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F-µ-Semiregular Modules

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Abstract:
Let R be an associative ring with identity and let M be a left R-module . As a generalization of µ-semiregular modules, we introduce an F-µ-semiregular module. Let F be a submodule of M and x∊M. x is called F-µ-semiregular element in M , if there exists a decomposition M=A⨁B, such that A is a projective submodule of R_x and $R_x \cap B \ll_\mu F$. M is called F-µ-semiregular if x is F-µ-semiregular element for each x∊M. A condition under which the module µ-semiregular is F-µ-semiregular module was given. The basic properties and some characterizations of the F-µ-semiregular module were provided.

Keywords: µ-small submodule, F-µ-semiregular module.

مقاسات شبه المنتظمة من النمط −µ−F

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الخلاصة
لتكن R حلقه ذات عنصر محايد وليكن M مقاسا ايسر معرف عليها . كتعميم لمقاسات شبه منتظمة من النمط −µ نقدم المقاسات شبه منتظمة من النمط −µ−F ، ليكن F مقاس جزئي من M وليكن x∊M ، يدعى العنصر x عنصر من النمط F−µ−semiregular اذا كان هنالك تحلل M=A⨁B بحيث ان A هو مقاس اسقاطي جزئي من R_x و $R_x \cap$ B $\ll_\mu$ F . ويسمى M مقاس شبه منتظم من النمط −µ−F , اذا كان كل عنصر في M هو عنصر من النمط F−µ−semiregular , هناك حاله يكون فيها المقاس شبه منتظم من النمط −µ هو مقاس شبه منتظم من النمط −µ−F . وقد تم اعطاء الخصائص الاساسية للمقاس شبه منتظم من النمط −µ−F. وقد تم اثبات بعض مكافئات المقاس شبه منتظم من النمط −µ−F.

1. Introduction

Throughout this paper, all rings have an identity and all modules will be unital left R-modules . Let M be a module and A be a submodule of M, then A is called small in M (denoted by A≪M) if M≠A+B, for any proper submodule B of M, see[1] and [2]. Z^*(M)={m∊M:mR≪E(M)}.A module M is called cosingular (non cosingular) module if Z^*(M)=M (Z^*(M) =0) [3] . As a generalization of small submodules, the concept of µ-small submodule was introduced in [4]; A submodule A of M is called µ-small submodule of M (denoted by A$\ll_\mu M$) if whenever M=A+X, $\frac{M}{X}$ is cosingular, then M=X. We write E(M), Rad(M), and Z(M), for the injective envelope, the Jacobson radical, and the singular

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submodule of M . respectively, for a left R-module M [3]. Define the submodule $Z^{*}(M)$ as a dual of singular submodule to be the set of all elements of m∊ M such that mR is a small module.

An R-module M is called µ-semiregular module if there exists a decomposition M=A⨁B, such that A is projective submodule of N and $N \cap B \ll_{\mu} M$ [5]. This concept leads us to introduce the following concept; Let M be an R-module F be a submodule of M ,and x∊M. x is called F-µ-semiregular element in M , if there exists a decomposition M=A⨁B, such that A is a projective submodule of R_x and $R_x \cap B \ll_{\mu} F$. M is called an F-µ-semiregular if m is F-µ-semiregular element for each m∊M.

In this paper, we investigate the basic properties of F-µ-semiregular module and give the condition under which the µ-semiregular module is an F-µ-semiregular module .

2. F-µ-semiregular module

Definition 2.1: Let M be an R-module F be a submodule of M, and 0≠ x∊M . x is called **F-µ-semiregular element** in M . if there exists a decomposition M=A⨁B, such that A is a projective submodule of R_x and $R_x \cap B \ll_{\mu} F$.

M is called an **F-µ-semiregular module** if x is F-µ-semiregular element for each x∊M.

Example 2.2

1- Consider the module Z_6 *as* Z_6-module . One can easily show that Z_6 is F-µ-semiregular, for every submodule F of Z_6 .

2- Consider the module Z as Z-module . Claim that Z is not F-µ-semiregular, for every proper submodule F of Z. Let R_x be nonzero submodule of Z.

