DOI: 10.24996/ijs.2021.62.5.25

ISSN: 0067-2904

FI-⊕-J-supplemented modules

Abdlkareem Ali, Wasan Khalid*
Department of mathematics, College of Science, University of Baghdad, Baghdad, Iraq

Received: 15/4/2020 Accepted: 21/8/2020

Abstract
A Module M is called cofinite J- Supplemented Module if for every cofinite submodule L of M, there exists a submodule N of M such that M=L+N with $N \cap L \ll_J N$. Initially, we gave the main properties of cof-J-supplemented modules. An R-module M is called fully invariant-J-supplemented if for every fully invariant submodule N of M, there exists a submodule K of M, such that M = N + K with N ∩ K $\ll_J$K. A condition under which the direct sum of FI-J-supplemented modules is FI-J-supplemented was given. Also, some types of modules that are related to the FI-J-supplemented module were discussed.

Keywords: cofinitely J-supplemented modules , fully invariant J-supplemented modules , fully invariant ⊕-J-supplemented modules **.**

المقاسات المكملة من النمط FI-⊕ -J

وسن خالد* , عبدالكريم علي
قسم الرياضيات , كلية العلوم , جامعة بغداد , بغداد , العراق

الخلاصة
لتكن R حلقة اي ذات عنصر محايد وليكن M مقاساً ايسر معرف عليها . سوف نقدم تعريف المقاس المكمل من النمط cof-J كتعميم لمقاس مكمل من النمط J. يتم إعطاء الخصائص الرئيسية للمقاسات المكملة من النمط cof-J والعديد من خصائص هذه المقاسات . كما نقدم أيضًا المقاسات المكملة من النمط FI-J ونعطي توصيفًا للمقاسات المكملة من النمط FI-J. الشروط التي بموجبها يكون المجموع المباشر للمقاسات المكملة من النمط FI-J ايضا مقاسات مكملة من النمط FI-J ، يتم دراسة بعض أنواع المقاسات ذات الصلة بالمقاس المكمل من النمط FI-J ، نناقش العلاقة بينها.

1. Introduction

Throughout this paper **,** an arbitrary associative ring with identity is denoted by R and all modules are unitary left R-modules . Assume that C and D are submodules of M, a submodule C is called small submodule of M (C≪ M), if whenever M = C + D , we have M = D [1]. A submodule C of a module M is called J-small submodule of M ($C \ll_J M$) if whenever M = C + D , with J($\frac{M}{D}$) $= \frac{M}{D}$ implies M = D , were J(M) denotes the Jacobson radical of M [2]. A submodule C is a supplement of D in M if C is minimal with respect to M = C + D . Equivalently, M = C + D with $C \cap D \ll C$ [3]. A module M is called supplemented module if every submodule of M has a supplement in M [4] . A Submodule C is called J-supplement of D in M if M = C + D and $\cap D \ll_J C$. M is called J-supplemented if every submodule of M has J-supplement in M [2]. A module M is called ⊕-supplemented module if every submodule of M has a direct summand supplement in M [5]. A Submodule C is called a ⊕-Jacobson-supplement of D in M (for short ⊕-J-supplement) if M = C + D , and C is a direct summand of M

*Email: wasankhalid222@gmail.com

with C ∩ D $\ll_J$ C **.** It is called a ⊕-J-supplemented if every submodule of M has a ⊕-J-supplement in M [6] **.** A submodule C of a module M is called cofinite submodule of M if $\frac{M}{C}$ is finitely generated . A module M is called cofinitely supplemented if every cofinite submodule of M has supplement submodule [7]. As a generalization of cofinitely supplemented , we define the cofinitely J-supplemented (for short cof-J-supplemented) as follows . A module M is called cof-J-supplemented module if for every cofinite submodule C of M , there exists a submodule D of M such that M = C + D and $C \cap D \ll_J D$. A submodule C of a module M is called a fully invariant submodule if $f(C) \subseteq C$ for every $f \in End_R(M)$ [8].

In section 2, we prove some properties of cof-J-supplemented and we show that any factor module of cof-J-supplemented module is cof-J-supplemented and any finite sum of cof-J-supplemented is cof-J-supplemented .

In section 3, we introduce the concept of fully invariant J-supplemented modules (for short FI-J-supplemented) as a generalization of J-supplemented, as follows . The module M is said to be FI-J-supplemented , if for every fully invariant submodule C of M , there exists a submodule D of M such that M = C + D and $C \cap D \ll_J D$. Clearly, the supplemented modules are J-supplemented and the J-supplemented modules are FI-J-supplemented . As a generalization of a ⊕-J-supplemented module , we introduce the concept of fully invariant ⊕-J-supplemented modules (FI-⊕-J-supplemented) . A module M is called FI-⊕-J-supplemented , if for every fully invariant submodule C of M , there exists a direct summand D of M such that M = C + D and $C \cap D \ll_J D$. Clearly, FI-⊕-J-supplemented modules are FI-J-supplemented .

