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An Approximate Solution of the Space Fractional-Order Heat Equation by the Non-Polynomial Spline Functions

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Abstract
The linear non-polynomial spline is used here to solve the fractional partial differential equation (FPDE). The fractional derivatives are described in the Caputo sense. The tensor products are given for extending the one-dimensional linear non-polynomial spline S_1 to a two-dimensional spline $S_1 \otimes S_2$ to solve the heat equation. In this paper, the convergence theorem of the method used to the exact solution is proved and the numerical examples show the validity of the method. All computations are implemented by Mathcad15.

Keywords: Caputo derivative, non-polynomial spline, tensor product, fractional heat equation.

حل تقريبي لمعادلة الحرارة بترتيب كسور الفضاء بواسطة دوال الثلمة اللاخطية

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الخلاصة
الهدف من هذا البحث هو حل المعادلات التفاضلية الجزئية الكسرية والمشتقات الكسرية باستخدام Caputo . وتم استخدام منتجات الموتر (Tensor product) لتوسيع النظام من نظام احادي البعد الى نظام ثنائي البعد لحل معادلة الحرارة .وايضا تم في هذا البحث اثبات نظرية التقارب للطريقة مع الحل المضبوط والامثلة العددية توضح صحة الطريقة .وتم تنفيذ جميع الحسابات بواسطة برنامج (Mathcad15).

1- Introduction

Fractional calculus is the study of fractional order derivatives and integrals. It gained extensive attention form the researchers in the last few decades. It has exceptional applications in diverse fields of science and engineering. Spline functions are piecewise polynomials of degree n that are joined together at the break points with $n-1$ continuous derivatives. A piecewise non-polynomial spline function is a blend of trigonometric, as well as polynomial basis functions, which form a complete extension. In [1], Saeah solved linear Volterra integral equations using non-polynomial spline function. In [2], Batool used the variational iteration method to solve partial integro differential equations of fractional order. In [3], Ghulam *et* al. studied the application of Caputo K- fractional derivatives. In [4], a method based the fractional shifted Legendre polynomials was applied to solve non-homogeneous space and time fractional partial differential equations (FPDEs), in which space and time fractional derivatives are described in the Caputo sense. In [5], some applications of the non-polynomial spline approach to the solution of the Burgers' equation were studied. In [6], the non-

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polynomial spline methods were used for the solution of a system of obstacle problems. The main objective of the present paper lies on introducing a new approximate solution of time-space fractional heat equations by using the linear non-polynomial spline method.

2- Tensor Product of Two-dimensional Problems

The treatment of high-dimensional problems, such as heat equation, can be approached by concepts of tensor product approximation.

Let $\mathbb{R}$ be a regain such that $\mathbb{R} = \{(x,y) \,|\, a \le x \le b, c \le y \le d\}$.

The method of finding two dimensional functions $g(x,y)$ is applied in a tensor product space $S_1 \otimes S_2$, such that S_1 and S_2 are two spline spaces, namely $s_1 = \text{span}\{\Phi_1, \Phi_2, \dots, \Phi_p\}$ and $s_2 = span\{\Psi_1, \Psi_2, \dots, \Psi_q\}$, given in

$g(x,y) \in (S_1 \otimes S_2)$, where

$$g(x,y) = \sum_{p=1}^{m_1} \sum_{q=1}^{m_2} C_{p,q} \Psi_q(y) \Phi_p(x) \quad \dots \quad (\mathbf{1})$$

where Ψ_q and Φ_p may be considered as the basis for the generalized spline functions, so that m_1, m_2 are integer numbers.