Since Z is an indecomposable module , then {0} is only projective summand of Z contained in R_x . Z={0}⨁ Z and R_x ∩Z=R_x is not µ-small in F, so Z is not F-µ-semiregular . ∀ F ⊊ Z.

Note : Every F-µ-semiregular module is µ-semiregular module, but the converse is not true in general. For example, for Q as Z-module , let Z be a submodule of Q Q={0}⨁Q. Since Q is indecomposable, then {0} is only projective summand of Z and Q∩Z=Z is not µ-small in Z .Then Q is not F-µ-semiregular but is µ-semiregular.

Recall that a submodule A of M is called **coclosed submodule** of M (denoted by **A** $\leq_{cc}$ **M**) if whenever $\frac{A}{X} \ll \frac{M}{X}$ for some submodule X of M implies that X=A [6].

Definition 2.3: [4]. Let M be an R-module and let A be a submodule of M , then we say that A is a **µ-coclosed submodule** of M denoted by $\left(\boldsymbol{A \leq_{\mu cc} M}\right)$, if whenever $\frac{A}{X}$ is cosingular and X $\leq_{\mu ce}$ A in M for some submodule X of A, we have X=A.

The followings are some properties of µ-coclosed submodule [4].

Remark 2.4: [4]. 1- Let M be an R-module and A be a coclosed submodule of M, then A is a µ-coclosed in M.

2- Let M be a cosingular R-module and A be submodule of M, then A is a µ-coclosed if and only if it is coclosed in M.

3- Every direct summand of an R-module M is µ-coclosed.

Proposition 2.5 : [4]. Let A be a µ-coclosed submodule of an R-module M. If X≤ A≤ M and X $\ll_{\mu} M$, then $X \ll_{\mu} A$.

Remark 2.6 : Let M be a µ-semiregular R-module. If F is µ-coclosed submodule of M, then F is F-µ-semiregular.

Proof : Let 0≠x ∊ F and R_x ⊆ F , then R_x ⊆ M . Since M is µ-semiregular module , then ∃ is a projective submodule A of R_x such that M=A⨁B and $B \cap R_x \ll_{\mu} M$. Now, F = F ∩ M = F∩(A⨁B) = A⨁(F∩B) (modular law) . F ∩ B ∩ R_x ≤ F , but F ∩ B ∩ R_x= B ∩ $R_x \ll_{\mu}$ M . Since F is µ-coclosed , then{by prop.(2.5)} B ∩ $R_x \ll_{\mu} F$. Thus, F is F-µ-semiregular.

Remark 2.7: Let M be an R-module . If M is {0}-µ-semiregular, then M is µ-semiregular.

Proof : Suppose that M is {0}-µ-semiregular and let R_x be a submodule of M.

Since M is {0}-µ-semiregular , then there exists a projective submodule P of R_x,

such that M=P⨁S and $R_x \cap S = \{0\} \ll_{\mu} \{0\}$. Thus, M=$R_x \oplus S$ and $R_x \cap S = \{0\} \ll_{\mu} M$. Then, M is µ-semiregular.

Remark 2.8: Let M be an R-module and F and L be submodules of M such that F≤L.

If M is F-µ-semiregular , then M is L-µ-semiregular.

Proof : It is clear.

Remark 2.9 : Let M be an R-module and F and L be submodules of M such that F≤L and F is μ-coclosed in M. If M is L-μ-semiregular , then M is F-μ-semiregular.

Proof : Let M be L-μ-semiregular, F ≤ L, and x∈ M. Since M is L-μ-semiregular, then there exists M=A⊕B, where A is projective of R_x and $R_x \cap B \ll_\mu L$.

Since F is μ-coclosed and $R_x \cap B \ll_\mu L$, then {by prop.(2.5)} $R_x \cap B \ll_\mu F$.

Thus, M is F-μ-semiregular.

Remark 2.10: Let M be an R-module and K be a submodule of M. If M is F-μ-semiregular, then K is (K∩F)-μ-semiregular, when K∩F is μ-coclosed.

Proof : Let 0≠ R_x≤ K, then R_x≤M . Since M is F-μ-semiregular, then ∃ A is a direct summand submodule of M , where A is projective of R_x and A≤R_x≤K , then M=A⊕B and B∩$R_x \ll_\mu F$. Hence, K=K∩M=K∩(A⊕B) , since A ≤K and by modular law, K=A⊕(K∩B).