2. Cofinitely J-supplemented modules

This section is devoted to introduce the cofinitely J-supplemented modules as a generalization of J-supplemented modules , and illustrate this concept by remarks and properties .

Definition(2.1): A module M is called cofinitely J-supplemented module (for short cof-J-supplemented) if for every cofinite submodule L of M , there exists a submodule N of M such that M = L + N and $N \cap L \ll_J N$.

Remark(2.2): It is clear that every J-supplemented module is cof-J-supplemented . The converse in general is not true . For instance , Q as Z-module is cof-J-supplemented module , but Q is not J-supplemented .

Proposition(2.3): Let M be a finitely generated R-module. Then M is J-supplemented module if and only if M is cof-J-supplemented**.**

Proof: Let L be a submodule of M. Since M is a finitely generated R-module , then $\frac{M}{L}$ is finitely generated , hence L is a cofinite submodule of M . But M is cof-J-supplemented , therefore L is J-supplemented in M. Thus M is J-supplemented module . The converse is clear .

Proposition(2.4): Let M be a cof-J-supplemented module , and let B be a submodule of M , then $\frac{M}{B}$ is a cof-J-supplemented **.**

Proof: Let B be a submodule of M and let $\frac{K}{B}$ be any cofinite submodule of $\frac{M}{B}$, then $\frac{M}{K} \cong \frac{\frac{M}{B}}{\frac{K}{B}}$. Therefore $\frac{M}{K}$ is finitely generated , then K is cofinite submodule of M . Since M is a cof-J-supplemented, then there exists a submodule C of M such that M = K + C , K ∩ C $\ll_J$ C . Now, $\frac{M}{B} = \frac{K+C}{B} = \frac{K}{B} + \frac{C+B}{B}$, $\frac{K}{B} \cap \frac{C+B}{B} = \frac{K \cap (C+B)}{B} = \frac{B+(K \cap C)}{B}$ (by modular law), but K ∩ C $\ll_J$ C , then $\frac{K}{B} \cap \frac{C+B}{B} \ll_J \frac{C+B}{B}$ [2] . Therefore $\frac{M}{B}$ is a cof-J-supplemented.

The converse in general is not true . For example, Z as Z-module $\frac{Z}{3Z} \cong Z_3$ is cof- J-supplemented, but Z is not cof-J-supplemented **.**

Corollary(2.5): The homomorphic image of a cof-J-supplemented module is a cof-J-supplemented module **.**

Proof: Since every homomorphic image is isomorphic to a quotient module .

Corollary(2.6): The direct summand of a cof-J-supplemented module is cof-J-supplemented **.**

Proof : Clear **.**

Proposition(2.7): Let $M = M_1 \oplus M_2$, then M_1 and M_2 are cof-J-supplemented modules if and only if M is cof-J-supplemented **.**

Proof : ($\Longrightarrow$) Let L be a cofinite submodule of M , then M = L+ M_1+ M_2 . Now , $\frac{M_2}{M_2 \cap (L+M_1)} \cong \frac{M_2+L+M_1}{L+M_1} = \frac{M}{L+M_1} \cong \frac{\frac{M}{L}}{\frac{L+M_1}{L}}$, which is finitely generated , hence $M_2 \cap (L + M_1)$ is a cofinite submodule of M_2. Since M_2 is a cof-J-supplemented , then there exists a submodule H of M_2 such that $M_2 = H + [\,M_2 \cap (L + M_1)]$ with $H \cap (L + M_1) \ll_J H$. We have $M = L + M_1 + M_2 = L + M_1 + M_2 \cap (L + M_1) + H = M_1 + L + H$ and since $M_1 \cap (L + H)$ is a cofinite submodule of M_1 and M_1 is a cof-J-supplemented , then there exists a submodule G of M_1 such that $M_1 = G + [\,M_1 \cap (L + H)]$ and $G \cap (L + H) \ll_J H$. Then $M = G + M_1 \cap (L + H) + L + H = L + H + G$ and $(H + G) \cap L \leq [H \cap (L + M_1)] + [\,G \cap (L + H)] \ll_J H + G$. Therefore M is a cof-J-supplemented module .

($\Longleftarrow$) by Corollary(2.4).