The interpolation condition is

$g(x_i, y_j) = f_{ij} \quad , i=0,1,2,\dots,m_1 \; , j=0,1,2,\dots,m_2$

hence

$\sum_{p=1}^{m_1} \sum_{q=1}^{m_2} C_{p,q} \Psi_q(y_j) \Phi_p(x_i) = f_{i,j} \; , i = 1,2,\dots,m_1 \; j = 1,2,\dots,m_2$

In the matrix form,

we solve the system $A\,C = F$ ... $(\mathbf{2})$ We find the coefficients $c_{p,q}$ which are unknown of the function g, which is given in eq(1), where $A = \Psi \otimes \Phi$ is tensor product of two matrices Ψ and Φ.

$$\Psi = \begin{bmatrix} a_{1,1} & \cdots & a_{1,m_1} \\ \vdots & \ddots & \vdots \\ a_{m_2,1} & \cdots & a_{m_2,m_1} \end{bmatrix}, \quad \Phi = \begin{bmatrix} b_{1,1} & \cdots & b_{1,m_1} \\ \vdots & \ddots & \vdots \\ b_{m_2,1} & \cdots & b_{m_2,m_1} \end{bmatrix}$$

$$A = \Psi \otimes \Phi = \begin{bmatrix} a_{1,1}\Phi & \cdots & a_{1,m_1}\Phi \\ \vdots & \ddots & \vdots \\ a_{m_2,1}\Phi & \cdots & a_{m_2,m_1}\Phi \end{bmatrix}$$

$$C = [c_{1,1} \; \dots \; c_{m_1,1} \; c_{1,2} \; \dots \; c_{m_{1,},2} \; \dots \; c_{m_1,m_2}]^T$$

$$F = [f_{1,1} \; \dots \; f_{m_1,1} \; f_{1,2} \; \dots \; f_{m_{1,},2} \; \dots \; f_{m_1,m_2}]^T$$

3- Non- Polynomial Spline Function for Caputo derivative

Definition (3.1) [7]

The Caputo fractional derivative of the fractional order $0 < \alpha < 1$ is defined as

$$D_x^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-t)^{-\alpha} \frac{d\,f(t)}{dt} dt$$

Now consider the partition $\Delta = \{t_0, t_1, t_2, \dots, t_n\}$ of $[a,b] \subset R$. Let $S(\Delta)$ denotes the set of piecewise polynomials on subinterval $I_i = [t_i, t_{i+1}]$ of partition Δ. Let $u(t)$ be the exact solution. This new method provides an approximation, not only for $u(t_i)$ at the knots, but also for $u^{(n)}(t_i)$,$n =$ 1,2, ..., at every point in the range of integration . Also, C^∞ of the differentiability of the trigonometric part of non-polynomial splines compensates for the loss of smoothness inherited by the polynomial [1] . The non-polynomial spline function is obtained by the segment Pi(t). Each non- polynomial spline of the n order Pi (t) has the form

$$p_i(t) = a_i \cos k(t-t_i) + b_i\, sink(t-t_i) + \dots + y_i(t-t_i)^{n-1} + z_i$$

where $a_i, b_i, \dots, y_i$ and z_i are constants and k is the frequency of the trigonometric functions which will be used to raise the accuracy of the method.

In this paper, the linear non-polynomial spline function will be used for solving fractional partial differential equations.

Definition (3.1) [1]: Linear Non-Polynomial Spline Function

The form of the linear non-polynomial spline function is

$$p_i(t) = a_i \cos k(t-t_i) + b_i\, sink(t-t_i) + c_i(t-t_i) + d_i$$

where a_i, b_i, c_i and d_i are constants to be determined.

