F-µ-Semiregular Modules

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Abstract:
Let R be an associative ring with identity and let M be a left R-module. As a generalization of µ-semiregular modules, we introduce an F-µ-semiregular module. Let F be a submodule of M and xœM. x is called F-µ-semiregular element in M, if there exists a decomposition M=A⨁B, such that A is a projective submodule of Rx and Rx∩B ≪µ, F. M is called F-µ-semiregular if x is F-µ-semiregular element for each xœM. A condition under which the module µ-semiregular is F-µ-semiregular was given. The basic properties and some characterizations of the F-µ-semiregular module were provided.

Keywords: µ-small submodule, F-µ-semiregular module.

1. Introduction
Throughout this paper, all rings have an identity and all modules will be unital left R-modules. Let M be a module and A be a submodule of M, then A is called small in M (denoted by A<<M) if M≠A+B, for any proper submodule B of M, see[1] and [2]. Z*(M)={mœM:mR<<E(M)}. A module M is called cosingular (non cosingular) module if Z*(M)=M (Z*(M)=0) [3]. As a generalization of small submodules, the concept of µ-small submodule was introduced in [4]: A submodule A of M is called µ-small submodule of M (denoted by A<<µ, M) if whenever M=A+X, M/X is cosingular, then M=X. We write E(M), Rad(M), and Z(M), for the injective envelope, the Jacobson radical, and the singular submodules of M, respectively.

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module of $M$ , respectively, for a left $R$-module $M$ [3]. Define the submodule $Z^s(M)$ as a dual of singular submodule to be the set of all elements of $m \in M$ such that $mR$ is a small module.

An $R$-module $M$ is called $\mu$-semiregular module if there exists a decomposition $M=A \oplus B$, such that $A$ is projective submodule of $N$ and $N \cap B \ll_{\mu} M$ [5]. This concept leads us to introduce the following concept; Let $M$ be an $R$-module $F$ be a submodule of $M$ ,and $x \in M$. $x$ is called $F$-$\mu$-semiregular element in $M$ , if there exists a decomposition $M=A \oplus B$, such that $A$ is a projective submodule of $R_x$ and $R_x \cap B \ll_{\mu} F$. $M$ is called an $F$-$\mu$-semiregular if $m$ is $F$-$\mu$-semiregular element for each $m \in M$.

In this paper, we investigate the basic properties of $F$-$\mu$-semiregular module and give the condition under which the $\mu$-semiregular module is an $F$-$\mu$-semiregular module.

2. $F$-$\mu$-semiregular module

**Definition 2.1**: Let $M$ be an $R$-module $F$ be a submodule of $M$, and $0 \neq x \in M$. $x$ is called $F$-$\mu$-semiregular element in $M$, if there exists a decomposition $M=A \oplus B$, such that $A$ is a projective submodule of $R_x$ and $R_x \cap B \ll_{\mu} F$.

$M$ is called an $F$-$\mu$-semiregular module if $x$ is $F$-$\mu$-semiregular element for each $x \in M$.

**Example 2.2**

1- Consider the module $Z_6$ as $Z_6$-module. One can easily show that $Z_6$ is $F$-$\mu$-semiregular, for every submodule $F$ of $Z_6$.

2- Consider the module $Z$ as $Z$-module. Claim that $Z$ is not $F$-$\mu$-semiregular, for every proper submodule $F$ of $Z$. Let $R_F$ be non-zero submodule of $Z$.

Since $Z$ is an indecomposable module, then $\{0\}$ is only projective summand of $Z$ contained in $R_x$. $Z=\{0\} \oplus Z$ and $R_x \cap Z=R_x$ is not $\mu$-small in $F$, so $Z$ is not $F$-$\mu$-semiregular , $\forall F \notin Z$.

Note: Every $F$-$\mu$-semiregular module is $\mu$-semiregular module, but the converse is not true in general. For example, for $Q$ as $Z$-module, let $Z$ be a submodule of $Q$. $Q=\{0\} \oplus Q$. Since $Q$ is indecomposable, then $\{0\}$ is only projective summand of $Z$ and $Q \cap Z=0$ is $\mu$-small in $Z$. Then $Q$ is not $F$-$\mu$-semiregular but is $\mu$-semiregular.