To show that the arbitrary sum of a cof-J-supplemented is cof-J-supplemented **,** we need the following standard lemma **.**

Lemma(2.8): Let L and N are submodules of a module M such that L is a cof-J-supplemented , N is cofinite submodule in M, and L + N has J-supplement H in M **.** Then L ∩ (H + N) has J-supplement G in L . Moreover, H + G is J-supplement of N in M **.**

Proof : Let H be J-supplement of L + N in M . Then M = (L + N) + H and (L + N) ∩ H $\ll_J H$. Now , $\frac{L}{L \cap (N+H)} \cong \frac{L+N+H}{N+H} = \frac{M}{N+H} \cong \frac{\frac{M}{N}}{\frac{N+H}{N}}$, which is finitely generated , hence L ∩ (N +H) is cofinite in L **.** But L is a cof-J-supplemented , then there exists a submodule G of L such that L = [L ∩ (H + N)] + G and (H + N) ∩ G $\ll_J G$**.** To prove that H + G is J-supplement of N in M , we have M = L +N + H = L ∩ (H + N) + G + N + H = N + H + G , then M = N + (H + G) . One can easily show that N ∩ (H + G) ⊆ [(G + N) ∩ H + (H + N) ∩ G] $\ll_J H + G$. Therefore H + G is J-supplement of N in M **.**

3. FI-J-supplemented and FI-⊕-J-supplemented modules

In this section , the concept of FI-J-supplemented modules as a generalization of J-supplemented and some properties of this type of modules are given . Also, as a generalization of FI-J-supplemented modules, FI-⊕-J-supplemented modules are introduced .

Definition(3.1): An R-module M is called fully invariant-J-supplemented (for short FI-J-supplemented) if for every fully invariant submodule N of M , there exists a submodule K of M, such that M = N + K and N ∩ K $\ll_J$K **.**

Examples(3.2):

(1) Every semi simple is FI-J-supplemented , for example Z_6 as Z-module is FI-J-supplemented .

(2) Q as Z-module is not FI-J-supplemented, by [2, proposition(2.5)]

(3) It is clear that every J-supplemented is FI-J-supplemented.

The following proposition gives a condition under which the J-supplemented and FI-J-supplemented are equivalent .

Proposition(3.3): Let M be a duo module . Then M is J-supplemented if and only if M is FI-J-supplemented **.**

Proof: Clear .

Proposition(3.4): Let M be FI-J-supplemented module and let N be fully invariant submodule of M. Then the factor $\frac{M}{N}$ is FI-J-supplemented **.**

Proof: Let M be FI-J-supplemented, and let $\frac{B}{N}$ be any fully invariant submodule of $\frac{M}{N}$, then B is a fully invariant submodule in M, by [9, Lemma(2.2)] . Since M is FI-J-supplemented module , then there exists a submodule C of M such that M = C + B , C ∩ B $\ll_J$ C and $\frac{M}{N} = \frac{B+C}{N} = \frac{B}{N1} + \frac{C+N}{N}$. Let ($\frac{B}{N} \cap \frac{C+N}{N}$) + $\frac{V}{N} = \frac{C+N}{N}$ with J ($\frac{C+N}{V}$) = $\frac{C+N}{V}$, $\frac{B \cap (C+N)}{N} = \frac{N+(B \cap C)}{N}$ (by modular law), then $\frac{N+(B \cap C)}{N} + \frac{V}{N} = \frac{C+N}{N}$, and N + (B ∩ C) + V = C + N , and N ⊆ V , then (B ∩ C) + V = C + N, and J($\frac{C+N}{V}$) = $\frac{C+N}{V}$. But B ∩ C $\ll_J$ C ⊆ C+ N and by [2, Proposition(2.6(4))] , B ∩ C $\ll_J$ C + N , thus V = C + N and $\frac{V}{N} = \frac{C+N}{N}$. Then $\frac{C+N}{N}$ is J-supplementl of $\frac{B}{N}$ in $\frac{M}{N}$, $\frac{C+N}{N}$ is FI-J-supplement of $\frac{B}{N}$ in $\frac{M}{N}$. Therefore $\frac{M}{N}$ is FI-J-supplemented .

Proposition (3.5) : Let M_1 and U be fully invariant submodules of M , and let M_1 be FI- J-supplemented module . If M_1+ U has FI- J–supplement in M, then so does U .