Definition (3.2) [8]
Let $H_1$and H_2 be two Hilbert spaces. If $H_1$and H_2 have the orthonormal bases $\{\Phi_k\}$ and $\{\Psi_k\}$, respectively , then $\{\Phi_k \otimes \Psi_k\}$ is an orthonormal basis for $H_1 \otimes H_2$. In particular , the Hilbert dimension of the tensor product is the product of the Hilbert dimensions.
Definition (3.3) [9]
A set of functions $\{\Phi_1(x), \Phi_2(x), \dots\}$ is an orthogonal set of functions on the interval $[a,b]$ if any two functions in the set are orthogonal to each other, so that
$(\Phi_n, \Phi_m) = \int_a^b \Phi_n(x)\Phi_m(x)\,dx = 0 \qquad n \neq m$

In the following theorem, the convergence of the method to the exact solution using a tensor product is given.
Theorem (3.4)
Non-polynomial solutions of the two variables x,t given in equation (1) converge to the exact solution $u(x,t)$.
Proof: The set of basis of linear non-polynomial spline functions $\{\cos(x), \sin(x), x, 1\}$ is orthogonal basis, then $S_1 = span\{\cos(t), \sin(t), t, 1\}$ and $S_2 = span\{\cos(x), \sin(x), x, 1\}$
$S_1 \otimes S_2 = C_{rm} = \langle u(x,t), \Phi_r(x)\Psi_m(t)\rangle$
$r = 1,2,3,4$, $m = 1,2,3,4$, where $<.>$ represents an inner product and $\Phi_r(x)\Psi_m(t)$ forms the orthonormal basis.
Let $\Phi_r(x)\Psi_m(t) = \alpha(x,t)$ and
define $S_n$to be the partial sum of $\gamma_j\alpha(x,t)$,
where $\gamma_j = \langle u(x,t),\alpha(x,t)\rangle$, j=1,2, . . . ,n
i.e. $S_n = \sum_{j=1}^{n}\gamma_j\alpha(x,t)$,
T. p (S_n is Cauchy sequence in Hilbert space).
Let S_m be the arbitrary partial sum with $n \geq m$
$$\langle u(x,t), S_n\rangle = \langle u(x,t), \sum\gamma_j\,\alpha(x,t)\rangle$$
$$= \sum_{j=1}^{n}\gamma_j\langle u(x,t),\alpha(x,t)\rangle$$
$$= \sum_{j=1}^{n}|\gamma_j|^2$$
$$\|S_n - S_m\|^2 = \left\|\sum_{j=m1}^{n}\gamma_j\alpha(x,t)\right\|^2$$
$$= \left\langle\sum_{j=m+1}^{n}\gamma_j\,\alpha(x,t), \sum_{j=m+1}^{n}\gamma_j\,\alpha(x,t)\right\rangle$$
$$= \sum_{j=m+1}^{n}|\gamma_j|^2 \;, for\; n > m$$
Hence, $\left\|\sum_{j=m+1}^{n}\gamma_j\,\alpha(x,t)\right\|^2 \longrightarrow 0$ as $n,m \to \infty$
and $\{S_n\}$ is a Cauchy sequence and it converges to s.
To prove that $u(x,t) = s$
$$\langle s - u(x,t),\alpha(x,t)\rangle = \langle s,\alpha(x,t)\rangle - \langle u(x,t),\alpha(x,t)\rangle$$
$$= \langle \lim_{n\to\infty} S_n ,\alpha(x,t)\rangle - \gamma_j$$
$$= \gamma_j - \gamma_j = 0$$
Hence, $s = u(x,t)$ and $\sum_{j=1}^{n}\gamma_j\,\alpha(x,t)$ converges to $u(x,t)$, which completes the proof .

4- Solution of Fractional Heat Equation Using Tensor Product of Linear Non-Polynomial Spline