Recall that a submodule $A$ of $M$ is called **coclosed submodule** of $M$ (denoted by $A \leq_{cc} M$) if whenever $\frac{A}{x} \ll_{\mu} \frac{M}{x}$ for some submodule $X$ of $M$ implies that $X=A$ [6].

**Definition 2.3**: [4]. Let $M$ be an $R$-module and let $A$ be a submodule of $M$, then we say that $A$ is a $\mu$-coclosed submodule of $M$ denoted by $(A \leq_{\mu cc} M)$, if whenever $\frac{A}{x}$ is cosingular and $X \leq_{\mu cc} A$ in $M$ for some submodule $X$ of $A$, we have $X=A$.

The followings are some properties of $\mu$-coclosed submodule [4].

**Remark 2.4**: [4].
1- Let $M$ be an $R$-module and $A$ be a coclosed submodule of $M$, then $A$ is a $\mu$-coclosed in $M$.

2- Let $M$ be a cosingular $R$-module and $A$ be submodule of $M$, then $A$ is a $\mu$-coclosed if and only if it is coclosed in $M$.

3- Every direct summand of an $R$-module $M$ is $\mu$-coclosed.

**Proposition 2.5**: [4]. Let $A$ be a $\mu$-coclosed submodule of an $R$-module $M$. If $X \leq A \leq M$ and $X \ll_{\mu} M$, then $X \ll_{\mu} A$.

**Remark 2.6**: Let $M$ be a $\mu$-semiregular $R$-module. If $F$ is $\mu$-coclosed submodule of $M$, then $F$ is $F$-$\mu$-semiregular.

**Proof**: Let $0 \neq x \in F$ and $R_x \subseteq F$, then $R_x \subseteq M$. Since $M$ is $\mu$-semiregular module, then $\exists$ is a projective submodule $A$ of $R_x$ such that $M=A \oplus B$ and $B \cap R_x \ll_{\mu} M$. Now, $F=F \cap M=F \cap (A \oplus B)$ $=A \oplus (F \cap B)$ (modular law ). $F \cap B \cap R_x \subseteq F$, but $F \cap B \cap R_x=F \cap R_x \ll_{\mu} M$. Since $F$ is $\mu$-coclosed , then{by prop.(2.5)} $B \cap R_x \ll_{\mu} F$. Thus, $F$ is $F$-$\mu$-semiregular.

**Remark 2.7**: Let $M$ be an $R$-module. If $M$ is $\{0\}$-$\mu$-semiregular, then $M$ is $\mu$-semiregular.

**Proof**: Suppose that $M$ is $\{0\}$-$\mu$-semiregular and let $R_x$ be a submodule of $M$.

Since $M$ is $\{0\}$-$\mu$-semiregular, then there exists a projective submodule $P$ of $R_x$, such that $M=P \oplus S$ and $R_x \cap S=\{0\} \ll_{\mu} \{0\}$. Thus, $M=R_x \oplus S$ and $R_x \cap S=\{0\} \ll_{\mu} M$. Then, $M$ is $\mu$-semiregular.

**Remark 2.8**: Let $M$ be an $R$-module and $F$ and $L$ be submodules of $M$ such that $F \subseteq L$.

If $M$ is $F$-$\mu$-semiregular , then $M$ is $L$-$\mu$-semiregular.

**Proof**: It is clear.
Remark 2.9: Let M be an R-module and F and L be submodules of M such that F ≤ L and F is µ-coclosed in M. If M is L-µ-semiregular, then M is F-µ-semiregular.

Proof: Let M be L-µ-semiregular, F ≤ L, and x ∈ M. Since M is L-µ-semiregular, there exists M = A ⊕ B, where A is projective of R_x and R_x ∩ B ≪_µ L.

Since F is µ-coclosed and R_x ∩ B ≪_µ L, then \{by prop.(2.5)\} R_x ∩ B ≪_µ F.