Proof : Since M_1+ U has FI- J–supplement in M , then there exists a fully invariant X ⊆ M , such that X + (M_1+ U) = M , and X ∩ (M_1+ U) $\ll_J$ X . Since M_1 is FI- J-supplemented module , then there exists Y ⊆ M_1 such that (X + U) ∩ M_1 + Y = M_1 and (X + U) ∩Y $\ll_J$ Y . Thus we have M = M_1+ U + X = (X + U) ∩ M_1 + Y + U + X = X + U + Y , and (X + U) ∩Y $\ll_J$ Y , that is Y is FI- J–supplement of X + U in M . It is clear that (X + Y) + U = M , so it suffices to show that (X + Y) ∩ U $\ll_J$ X + Y since Y + U ⊆ M_1 + U , then X ∩ (Y + U) ⊆ X ∩(M_1 + U) $\ll_J$ X and X ∩ (Y + U) $\ll_J$ X by[2, Proposition(2.6(1))] . Thus by [2 ,Proposition(2.6(4))] , (X + Y) ∩ U ⊆ X ∩(Y + U) + Y ∩ (X + U) $\ll_J$ X + Y .

Proposition (3.6) : Let M = M_1⊕ M_2 , then $M_1$and M_2 are FI-J–supplemented modules if and only if M is FI-J–supplemented module.

Proof : (⟹) Let K be a fully invariant submodule of M , then since M_1+ M_2 + K = M , it trivially has FI- J-supplement in M, by Proposition (3.5), then M_2 + K and K have FI-J-supplement in M. Also, by Proposition (3.5) again, K has FI- J–supplement in M , so M is FI- J–supplemented module **.**

(⟸) $M_2 \cong \frac{M}{M_1}$, since M is FI-J-supplemented module, and by Proposition (3.4), $\frac{M}{M_1}$ is FI- J-supplemented module . Thus M_2 is FI-J–supplemented module . Similarly, M_1 is FI-J–supplemented modulel **.**

Corollary(3.7): Let $M = \oplus_{i=1}^{n} M_i$ be a direct sum of FI-J-supplemented R-modules . Then M is FI-J-supplemented.

Proof: Let n be any positive integer and let M_i be FI-J-supplemented R-module for each $1 \leq i \leq n$. Let $M = M_1 \oplus \ldots \oplus M_n$. To prove that M is FI-J-supplemented R-module, it is sufficient by the induction on n to prove that this is the case when $n = 2$. Thus suppose that $n = 2$.

Let A be any fully invariant submodule of M, then $= M_1 + M_2 + A$. Then since $M_1$and M_2 are FI-J-supplemented , then by proposition(3.5) , we have $M_2 + A$ has FI-J-supplement in M, and by proposition(3.5) again, A has FI-J-supplement in M . Therefore $M = M_1 \oplus M_2$ is FI-J-supplemented .

Corollary (3.8) : Let M = M_1 ⊕ M_2 be a duo module , N and L are fully invariant submodules of M_1 , if N is FI- J-supplement of L in M_1 , then N ⊕ M_2 is FI- J-supplement of L in M **.**

Proof : Let N be FI-J–supplement of L in M_1 , then M_1 = N + L and N ∩ L $\ll_J$ N. Since M = M_1⊕ M_2 , then M = (N + L) ⊕ M_2 , hence M = L+(N ⊕ M_2). But (N ⊕ M_2) ∩ L = (N ⊕ M_2) ∩ M_1 ∩ L = N ∩ L $\ll_J$ N, then N ∩ L $\ll_J$ N ⊕ M_2 [2] , hence N ⊕ M_2 is FI- J-supplement of L in M **.**

Proposition (3.9) : Let U and V are fully invariant submodules of an R–module M and let V be FI- J-supplement of U in M . If K $\ll_J$ M , then V is FI-J-supplement of U + K in M **.**

Proof : Let V be FI- J-supplement of U in M , then M = V + U and V ∩ U $\ll_J$ V . Let V + (U + K) = M , and let V ∩ (U + K) + X = V , with J ($\frac{V}{X}$) = $\frac{V}{X}$, M = V + (U + K) = V ∩ (U + K) + X + (U + K) = X + (U + K) = (U + X) + K, since $\frac{M}{U+X} = \frac{V+(U+K)+X}{U+X} = \frac{V+(U+X)}{(U+X)} \cong \frac{V}{V\cap(U+X)} = \frac{V}{X+(U\cap V)}$, by the second isomorphism and modular law . Since J($\frac{V}{X}$) = $\frac{V}{X}$, we get J($\frac{V}{X+(U\cap V)}$) = $\frac{V}{X+(U\cap V)}$ [2 Proposition(2.2)] , hence J($\frac{M}{U+X}$) = $\frac{M}{U+X}$. Since K $\ll_J$ M then M = U + X , but M = U + V, X ⊆ V, and J($\frac{V}{X}$) = $\frac{V}{X}$, then V = X **.** Thus V is FI-J-supplement of U + K in M **.**

Proposition (3.10) : Let M be any R–module, V be FI-J–supplement of W in M , and K be fully invariant of M such that K ⊆ V .Then K $\ll_J$ M if and only if K $\ll_J$ V .