The non-polynomial spline method will be used here to approximate the solution of the failure probability density function (FPDF):
$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,t) = f(x,t,u_x,u_{xx}),\; 0 \leq x \leq 1, 0 \leq \alpha \leq 1, t > 0 \qquad \dots(\mathbf{3})$$
with boundary condition: $u(0,t) = u(1,t) = 0$, $t > 0$ …(**4**)
and initial condition: $u(x,0) = g(x), 0 \leq x \leq 1$ …(**5**)
where $g(x)$ is a given function.
By equation (2), the function $g(x,t)$ is replaced by $z(x,t)$, which is used to approximate the solution of the FPDE. Hence,
$$z(x,t) = \sum_{i=0}^{m_1}\sum_{j=0}^{m_2} C_{i,j}\Psi_j(t)\Phi_i(x) \qquad \dots(\mathbf{6})$$
where $\Phi_i(x)$,$i = 0,1,2,\dots,m_1$ are the basis of the linear non-polynomial spline function and $\Psi(t)$,$j = 0,1,2,\dots,m_2$ are the basis of the linear non-polynomial spline function.
From the initial condition given by equation (5), one my get:

$$\sum_{i=0}^{m_1}\sum_{j=0}^{m_2} C_{i,j}\Psi_j(0)\Phi_i(x) = g(x) \quad \dots (7)$$

We substitute the knot points for the x-axis to get an equation for each knot point, and form the boundary condition given in equation (6), we have

$$\sum_{i=0}^{m_1}\sum_{j=0}^{m_2} C_{i,j}\Psi_j(t)\Phi_i(0) = \mu_1(t) \quad \dots (8)$$

and $\sum_{i=0}^{m_1}\sum_{j=0}^{m_2} C_{i,j}\Psi_j(t)\Phi_i(1) = \mu_2(t)$ ... (9)

Similarly , we substitute the knot points for the t-axis to get an equation for each knot point at $x = 0$ and $x = 1$. Form the boundary condition given in equation (4), we have

$$\sum_{i=0}^{m_1}\sum_{j=0}^{m_2} C_{i,j}\left(\frac{\partial^\alpha}{\partial t^\alpha}\Psi(t)\right)\Phi_i(x) = f\left(x, t, \sum_{i=0}^{m_1}\sum_{j=0}^{m_2} C_{i,j}\Psi_j(t)\Phi'_i(x), \sum_{i=0}^{m_1}\sum_{j=0}^{m_2} C_{i,j}\Psi_j(t)\Phi''_i(x)\right) \quad \dots (\mathbf{10})$$

We substitute the mesh points (x_i, t_j),for $i = 0,1,2,\dots,m_1$, $j = 0,1,2,\dots,m_2$, to get an equation for each pair $(i\,,j)$, for all $i = 0,1,2,\dots,m_1$, $j = 0,1,2,\dots,m_2$. Then, from equations (**7-10**), a system with unknown coefficients $C_{i,j}$ must be determined to compute equation (**6**).

5- Illustrative Example

To demonstrate the effectiveness of the proposed method, we consider here two test examples of one dimensional fractional heat equation problem. The software MathCad 15 is used to get the numerical results.

Example(1.1): Consider the homogeneous one-dimensional fractional heat equation

$\frac{\partial^\alpha}{\partial t^\alpha} z(x,t) = \frac{1}{2}x^2 \frac{d^2}{dx^2} z(x,t)$, $0 \le x \le 1, 0 \le \alpha \le 1, t > 0$

subject to the boundary condition : $z(0,t) = 0$, $z(1,t) = e^t$

and the initial condition : $z(x,0) = x^2$.

The exact solution for $\alpha = 0.9$ is given by: $\boldsymbol{u(x,t) = x^2 e^t}$.

Let $\Delta_1$be a partition for the x-axis, such that: $\Delta_1: 0 = x_0 < x_1 < x_2 < x_3 = 1$, then $x_0 = 0, x_1 = \frac{1}{3}, x_2 = \frac{2}{3}, x_3 = 1$, the mesh points for the x-axis. Let Δ_2 be a partition for the t-axis such that: $\Delta_2: 0 = t_0 < t_1 < t_2 < t_3 = 0.03$,

then $t_0 = 0, t_1 = 0.01, t_2 = 0.02, t_3 = 0.03$, the mesh points for t-axis.

