D-operators such that \(T_k \to T\). It is enough to show that \(T\) is D-operator.
Since \(T_k \to T\) then \(T_k^* \to T^*\) and \((T_k^D) \to T^D\). Hence, \(T_k^* T_k^D \to T^* T^D\), then we get that
\[
\left(T_k^* T_k^D\right)^2 \to (T^* T^D)^2
\]
(1)

On the other hand, we obtain that \(T_k^* T_k \to T^* T\) and \((T_k^D) \to (T^D)^2\). Hence, \(T_k^* (T_k^D) \to T^* (T^D)^2\)
(2)

Therefore, from equations (1) and (2), we conclude that
\[
\|T^* (T^D)^2 - (T^* T^D)^2\| = \|T^* (T^D)^2 - T_k^* (T_k^D)^2 + T_k^* (T_k^D)^2 - (T^* T^D)^2\|
\leq \|T^* (T^D)^2 - T_k^* (T_k^D)^2\| + \|T_k^* (T_k^D)^2 - (T^* T^D)^2\|
= \|T^* (T^D)^2 - T_k^* (T_k^D)^2\| + \|T_k^* (T_k^D)^2 - (T^* T^D)^2\|
\to 0 \quad \text{as} \quad k \to \infty.
\]

Hence, \(T^* (T^D)^2 = (T^* T^D)^2\). Thus \(T \in [D]\)

**Proposition 2.4:**

Let \(S, T \in [D]\). If \([T, S] = [T, S^*] = 0\), then \(ST \in [D]\).

**Proof:**

Since \([T, S] = [T, S^*] = 0\), then by lemma(1.1) (d) we have \([T, S^D] = [T^D, S] = [T^D, S^*] = [T^*, S^D] = 0\).

Moreover, since \(S, T \in [D]\), then
\[
(T^* (T^D)^2 = (T^* T^D)^2 \quad \text{and} \quad S^* (S^D)^2 = (S^* S^D)^2.
\]

Therefore,
\[
(ST)^2((ST)^D)^2 = (ST)^* (ST)^D (ST)^D
= T^* S^* T^* S^D T^D S^D T^D S^D.
= T^* T^D S^D S^* S^D T^D S^D.
= T^* S^* T^* S^D T^D S^D.
= T^* S^* T^* S^D T^D S^D.
= (ST)^* (ST)^D (ST)^D.
= ((ST)^D)^2.
\]

Thus \(ST \in [D]\)

**Proposition 2.5:**

Let \(S, T \in [D]\). If \(ST = TS = 0\), then \(S + T \in [D]\).

**Proof:**

Since \(S, T \in [D]\), then \((S + T)^2 = (T^* T^D)^2 \quad \text{and} \quad S^* (S^D)^2 = (S^* S^D)^2\). Since \(ST = TS = 0\), then \(S^* T^* = T^* S^* = 0\), and by lemma (1.1)(e) we have \((S + T)^D = S^D + T^D\). Hence
\[
(S + T)^2((S + T)^D)^2 = (S + T)^* (S + T)^D S^D + T^D ((S + T)^D)^2
= (S + T)^* (S^D + T^D)^2
= (S^* S^D)^2 + (T^* T^D)^2
= (S + T)^* (S^* S^D + T^* T^D)
= (S + T)^* (S^D + T^D)^2
= (S + T)^* (S + T)^D
\]

Thus \(S + T \in [D]\)

The following example shows that the propositions (2.4) and (2.5) are not necessarily true in general.

**Example 2.6:**
1) Let \( S = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \) and \( T = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \).

Therefore, \( S^D = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \) and \( T^D = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \).

It can be easily checked that \( S, T \in [D] \) and \( ST \not\in TS \). Note that, \((ST)^D = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \).

But, it is easy to compute that \( ST \not\in [D] \).

2) Let \( T = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} \). Clearly \( T \in [D] \), but \( T + I \not\in [D] \).

The following corollary is a straightforward result from proportion (2.4).

**Corollary 2.7:**

If \( T \in [D] \), then \( T^n \in [D] \) for all positive integers \( n \).

**Theorem 2.8:**

Let \( T_1, T_2, ..., T_n \in [D] \), then

1) \( T_1 \oplus T_2 \oplus ... \oplus T_n \in [D] \).

2) \( T_1 \oplus T_2 \oplus ... \oplus T_n \in [D] \).

**Proof:**

1) Since \( T_i \in [D] \) \( \forall i = 1, 2, ..., n \), then

\[
T_i^2 (T_i^D)^2 = (T_i^D)^2 \]

Hence,

\[
(T_1 \oplus T_2 \oplus ... \oplus T_n)^2 \]

\[
= (T_1^2 \oplus T_2^2 \oplus ... \oplus T_n^2)^2 \]

\[
= (T_1^2 \oplus T_2^2 \oplus ... \oplus T_n^2)^2 \oplus (T_2^D)^2 \oplus ... \oplus (T_n^D)^2 \]

\[
= (T_1^2 \oplus T_2^2 \oplus ... \oplus T_n^2)^2 \oplus (T_2^D)^2 \oplus ... \oplus (T_n^D)^2 \]

\[
= T_1 T_2^D T_1 T_2 \oplus T_2 T_2^D T_2 T_1 \oplus ... \oplus T_n T_n^D T_n T_n \]

\[
= (T_1^D T_1^D \oplus T_2^D T_2^D \oplus ... \oplus T_n^D T_n^D) \]

\[
= (T_1^D T_1^D \oplus T_2^D T_2^D \oplus ... \oplus T_n^D T_n^D)^2 \]

2) Let \( x_1, x_2, ..., x_n \in H \), then

\[
T_1^2 (T_1^D)^2 (x_1 \oplus x_2 \oplus ... \oplus x_n) \]

\[
= (T_1^2 \oplus T_2^2 \oplus ... \oplus T_n^2)^2 (x_1 \oplus x_2 \oplus ... \oplus x_n) \]

\[
= (T_1^2 \oplus T_2^2 \oplus ... \oplus T_n^2)^2 (x_1 \oplus x_2 \oplus ... \oplus x_n) \]

\[
= (T_1^2 T_2^D)^2 (x_1 \oplus x_2 \oplus ... \oplus x_n) \]

\[
= (T_1^D T_1^D) (x_1 \oplus x_2 \oplus ... \oplus x_n) \]

In the following theorem, we compute the Drazin invertible operator for some special matrix.