Proof : (⟹) Let K + X = V with J($\frac{V}{X}$) = $\frac{V}{X}$. Since V + W = M and V ∩ W $\ll_J$ V , then M= (K + X) + W. Hence M = K + (X + W) to show that J($\frac{M}{X+W}$) = $\frac{M}{X+W}$, since $\frac{M}{X+W} = \frac{V+(X+W)}{(X+W)} \cong \frac{V}{V\cap(X+W)} = \frac{V}{X+(V\cap W)}$ by the second isomorphism and modular law. But J($\frac{V}{X}$) = $\frac{V}{X}$, then we get

J($\frac{V}{X+(V\cap W)}$) = $\frac{V}{X+(V\cap W)}$ [2, Proposition(2.2)] . Hence J($\frac{M}{X+W}$) = $\frac{M}{X+W}$. Since K $\ll_J$ M then M = X + W . Now M = V + W, X ⊆ V, and J($\frac{V}{X}$) = $\frac{V}{X}$, then V = X . Hence K $\ll_J$ V .
(⟸) Clearly by [2 ,Proposition(2.6(4)] **.**
Proposition (3.11) : Let M be any R–module and let V be FI-J-supplement of U in M , K and T are fully invariant submodules of M such that K , T ⊆ V. Then T is FI-J-supplement of K in V if and only if T is FI-J-supplement of U + K in M **.**
Proof : (⟹) Let T be FI-J-supplement of K in V , then V = T + K and T ∩ K $\ll_J$ T . Let (U + K) + L = M for L ⊆ T with J($\frac{T}{L}$) = $\frac{T}{L}$. Now K + L ⊆ V. Since $\frac{V}{K+L} = \frac{T+(K+L)}{K+L} \cong \frac{Tl}{T\cap(K+L)} = \frac{T}{L+(K\cap T)}$ by the second isomorphism and modular law) , and J($\frac{T}{L}$) = $\frac{T}{L}$, we get J($\frac{T}{L+(K\cap T)}$) = $\frac{T}{L+(K\cap T)}$ [2] . Hence J($\frac{V}{K+L}$) = $\frac{V}{K+L}$ and because V is FI-J-supplement of U in M, then M = U + V, and by [2] , K + L = V. Since L ⊆ T and T is FI-J-supplement of K in V, then T = L **.** Hence T is FI-J-supplement of U + K in M.
(⟸) Let T is FI-J-supplement of U + K in M .Then T + (U + K) = M and T ∩ (U + K) $\ll_J$ T. Let T + K = V . Since T ∩ K ⊆ T ∩ (U + K) $\ll_J$ T , then by [2 ,Proposition(2.6(1))] , T ∩ K $\ll_J$ T . Hence T is FI-J-supplement of K in V **.**

Let U , V be submodules of a module M . We will say that U and V are mutual FI-J-supplements , if U is FI-J-supplement of V in M and V is FI-J-supplement of U in M .
Corollary (3.12) : Let M be any R–module and let U and V be mutual FI-J-supplements in M . Let L be
FI-J-supplement of S in U and T be FI-J-supplement of K in V , then L + T is FI-J-supplement of K + S in M .
Proof: Since U = S + L and V is FI-J-supplement of U in M , then by Proposition(3.11), T is FI-J-supplement of S + L + K in M and then (S + L + K) ∩T $\ll_J$ T. Since V = K + T and U is FI-J-supplement of V in M , then by Proposition (3.11) , L is J–supplement of S + K + T in M and then (S + K + T) ∩ L $\ll_J$ L . Because U = S + L , V = K + T , and M = U + V , then we have M = S + L + K + T = S + K + L + T . Then by [2 ,Proposition(2.6(2)] , (S + K) ∩ (L + T) ⊆ L ∩ (S + K + T) + T ∩ (S + K + L) $\ll_J$ L + T . And since L and T are fully invariant in M , then L + T is fully invariant in M [10] . Therefore L + T is FI-J-supplement of K + S in M **.**
Definition(3.13): An R-module M is called fully invariant ⊕-J-supplemented (for short FI-⊕-J-supplemented) if for every fully invariant submodule N of M , there exists a direct summand K of M, such that M = N + K and N ∩ K $\ll_J$K **.**
Examples(3.14):
(1) It is clear that every FI- ⊕-J-supplemented is FI- J-supplemented . But the converse in general is not true , for example Z as Z-module .
(2) Z_6 as Z-module is FI-⊕-J-supplemented .
(3) It is clear that every a ⊕-J-supplemented is FI- ⊕-J-supplemented .
(4) Q as Z-module is not FI-⊕-J-supplemented .