Thus, M is F-µ-semiregular.

Remark 2.10: Let M be an R-module and K be a submodule of M. If M is F-µ-semiregular, then K is (K ∩ F)-µ-semiregular, when K ∩ F is µ-coclosed.

Proof: Let 0 ≠ R_x ≤ K, then R_x ≤ M. Since M is F-µ-semiregular, then ∃ A is a direct summand submodule of M, where A is projective of R_x and A ≤_µ R_x ≤ K, then M = A ⊕ B and R_x ∩ B ≪_µ F. Hence, K = K ∩ M = K ∩ (A ⊕ B), since A ≤ K and by modular law, K = A ⊕ (K ∩ B).

Hence (K ∩ B) ∩ R_x ≤ F, but K ∩ R_x ≤ K ∩ F ≤ F and K ∩ F is µ-coclosed, hence (K ∩ B) ∩ R_x ≪_µ K ∩ F. Then K is (F ∩ K)-µ-semiregular.

Proposition 2.11: Let M be an F-µ-semiregular and K be a submodule of M such that F ≤ K. Then K is F-µ-semiregular.

Proof: Let 0 ≠ R_x be a cyclic submodule of K. Since M is F-µ-semiregular, then M = A ⊕ B, where A is a projective submodule of R_x and R_x ∩ B ≪_µ F. Since A ≤ K, then by the modular law, we have K = A ⊕ (B ∩ K). But R_x ∩ (B ∩ K) = R_x ∩ B ≪_µ F. Therefore, K is F-µ-semiregular.

Recall that M is called µ-semi hollow module if every finitely generated proper submodule of M is µ-small submodule of M [5].

Remark 2.12: Let M be an R-module, where M is µ-semi hollow module. Then M is M-µ-semiregular module.

Proof: Let 0 ≠ R_x be a cyclic submodule of K. Since M is F-µ-semiregular, then M = A ⊕ B, where {0} is a projective submodule of R_x and R_x ∩ M = R_x ≪_µ M \{since M is µ-semi hollow\}. Thus M is M-µ-semiregular.

Proposition 2.13: Every semisimple projective R-module M is F-µ-semiregular, for every submodule F of M.

Proof: Let 0 ≠ R_x be a cyclic submodule of K. Since M is semisimple, then M = R_x ⊕ B, for some submodule B of M. Since M is a projective, then R_x is a projective submodule of R_x and R_x ∩ B = 0 ≪_µ F. So, M is F-µ-semiregular module.

Proposition 2.14: Let M be an indecomposable R-module and F be a proper submodule of M. If M is F-µ-semiregular, then M is a projective module.

Proof: Let M be an indecomposable R-module, and let F be a proper submodule of M. Since M is F-µ-semiregular, then for every 0 ≠ x ∈ M, there exists a decomposition M = A_x ⊕ B_x, where A_x is a projective submodule of R_x and B_x ∩ R_x ≪_µ F. But M is an indecomposable R-module. Then for every x ∈ M, either A_x = 0 or A_x = M. If A_x = 0, for every x ∈ M, then B_x = M, for every x ∈ M, and hence B_x ∩ R_x = M ∩ R_x = R_x ≪_µ F, for every x ∈ M and M=F. Which is a contradiction.

Then, there exists x_0 ∈ M such that M = A_{x_0}, and hence M is projective.

Recall that a submodule A of M is called a fully invariant if g (A) ≤ A, for g ∈ End(M), and M is called duo module if every submodule of M is fully invariant [7].

Proposition 2.15: Let M be an R-module and F be a fully invariant submodule of M, then for any submodule N of M, the following conditions are equivalent:

1. There exists a decomposition M = A ⊕ B such that A is projective submodule of N and N ∩ B ≪_µ F.

2. There exists a homomorphism α : M → N such that α² = α, α (M) is a projective and (I- α) (N) ≪_µ F.

3. N can be written as N = A ⊕ S, where A is a projective summand of M and S ≪_µ F.

Proof: 1 ⇒ 2: Let N be a submodule of M, then by our assumption, M = A ⊕ B, where A is projective submodule of N and N ∩ B ≪_µ F. By the modular law, we have that N = A ⊕ (B ∩ N).