By solving the linear system 16 unknown coefficients, we get

$c_{00} = -4.095$, $c_{01} = -2.328$, $c_{02} = -0.418$, $c_{03} = 2.298$,

$c_{10} = 13.641, c_{11} = 7.887 \times 10^{-3}, c_{12} = 1.409, c_{13} = -14.623$,

$c_{20} = -14.641, c_{21} = 2.173 \times 10^{-6}, c_{22} = -1.454, c_{23} = 15.382$,

$c_{30} = 4.095, c_{31} = 2.328 , c_{32} = 0.418, c_{33} = -2.298$

Then, the approximate solution is: $z(x,t) = (-4.095\cos t - 2.328\sin t - 0.418\,t + 2.298)\cos x + (13.641\cos t + 7.887 \times 10^{-3}\sin t + 1.409\,t - 14.623)\sin x + (-14.382\cos t + 2.173 \times 10^{-6}\sin t - 1.454\,t + 15.382)x + (4.095\cos t + 2.328\sin t + 0.418\,t - 2.298)$

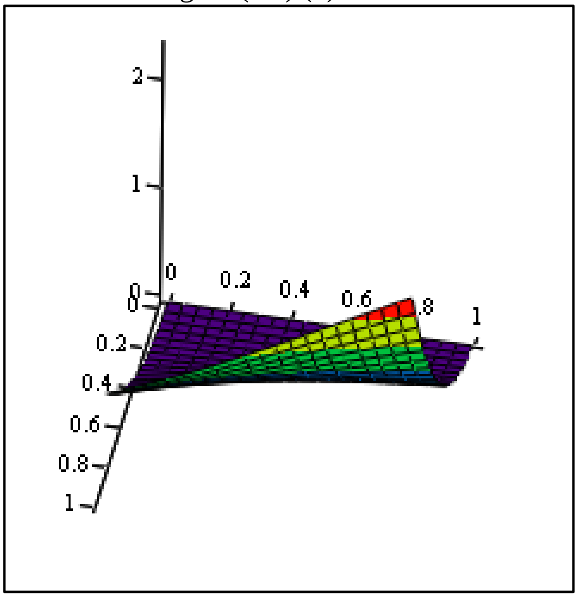
Table (1.1) illustrates the absolute error between the exact solution and the non-polynomial spline approximate solution.

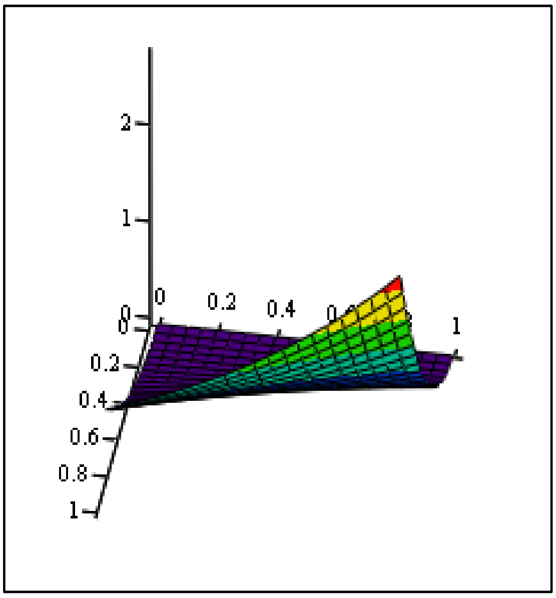
Table (1.1)-The exact and approximate solutions of example (1.1) when $\alpha = 0.9$.

x	t	$u(x,t)$	$z(x,t)$	$\lvert u(x,t) - z(x,t)\rvert$
0	0	0	0	0
	0.01	0	0	0
	0.02	0	0	0
	0.03	0	0	0
$\frac{1}{3}$	0	0.111	0.111	1.706×10^{-4}
	0.01	0.112	0.112	1.876×10^{-5}
	0.02	0.113	0.114	2.073×10^{-4}