The following proposition gives a condition under which the ⊕-J-supplemented and FI- ⊕-J-supplemented are equivalent .
Proposition(3.15): Let M be a duo module. Then M is a ⊕-J-supplemented if and only if M is FI-⊕-J-supplemented module **.**
Proof: We have to show that M is a ⊕-J-supplemented module . Let A be a submodule of M . Since M is a duo module , then A is a fully invariant submodule of M . But M is FI-⊕-J-supplemented module . Hence A has a ⊕-J-supplementl in M . Therefore M is a ⊕-J-supplemented module . The converse is clear **.**
Proposition(3.16): Let M be an R-module . Then M is FI-⊕-J-supplemented module if and only if for every fully invariant submodule N of M , there exists a direct summand K of M such that M = N + K and N ∩ K $\ll_J$M **.**
Proof: See [2, Proposition(2.7)] **.**

Proposition(3.17): Let M be FI-⊕-J-supplemented module and let A be fully invariant submodule of M . Then the factor $\frac{M}{A}$ is FI-⊕-J-supplementedl module **.**

Proof: Let $\frac{B}{A}$ be any fully invariant submodule of $\frac{M}{A}$. Then B is a fully invariant submodule of M by [9, Lemma(2.1)]. Since M is FI-⊕-J-supplemented module, then there exists a direct summand C of M such that M = C + B , C ∩ B $\ll_J$ C, M = $C \oplus \acute{C}$, $\acute{C} \leq M$ and $\frac{M}{A} = \frac{B+C}{A} = \frac{B}{A} + \frac{C+A}{A}$. Let ($\frac{B}{A} \cap \frac{C+A}{A}$) + $\frac{Vl}{A} = \frac{C+A}{A}$ with J ($\frac{C+A}{V}$) = $\frac{C+A}{V}$, $\frac{B \cap (C+A)}{A} = \frac{A+(B \cap C)}{A}$ by the modular law, then $\frac{A+(B \cap C)}{A} + \frac{V}{A} = \frac{C+A}{A}$, A + (B ∩ C) + V = C + A , and A ⊆ V. Then (B ∩ C) + V = C + A , and J($\frac{C+A}{V}$) = $\frac{C+A}{V}$. But B ∩ C $\ll_J$ C ⊆ C + A and by [2 ,Proposition(2.6(4))] , B ∩ C $\ll_J$ C + A , thus V = C + A and $\frac{V}{A} = \frac{C+A}{A}$. Then $\frac{C+A}{A}$ is J-supplement of $\frac{B}{A}$ in $\frac{M}{A}$. Since A is a fully invariant submodule of M and M = $C \oplus C'$, then $\frac{M}{A} = \frac{C+A}{A} \oplus \frac{C'+A}{A}$, and $\frac{C+A}{A}$ is a direct summand of $\frac{M}{A}$ by [5, Lemma(5.4)] .Therefore **,** $\frac{C+A}{A}$ is a ⊕-J-supplement of $\frac{B}{A}$ in $\frac{M}{A}$, hence $\frac{C+A}{A}$ is FI-⊕-J-supplement of $\frac{B}{A}$ in $\frac{M}{A}$ .Then $\frac{M}{A}$ is FI-⊕-J-supplemented **.**

The converse is not true in general . For example Z as Z-module, $\frac{Z}{6Z} \cong Z_6$ is FI-⊕-J-supplemented but Z is not FI-⊕-J-supplemented **.**

Proposition(3.18): Let M_1 and K are fully invariant submodules of M , and let M_1 be FI-⊕-J-supplemented module . If M_1+ K has FI- ⊕-J-supplement in M , then so does K **.**

Proof: Since M_1+ K has FI-⊕-J-supplement in M, then there exists a direct summand fully invariant X of M, such that (M_1+ K) + X = M, and (M_1+ K) ∩ X$\ll_J$ X. Since M_1 is FI-⊕-J-supplemented module, then there exists a direct summand Y of M_1 such that (X+ K) ∩ $M_1 + Y = M_1$ and (X+ K) ∩ $Y \ll_J Y$. We have M = M_1+ K + X= $(X+K) \cap M_1 + Y + K + X = Y + K + X$, then M = Y+ K + X, and (X+ K) ∩ $Y \ll_J Y$, that is Y is FI-⊕-J-supplement of X + K in M. Next , we show that X + Y is FI-⊕-J-supplement of K in M. It is clear that M = K + (X +Y) , so it suffices to show that (X + Y) ∩ K $\ll_J$ X +Y. Since $Y + K \subseteq M_1 + K$, then $X \cap (Y+K) \subseteq X \cap (M_1 + K) \ll_J X$, and by [2,Proposition(2.6(1))] then $X \cap (Y+K) \ll_J X$ .Thus by [2, Proposition(2.6(5))] , $(X+Y) \cap K \subseteq X \cap (Y+K) + Y \cap (X+K) \ll_J X + Y$.