Let α : M → A be the projection map. It is clear that α² = α and α (M) is projective. Now, consider the map (I- α). It is clear that (I- α) : M → B and (I- α) (N) ≤ B. Now let x ∈ (I- α) (N)
N. But α (x) = α (m) + m = α (m) + m - α (m) = α (m) + (I - α) (m). Thus M = α (M) + (I - α) (M). Now, let x ∈ α (M) ∩ (I - α) (M), then x = α (m 1) and x = (I - α) (m 2) for some m 1, m 2 ∈ M.

So, α (x) = α (m 1) = α (m 2) = 0, then α (m 1) = 0 and hence x = 0. α (M) is projective. Let d ∈ N ∩ (I - α) (M), then d ∈ N and d ∈ (I - α) (M). Since d ∈ (I - α) (M), then d = (I - α) (m), where m ∈ N. Now, d = m - α (m) and hence m ∈ N, so d ∈ (I - α) (N). Thus N ∩ (I - α) (M) ≤ (I - α) (N) ≤ μ F.

1⇒3: Let N be a submodule of M, then by our assumption, M = A ⊕ B, where A is a projective submodule of N and N ∩ B ≤ μ F. By the modular law, N = A ⊕ (N ∩ B), where A is a projective summand of M and N ∩ B ≤ μ F.

3⇒1: Let N be a submodule of M, then by our assumption, N = A ⊕ S, where A is a projective summand of M and S ≤ μ F. Hence M = A ⊕ B, for some submodule B of M. By the modular law, N = A ⊕ (N ∩ B). Let P : M → B be the projection map. Claim that P(N) = P(A) ⊕ P(S) = P(S). On the other hand, P(N) = P(A) ⊕ P(N ∩ B) = P(N ∩ B). Thus N ∩ B = P(N ∩ B) = P(S) ≤ μ F. But F is a fully invariant submodule of M, therefore P(F) ≤ μ F and hence N ∩ B ≤ μ F.

Corollary 2.16: Let F be a fully invariant submodule of an R-module, then the following statements are equivalent:

1- M is F-μ- semiregular.

2- For every finitely generated submodule N of M there exists a homomorphism γ : M → N, such that γ² = γ, γ (M) is a projective and (I - γ) (N) ≤ μ F.

3- For every finitely generated submodule N of M there exists a decomposition M = A ⊕ B such that A is projective submodule of N and N ∩ B ≤ μ F.

4- For every finitely generated submodule N of M, N can be written as N = A ⊕ S, where A is a projective summand of M and S ≤ μ F.

Proof: It is clear.

Corollary 2.17: Let M be an R-module and F be a fully invariant submodule of M, then for every 0 ≠ x ∈ M, the following statements are equivalent:

1- x is F-μ-regular element.
2- R_x can be written as R_x = A ⊕ S, where A is a projective summand of M, and S ≤ μ F.

Proof: It is clear.

Corollary 2.18: Let M be an R-module and F be a fully invariant submodule of M, then the following statements are equivalent:

1- M is an F-μ-regular module.
2- For 0 ≠ x ∈ M, R_x can be written as R_x = A ⊕ S, where A is a projective summand of M, and S ≤ μ F.

Proof: It is clear.

Proposition 2.19: Let R be an indecomposable ring, M is an R-module, and x ∈ M, then R_x is F-μ-semiregular if and only if either R_x is projective summand of M or R_x ≤ μ F, where F is a fully invariant submodule of M.

Proof: Let 0 ≠ x ∈ M and assume that x is F-μ-semiregular, then by (cor. 2.17), R_x = A ⊕ B, where A is projective summand of M and B ≤ μ F. Let : R → R_x be defined by φ(x) = rx, ∀ r ∈ R. φ be an epimorphism, and p : R_x → A be the projection homomorphism, then, clearly, ρ ◦ p = γ : R → A is an epimorphism.
Consider the following short exact sequence:

\[ 0 \rightarrow \text{Ker } \gamma R \xrightarrow{\gamma} A \rightarrow 0 \]

where \( \gamma \) is an inclusion homomorphism. Since \( A \) is projective, then by [8], the sequence splits, thus \( \text{Ker } \gamma = \text{Ker } (\rho, \varphi) \), and \( \gamma = R \rightarrow A \rightarrow 0 \).