	0.03	0.114	0.115	3.948×10^{-4}
$\frac{2}{3}$	0	0.444	0.444	2.563×10^{-4}
	0.01	0.449	0.449	2.385×10^{-4}
	0.02	0.453	0.454	7.1155×10^{-4}
	0.03	0.458	0.459	1.174×10^{-3}
1	0	1	1	2.478×10^{-4}
	0.01	1.01	1.01	2.41×10^{-4}
	0.02	1.02	1.02	2.343×10^{-4}
	0.03	1.03	1.03	2.297×10^{-4}

The approximation solution $z(x,t)$ is illustrated in Figure- (1.1) (a), while the exact solution is illustrated in Figure (1.1) (b).





z u

(a) **(b)**

Figure (1.1)- (a) the approximate surface $z(x,t)$ and (b) the exact surface $u(x,t)$ for example (1.1).

Example(1.2): Consider the nonhomogeneous one-dimensional fractional heat equation

$$\frac{\partial}{\partial t}u(x,t) = \frac{\partial^{1-\alpha_1}}{\partial t^{1-\alpha_1}}\frac{\partial^2}{\partial x^2}u(x,t) - \frac{\partial^{1-\alpha_2}}{\partial t^{1-\alpha_2}}u(x,t) + f(x,t)$$

$0 < x < 1, t > 0, 0 \leq \alpha \leq 1$

with boundary and initial conditions, respectively:

$u(0,t) = u(1,t) = 0, \quad 0 \leq \ t \leq 1$

$u(x,0) = 0, \quad 0 \leq x \leq 1$

where $f(x,t) = \left(2t + \frac{2\pi^2 t^{\alpha_1+1}}{\Gamma(2+\alpha_1)} + \frac{2\, t^{\alpha_2+1}}{\Gamma(2+\alpha_2)}\right)\sin(\pi x)$.

The exact solution [10] for $\alpha_1 = \alpha_2 = 0.5$

is given by : $\boldsymbol{u(x,t) = t^2 \sin(\pi x)}$.

Let Δ_1 be a partition for the x-axis, such that: $\Delta_1: 0 = x_0 < x_1 < x_2 < x_3 = 1$, then $x_0 = 0, x_1 = \frac{1}{3}, x_2 = \frac{2}{3}, x_3 = 1$, the mesh points for the x-axis. Let Δ_2 be a partition for the t-axis such that $\Delta_2: 0 = t_0 < t_1 < t_2 < t_3 = 0.03$,

then $t_0 = 0, t_1 = 0.01, t_2 = 0.02, t_3 = 0.03$, the mesh points for t-axis.

By solving the linear system 16 unknown coefficients, we get

$c_{00} = -4.23,\ c_{01} = 3.814,\ c_{02} = -0.565, c_{03} = 4.23$,

$c_{10} = -2.311, c_{11} = 2.083, c_{12} = -0.309, c_{13} = 2.311$,

$c_{20} = 5.102 \times 10^{-11}, c_{21} = 4.304 \times 10^{-15}, c_{22} = -1.141 \times 10^{-11}$,

$c_{23} = -5.12 \times 10^{-11}, c_{30} = 4.23, c_{31} = -3.814\ , c_{32} = 0.565, c_{33} = -4.23$.

Then, the approximate solution is:

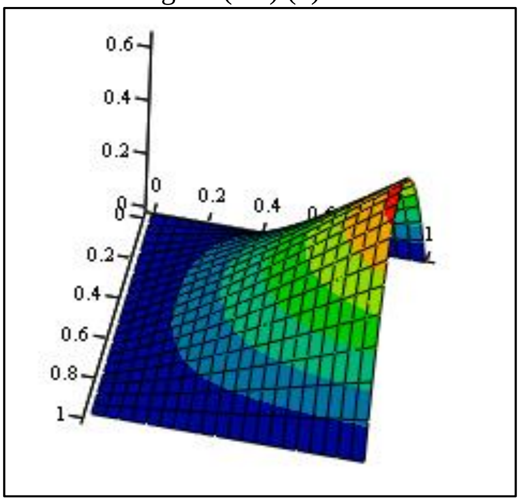
$z(x,t) =$
$(-4.23\cos t + 3.514\sin t - 0.565\, t + 4.23)\cos x + (-2.311\cos t + 2.083\sin t - 0.309\, t + 2.311)\sin x + (5.102 \times 10^{-11}\cos t + 4.304 \times 10^{-15}\sin t - 1.41 \times 10^{-11}\, t - 5.12 \times 10^{-11})x + (4.23\cos t - 3.814\ \sin t + 0.565\, t - 4.23\ ).$

Table- (1.2) illustrates the absolute error between the exact solution and the non-polynomial spline approximate solution.

Table (1.2)-The exact and approximate solutions of example (1.2) when $\alpha_1 = \alpha_2 = 0.5$.

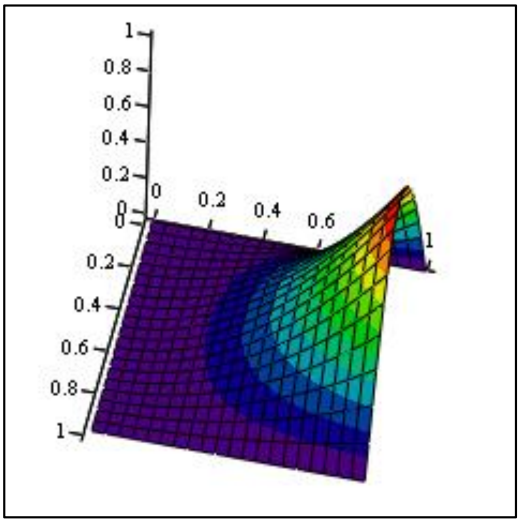
x	t	$z(x,t)$	$u(x,t)$	$\lvert u(x,t) - z(x,t) \rvert$
0	0	0	0	0
	0.01	0	0	0
	0.02	0	0	0
	0.03	0	0	0
$\frac{1}{3}$	0	-0.06×10^{-12}	0	0.06×10^{-12}
	0.01	4.042×10^{-3}	8.66×10^{-5}	0.3953×10^{-2}
	0.02	8.13×10^{-3}	3.464×10^{-4}	0.779×10^{-2}
	0.03	0.012	7.794	0.012
$\frac{2}{3}$	0	0.12×10^{-12}	0	0.12×10^{-12}
	0.01	4.039×10^{-3}	8.66×10^{-5}	0.3953×10^{-2}
	0.02	8.131×10^{-3}	3.464×10^{-5}	0.7784×10^{-2}
	0.03	0012	7.794×10^{-4}	0.011
1	0	-0.18×10^{-12}	0	0.18×10^{-12}
	0.01	-0.7877×10^{-5}	0	0.7877×10^{-5}
	0.02	-0.1574×10^{-4}	0	0.1574×10^{-4}
	0.03	-0.2359×10^{-4}	0	0.2359×10^{-4}

The approximation solution $z(x,t)$ is illustrated in Figure-(1.2) (a), while the exact solution is illustrated in Figure-(1.2) (b).



z

(a)



u

(b)

Figure (1.2)- (a) the approximate surface $z(x,t)$ and (b) the exact surface $u(x,t)$ for example (1.2).

6-Conclusions and future work

In this paper, the approximate solutions of the fractional-order heat equation are determined using the method of a linear non-polynomial spline. The results revealed the highest agreement with the exact solutions for the problems. The solutions for the numerical examples showed the validity of the proposed method. In addition, it is observed that the solutions of the fractional-order equation are convergent to the exact solution for the problem. In the future, the method of linear non-polynomial spline can be used to find the solution of other FPDEs that are frequently used in science and engineering.

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