Proposition(3.19): Let $M = M_1 \oplus M_2$, and M_1 andl M_2 are FI-⊕-J-supplemented modules if and only if M is FI-⊕-J-supplemented.

Proof: (⟹) Suppose that $M = M_1 \oplus M_2$, and M_1 and M_2 are FI-⊕-J-supplemented modules. Let K be a fully invariant submodule of M. Since $M_1 + M_2 + K = M$, it trivially has FI-⊕-J-supplement in M. By proposition(3.18) , then $M_2 + K$ has FI-⊕-J-supplement in M, and by proposition(3.18) again, K has FI-⊕-J-supplement in M , so M is FI-⊕-J-supplemented module.

(⟸) Suppose that $= M_1 \oplus M_2$, and M is FI-⊕-J-supplemented module. To show that M_1 and M_2 are FI-⊕-J-supplemented modules . Since $M_2 \cong \frac{M}{M_1}$ and M is FI-⊕-J-supplemented module, then by Proposition (3.17), $\frac{M}{M_1}$ is FI-⊕-J-supplemented module. Thus M_2 is FI-⊕-J-supplemented module . Similarity M_1 is FI-⊕-J-supplemented module **.**

Corollary (3.20) : Let M = $M_1 \oplus M_2$ be a duo module , and K and L are fully invariant submodules of M_1 . If K is FI-⊕-J-supplement of L in M_1 , then K ⊕ M_2 is FI-⊕-J-supplement of L in M .

Proof : Let K be FI-⊕-J-supplement of L in M_1 , then M_1 = K + L , K is a direct summand of M_1 and K ∩ L $\ll_J$ K. Since M = $M_1 \oplus M_2$, then M = (K + L) ⊕ M_2 , hence M = L+(K ⊕ M_2) but (K ⊕ M_2) ∩ L = (K ⊕ M_2) ∩ M_1 ∩ L = K ∩ L $\ll_J$ K . And by [2, Proposition(2.6(4))] , then K ∩ L $\ll_J$ K ⊕ M_2 , hence K ⊕ M_2 is FI-⊕-J-supplement of L in M **.**

Theorem(3.21): Let M be a module such that $M = M_1 \oplus M_2$ is a direct sum of submodules M_1 and M_2. Then M_2 is FI-⊕-J-supplemented module if and only if there exists a direct summand Y of M such that Y ⊆ M_2 , M = X + Y and X ∩ Y $\ll_J$ Y, for every fully invariant submodule $\frac{X}{M_1}$ of $\frac{M}{M_1}$.

Proof: (⟹) Let $\frac{X}{M_1}$ be any fully invariant submodule of $\frac{M}{M_1}$. Then X ∩ M_2 is fully invariant submodule of M_2 by[12, Lemma(2.3)] **.** Since M_2 is FI-⊕-J-supplemented module , then there exists a

direct summand Y of M_2 such that $M_2 = (X \cap M_2) + Y$ and $X \cap M_2 \cap Y = X \cap Y \ll_J Y$ .Clearly , Y is a direct summand of M and $M = M_1 + M_2 = M_1 + (X \cap M_2) + Y \subseteq M_1 + X + Y$, but $M_1 \subseteq X$, therefore M = X+ Y . So we get the result **.**
(⟸) To show that M_2 is FI-⊕-J-supplemented, let X be a fully invariant submodule of M_2**.** Then $\frac{X \oplus M_1}{M_1}$ is a fully invariant submodule of $\frac{M}{M_1}$ by [12, Lemma(2.3)] **.** By our assumption, there exists a direct summand Y of M such that $Y \subseteq M_2$, M =(X + M_1) + Y and $(X + M_1) \cap Y \ll_J Y$. Since $M_2 = M_2 \cap M = M_2 \cap [(X + M_1) + Y] = Y + [(X + M_1) \cap M_2] = Y + X + (M_1 \cap M_2) = X + Y$, by the modular law, and since $X \cap Y \subseteq (X + M_1) \cap Y \ll_J Y$, then by [2, Proposition(2.6(1))] we get $X \cap Y \ll_J Y$. Therefore Y is FI-⊕-J-supplement of X in M_2. Thus M_2 is FI-⊕-J-supplemented module **.**