Proof: Let \( M = \bigoplus_{i \in I} M_i \) be a direct sum of the submodule \( M_i \) of \( M \). If \( M \) is \( F-\mu \)-semiregular, then each \( M_i \) is \( F_i-\mu \)-semiregular, where \( F_i = F \cap M_i \).

Proposition 2.20: Let \( M = \bigoplus_{i \in I} M_i \) be a direct sum of the submodule \( M_i \) of \( M \). If \( M \) is \( F-\mu \)-semiregular, then each \( M_i \) is \( F_i-\mu \)-semiregular, where \( F_i = F \cap M_i \).

Proof: Let \( 0 \neq x_i \in M_i \), for some \( i \in I \). Since \( x_i \in M \) and since \( M \) is \( F-\mu \)-semiregular, then \( \exists \ A \leq R_{x_i} \), \( A \) is projective, and \( A \) is direct summand of \( M \), since \( M = A \oplus B \) for \( B \leq M \).

Also, \( R_{x_i} \cap B \ll \mu F \). Now, \( M_i \cap M = M_i \cap (A_i \oplus B_i) = A_i \oplus (M_i \cap B_i) \). Since \( A_i \) is a direct summand of \( M_i \), \( A_i \) is projective, and \( (B_i \cap M_i) \cap R_{x_i} \leq R_{x_i} \cap B_i \ll \mu F \) then \( (B_i \cap M_i) \cap R_{x_i} \ll \mu F \). But \( (B_i \cap M_i) \cap R_{x_i} \leq M_i \cap F \leq F \). Since \( M = \bigoplus_{i \in I} M_i \), then \( F = \bigoplus_{i \in I} (M_i \cap F) \).

Since \( M_i \cap F \) is a direct summand of \( F \) [4], then \( (B_i \cap M_i) \cap R_{x_i} \ll \mu M_i \cap F \). Thus, \( M_i \) is \( F_i-\mu \)-semiregular module.

Proposition 2.21: Let \( M_1 \) and \( M_2 \) be \( R \)-modules such that \( M = M_1 \oplus M_2 \) is a direct module. If \( M_1 \) is \( F_1-\mu \)-semiregular and \( M_2 \) is \( F_2-\mu \)-semiregular, then \( M = F_1 \oplus F_2-\mu \)-semiregular module.

Proof: Let \( N \) be a finitely generated submodule of \( M \). Since \( M \) is a direct module, then \( N = N \cap M_i \oplus N \cap M_2 \). Since \( N \) is finitely generated, then \( N \cap M_i \) and \( N \cap M_2 \) are finitely generated. Also, since \( M_i \) is \( F_i-\mu \)-semiregular, \( \forall i = 1,2 \), and \( N_i \) is finitely generated, \( \forall i = 1,2 \), then \( \exists \ A \) is a projective direct summand submodule of \( N_i \), such that \( M_i = A_i \oplus B_i \) and \( N_i \cap B_i \ll \mu F_i \). \( \forall i = 1,2 \). Thus, \( M = M_1 \oplus M_2 = (A_1 \oplus B_1) \oplus (A_2 \oplus B_2) = (A_1 \oplus A_2) \oplus (B_1 \oplus B_2) \).

Since \( A_1 \) and \( A_2 \) are projective, then \( (A_1 \oplus A_2) \) is projective.

Now, \( N \cap (B_1 \oplus B_2) = (N \cap M_1 \oplus N \cap M_2) \cap (B_1 \oplus B_2) = (N_1 \oplus B_1) \cap (N_2 \oplus B_2) \ll \mu F_1 \oplus F_2 \). Thus, \( M = F_1 \oplus F_2-\mu \)-semiregular module.

References