Hence (K ∩ B) ∩ R_x ≤ F, but K∩ R_x≤K∩F≤F and K∩ F is μ-coclosed , hence (K∩B)∩ $R_x \ll_\mu$ K∩F. Then K is (F∩K)-μ-semiregular.

Proposition 2.11: Let M be an F-μ-semiregular and K be a submodule of M such that F ≤ K. Then K is F -μ-semiregular.

Proof : Let 0 ≠ R_x be a cyclic submodule of K. Since M is F-μ-semiregular , then M=A⊕B, where A is a projective submodule of R_x and $R_x \cap B \ll_\mu F$. Since A ≤ K, then by the modular law, we have K=A⊕(B∩K). But $R_x \cap (B \cap K) = R_x \cap B \ll_\mu F$. Therefore, K is F-μ-semiregular.

Recall that M is called **μ-semi hollow module** if every finitely generated proper submodule of M is μ-small submodule of M [5] .

Remark 2.12 : Let M be an R-module , where M is μ-semi hollow module .Then M is M-μ-semiregular module.

Proof : Let 0≠ R_x be a proper cyclic submodule of M, then M={0}⊕M, where {0} is a projective submodule of R_x and $R_x \cap M = R_x \ll_\mu M$ {since M is μ-semi hollow}. Thus M is M-μ-semiregular.

Proposition 2.13 : Every semisimple projective R-module M is F-μ-semiregular , for every submodule F of M.

Proof : Let 0≠ R_x be a cyclic submodule of M. Since M is semisimple, then $M = R_x \oplus B$, for some submodule B of M. Since M is a projective , then R_x is a projective submodule of R_x and $R_x \cap B = 0 \ll_\mu F$. So, M is F-μ-semiregular module.

Proposition 2.14: Let M be an indecomposable R-module and F be a proper submodule of M .If M is F-μ-semiregular , then M is a projective module.

Proof : Let M be an indecomposable R-module , and let F be a proper submodule of M. Since M is F-μ-semiregular , then for every 0≠ x ∈ M, there exists a decomposition M=$A_x \oplus B_x$, where A_x is a projective submodule of R_x and $B_x \cap R_x \ll_\mu F$. But M is an indecomposable R-module. Then for every x ∈ M, either $A_x = 0$ or $A_x = M$. If $A_x = 0$, for every x ∈ M, then $B_x = M$, for every x ∈ M, and hence $B_x \cap R_x = M \cap R_x = R_x \ll_\mu F$, for every x ∈ M and M=F. Which is a contradiction.

Then, there exists $x_0 \in M$, such that $M = A_{x0}$, and hence M is projective.

Recall that a submodule A of M is called a **fully invariant** if g (A) ≤ A , for g ∈ End(M), and M is called **duo module** if every submodule of M is fully invariant [7] .

Proposition 2.15: Let M be an R-module and F be a fully invariant submodule of M, then for any submodule N of M, the following conditions are equivalent:

1- There exists a decomposition M = A ⊕ B such that A is projective submodule of N and N ∩ B $\ll_\mu$F.

2- There exists a homomorphism α : M ⟶ N such that α² = α, α (M) is a projective and (I-α)(N) $\ll_\mu$ F.

3- N can be written as N = A ⊕ S, where A is a projective summand of M and S $\ll_\mu$F.

Proof :1⟹2: Let N be a submodule of M, then by our assumption, M =A ⊕ B, where A is projective submodule of N and N ∩ B $\ll_\mu$F. By the modular law, we have that N = A ⊕ (B ∩ N). Let α : M⟶A be the projection map. It is clear that α ² = α and α (M) is projective. Now, consider the map (I– α). It is clear that (I- α) : M ⟶ B and (I- α) (N) ≤ B. Now let x ∈ (I-α) (N)

, then x = n - α (n) for some n∈N. But α (x) ∈A ≤ N , therefore x∈ N and hence x∈ N∩B$\ll_{\mu}$F. Thus (I - α) (N) ≤ N∩B $\ll_{\mu}$F.