**Theorem 2.9:**

Let \( T = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \), where \( a, b, c \) are non-zero complex numbers such that \( a \neq b \), then one of the following forms of Drazin invertible can be satisfied:

1) \( T^D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \).

2) \( T^D = \begin{pmatrix} 0 & -c/((a - b)b) \\ 0 & 1/b \end{pmatrix} \).

3) \( T^D = \begin{pmatrix} 1/a & c/((a - b)a) \\ 0 & 0 \end{pmatrix} \).

4) \( T^D = \begin{pmatrix} 1/a & -c/(ab) \\ 0 & 1/b \end{pmatrix} \).
Proof:
Let $T^D = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}$, where $m_1, m_2, m_3, m_4 \in \mathbb{C}$, then
\[ TT^D = T^D T \]
\[ T^D TT^D = T^D \]  
(1)
Thus, from eq (1), it is easy to conclude that
\[ \begin{pmatrix} am_1 + cm_3 \\ bm_3 \\ am_2 + cm_4 \\ bm_4 \end{pmatrix} = \begin{pmatrix} am_1 & cm_1 + bm_2 \\ am_3 & cm_3 + bm_4 \end{pmatrix}. \]
Therefore we get that
\[ am_1 + cm_3 = am_1 \]
\[ am_2 + cm_4 = cm_1 + bm_2 \]
(3)
(4)
From (3), we get $m_3 = 0$.
This implies that, from eq (2), the following matrix equation:
\[ \begin{pmatrix} am_1^2 & am_1 m_2 + m_4 (cm_1 + bm_2) \\ 0 & bm_4^2 \end{pmatrix} = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}. \]
Thus we get the following equations:
\[ m_1 (am_1 - 1) = 0 \]
\[ m_4 (bm_4 - 1) = 0 \]
\[ am_1 m_2 + m_4 (cm_1 + bm_2) = m_2 \]
(5)
(6)
(7)
Thus, from equations (5), (6) and (7), we obtain the following cases:
Case 1: If $m_1 = m_4 = 0$, then we have from eq (7) that $m_2 = 0$. Thus $T^D$ is the zero matrix.
Case 2: If $m_1 = 0$, $m_4 = \frac{1}{b}$, then we have from eq (4) that $m_2 = -c/((a-b)b)$. Thus $T^D = \begin{pmatrix} 0 & -c/((a-b)b) \\ 0 & 1/b \end{pmatrix}$.
Case 3: If $m_1 = 1/a$, $m_4 = 0$, then we have from eq (4) that $m_2 = c/((a-b)a)$. Thus $T^D = \begin{pmatrix} 1/a & c/((a-b)a) \\ 0 & 0 \end{pmatrix}$.
Case 4: If $m_1 = 1/a$, $m_4 = 1/b$, then we have from eq (4) that $m_2 = -c/(ab)$. Thus $T^D = \begin{pmatrix} 1/a & -c/(ab) \\ 0 & 1/b \end{pmatrix}$
Remark 2.10:
Note that, from theorem (2.9), the case (1) satisfies when $T$ is nilpotent matrix ($T^D = 0$) and case (4) satisfies when $T$ is invertible ($T^D = T^{-1}$).
Corollary 2.11:
Let $T = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}$, where $a, b, c$ are non-zero complex numbers such that $a \neq b$. If $T^D \neq 0$, then $T$ is not D-operator.
Proof:
We discuss case (2) in theorem (2.8) and the other cases can be proved similarly. Note that
\[ T = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}, T^* = \begin{pmatrix} a & 0 \\ c & b \end{pmatrix} \text{ and } T^D = \begin{pmatrix} 0 & -c/((a-b)b) \\ 0 & 1/b \end{pmatrix}. \]
Hence
\[ T^{*2} = \begin{pmatrix} a^2 & 0 \\ c(a+b) & b^2 \end{pmatrix} \text{ and } (T^D)^2 = \begin{pmatrix} 0 & -c/((a-b)b^2) \\ 0 & 1/b^2 \end{pmatrix}. \]
Thus
\[ T^{*2}(T^D)^2 = \begin{pmatrix} 0 & -ca^2/((a-b)b^2) \\ 0 & (-ca^2(a+b)/((a-b)b^2)) + 1 \end{pmatrix}. \]  
(8)
On the other hand,
\[ (T^{*}T^D)^2 = \begin{pmatrix} -c^2a/((a-b)b^2) & -ca/((a-b)b) \\ c/b & (-c^2a/((a-b)b^2)) + 1 \end{pmatrix}. \]  
(9)
Assume that $T$ is D-operator, then $T^{*2}(T^D)^2 = (T^{*}T^D)^2$. Therefore, from eqs (8) and (9), we obtain that $c/b=0$. Since $b \neq 0$, then $c = 0$, which is a contradiction. Hence $M$ cannot be D-operator.
Conclusions
The present paper discusses some elementary properties of a new class of operators, namely the D-operators. The D-operators is some generalization of normal operators. Some properties of normal operator may not be satisfied in D-operators, such as the property of the sum and the product of two D-operators, which we proved that it is not necessarily true.

References