2⇒1: Assume that there exists a homomorphism α : M ⟶ N such that α^2= α, α (M) is projective and (I-α) (N) $\ll_{\mu}$F . Claim that M = α (M) ⊕ (I- α) (M) to show that: Let m ∈ M, then m = m + α (m) - α (m) = α (m) + m - α (m) = α (m) + (I- α) (m). Thus M = α (M) + (I- α) (M). Now, let x ∈ α (M) ∩ (I- α) (M) , then x = α (m_1) and x = (I- α) (m_2) for some m_1,m_2 ∈ M.

So, α (x) = α (m_1) = α (m_2) - α (m_2) = 0, then α (m_1) = 0 and hence x = 0. α (M) is projective. Let d ∈N∩(I- α)(M), then d∈N and d ∈ (I- α) (M). Since d ∈(I- α) (M), then d = (I- α) (m), where m ∈ M. Now, d = m - α (m) and hence m ∈ N, so d ∈ (I- α) (N). Thus N ∩ (I- α) (M) ≤ (I- α) (N)$\ll_{\mu}$F.

1⇒3: Let N be a submodule of M, then by our assumption, M=A⊕B, where A is a projective submodule of N and N ∩ B $\ll_{\mu}$F. By the modular law , N = A ⊕ (N ∩ B), where A is projective summand of M, and N ∩ B $\ll_{\mu}$F.

3⇒1: Let N be a submodule of M, then by our assumption, N = A ⊕ S**,** where A is a projective summand of M and S $\ll_{\mu}$F. Hence M = A ⊕B , for some submodule B of M. By the modular law, N = A ⊕ (N ∩ B). Let P : M ⟶ B be the projection map. Claim that P(S)= P(N ∩ B) to show that: P(N) = P(A) ⊕ P(S) = P(S). On the other hand, P(N) = P(A) ⊕ P(N ∩ B) = P(N ∩ B). Thus N ∩ B = P (N ∩ B) = P(S) $\ll_{\mu}$ P(F). But F is a fully invariant submodule of M, therefore P(F) $\ll_{\mu}$F and hence N ∩ B$\ll_{\mu}$F.

Corollary 2.16: Let F be a fully invariant submodule of an R-module, then the following statements are equivalent:
1-M is F-μ- semiregular.
2-For every finitely generated submodule N of M there exists a homomorphism γ :M → N, such that γ^2 = γ, γ (M) is a projective and (I- γ) (N) $\ll_{\mu}$ F.
3- For every finitely generated submodule N of M there exists a decomposition M =A ⊕ B such that A is projective submodule of N and N∩B$\ll_{\mu}$ F.
4- For every finitely generated submodule N of M, N can be written as N =A⊕S , where A is projective summand of M and S $\ll_{\mu}$ F.
Proof : It is clear .

Corollary 2.17: Let M be an R-module and F be a fully invariant submodule of M, then for every 0≠x∈M, the following statements are equivalent:
1- x is F-μ-semiregular element.
2- R_x can be written as R_x=A⊕S, where A is a projective summand of M, and S$\ll_{\mu}$F.
Proof : It is clear.

Corollary 2.18 : Let M be an R-module and F be a fully invariant submodule of M, then the following statements are equivalent:
1- M is an F-μ-semiregular module.
2- For 0≠ x∈M , R_x can be written as R_x=A⊕S, where A is a projective summand of M, and S$\ll_{\mu}$F.
Proof : It is clear.

Proposition 2.19: Let R be an indecomposable ring, M is an R-module , and x ∈ M , then R_x is F-μ-semiregular if and only if either R_x is projective summand of M or R_x $\ll_{\mu}$ F, where F is a fully invariant submodule of M.

Proof:⇒) Let 0≠x∈M and assume that x is F-μ-semiregular , then by (cor.(2.17)), R_x=A⊕B, where A is projective summand of M and B$\ll_{\mu}$ F. Let : R⟶R_x be defined by φ(x)= rx , ∀ r∈ R. φ be an epimorphism, and ρ: R_x⟶A be the projection homomorphism , then, clearly, $\rho_\circ\varphi$=γ:R⟶A is an epimorphism.

Consider the following short exact sequence:

$0 \rightarrow \text{Ker}\,\gamma \xrightarrow{\iota} R \xrightarrow{\gamma} A \rightarrow 0$

where ι is an inclusion homomorphism . Since A is projective , then by [8] , the sequence splits , thus Ker γ is a direct summand of R. Now, Ker γ=Ker $(\rho_\circ\varphi)$={r ∈R ; $(\rho_\circ\varphi)$(r)=0}

={r∈R ; $\rho(\varphi$(r))=0}={r∈R ; and φ(r)∈B}=φ^{-1}(B). But R is indecomposable , then either φ^{-1}(B)=0 or φ^{-1}(B)=R . If φ^{-1}(B)=0 then B=0 , hence R_x =A is a projective summand of M. If φ^{-1}(B) = R , then B= $\varphi(\varphi^{-1}$(B))=φ(R)= R_x. Thus B=R_x, therefore $R_x \ll_\mu$ F.

Conversely, let x ∈ M . If R_x is a projective summand of M, then M= R_x⊕B, for some B≤M, hence R_x is a projective summand of R_x, and B ∩ R_x ={0}$\ll_\mu$ F. If $R_x \ll_\mu$ F, then M={0}⊕M , where {0} is projective summand of R_x and R_x ∩ M= $R_x \ll_\mu$ F.

Proposition 2.20: Let M=$\oplus_{i\in I}M_i$ be a direct sum of the submodule M_i of M. If M is F-μ-semiregular, then each M_i is F_i-μ-semiregular ,where F_i =F ∩ M_i.

Proof : Let 0≠ $x_i \in M_i$, for some i∈I. Since $x_i \in M$ and since M is F-μ-semiregular, then ∃ $A \leq R_{xi}$, A is projective, and A is direct summand of M , since M=A⊕B for B ≤ M.

Also, $R_{xi} \cap B \ll_\mu F$. Now, $M_i \cap M = M_i = M_i \cap (A_i \oplus B_i) = A \oplus (M_i \cap B_i)$.Since A_i is a direct summand of M_i. A_i is projective, and $(B_i \cap M_i) \cap R_{xi} \leq R_{xi} \cap B_i \ll_\mu F$ then

$(B_i \cap M_i) \cap R_{xi} \ll_\mu F$. But $(B_i \cap M_i) \cap R_{xi} \leq M_i \cap F \leq F$. Since M=$\oplus_{i\in I}M_i$, then F=$\oplus_{i\in I}$ $(M_i \cap F)$. Since $M_i \cap F$ is a direct summand of F [4], then $(B_i \cap M_i) \cap R_{xi} \ll_\mu M_i \cap F$. Thus, M_i is F_i-μ-semiregular module.

Proposition 2.21: Let M_1 and M_2 be R-modules such that M=$M_1 \oplus M_2$ is a duo module. If M_1 is F_1 -μ-semiregular and M_2 is F_2-μ-semiregular , then M is $F_1 \oplus F_2$-μ-semiregular module .

Proof: Let N be a finitely generated submodule of M. Since M is a duo module , then N= N∩M_1 ⊕ N ∩ M_2 . Since N is finitely generated, then N∩ M_1 and N ∩ M_2 are finitely generated. Also, since M_i is F_i -μ-semiregular, ∀ i=1,2 , and N_i is finitely generated .∀i=1,2, then ∃ is a projective direct summand submodule of N_i , such that $M_i = A_i \oplus B_i$. and $N_i \cap B_i \ll_\mu F_i$. ∀ i=1,2 . Thus, M = $M_1 \oplus M_2 = (A_1 \oplus B_1) \oplus (A_2 \oplus B_2) = (A_1 \oplus A_2) \oplus (B_1 \oplus B_2)$. Since A_1 and A_2 are projective , then $(A_1 \oplus A_2)$ is projective. Now , N ∩ $(B_1 \oplus B_2)$=(N∩ M_1 ⊕ N ∩ M_2) ∩ $(B_1 \oplus B_2)$=($N_1 \oplus B_1) \cap (N_2 \oplus B_2) \ll_\mu F_1 \oplus F_2$. Thus, M is $F_1 \oplus F_2$-μ-semiregular module.

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