Theorem(3.22): Let M_2 be a direct summand of FI-⊕-J-supplemented module M , such that for every direct summand K of M with M = K + M_2 , K ∩ M_2 is a direct summand of M . Then M_2 is FI-⊕-J-supplemented module **.**

Proof : Suppose that $M = M_1 \oplus M_2$ and let $\frac{N}{M_1}$ be a fully invariant submodule of $\frac{M}{M_1}$. Consider the fully invariant submodule N ∩ M_2 of M . Since M is FI-⊕-J-supplemented module , then there exists a direct summand K of M such that M = (N ∩ M_2)+ K and N ∩ M_2 ∩ K $\ll_J$ K . By [3, Lemma(1.2)] , M = (K ∩ M_2) + N . Since M = K + M_2 , then K ∩ M_2 is a direct summand of M by hypothesis, and by theorem(3.21), M_2 is FI-⊕-J-supplemented module **.**

Lemma(3.23): Let X and Y be fully invariant submodules of a module M such that X + Y has a ⊕-J-supplement H in M and X ∩ (H + Y) has a ⊕-J-supplement G in X . Then H + G is a ⊕-J-supplement of Y in M **.**

Proof: Let H be a ⊕-J-supplement of X + Y in M and let G be a ⊕-J-supplement of X ∩ (H + Y) in X . Then M = (X + Y) + H such that (X + Y) ∩ H $\ll_J$ H , X = [X ∩ (H + Y)] + G such that (H + Y) ∩ G $\ll_J$ G . Since M = X + Y + H = X ∩ (H + Y) + G + Y + H = Y + H + G , then M = Y + (H + G) . But G + Y ⊆ X + Y , then (G + Y) ∩ H ⊆ (X + Y) ∩ H $\ll_J$ H , and by [2, Proposition(2.6(1))], (G + Y) ∩ H $\ll_J$ H . Thus Y ∩ (H + G) ⊆ [(G + Y) ∩ H + (H + Y) ∩ G] $\ll_J$ H + G **.**`

Theorem(3.24): For any ring R, any finite direct sum of FI-⊕-J-supplemented R-modules is FI-⊕-J-supplemented **.**

Proof: Let n be any positive integer and let M_i be FI-⊕-J-supplemented R-module for each $1 \leq i \leq n$. Let $M = M_1 \oplus \dots \oplus M_n$. To prove that M is FI-⊕-J-supplemented R-module, it is sufficient by the induction on n to prove this is the case when $n = 2$. Thus suppose that $n = 2$.
Let X be any fully invariant submodule of M . Then $M = M_1 + M_2 + X$ so that $M_1 + M_2 + X$ has a ⊕-J-supplement 0 in M . Since M_2 is FI-⊕-J-supplemented , then $M_2 \cap (M_1 + X)$ has a ⊕-J-supplement H in M_2 such that H is a direct summand of M_2 . By lemma (3.23) , H is a ⊕-J-supplement of $M_1 + X$ in M . Since $M_1$is FI-⊕-J-supplemented, $M_1 \cap (X + H)$ has a ⊕-J-supplement K in M_1 such that K is a direct summand of M_1. Again by lemma(3.23) , H + K is a ⊕-J-supplement of X in M . Since H is a direct summand of M_2 and K is a direct summand of M_1, it follows that H + K = H ⊕ K is a direct summand of M . Thus $M = M_1 \oplus M_2$ is FI-⊕-J-supplemented **.**

References

1. Inoue , T. **1983**. Sum of hollow modules , *Osaka J. Math* , : 331-336
2. Kabban , Ali , Khalid.W. **2019**. On Jacobson-small submodules, *Iraqi Journal of science* 60(7) , 1584-1591.
3. Keskin , D. **2000**. On Lifting Modules , *Comm. Algebra*, : 3427-3440.
4. Wisbauer , R. **1991** . *Foundations of Module and Ring Theory* , Gordon and Breach , Philadelphia
5. Haramanci , A., Keskin , D and Smith, P. F.**1999**. On ⊕-supplemented modules , *Acta Math. Hungar*. , 83(1-2) ,161-169.
6. Ali, A., Khalid ,W. **2020**. ⊕ Jacobson supplemented modules , *J. Phys. Conf. Ser.,* 1-14.
7. Alizade , R., Bilhan , G., Smith , P . F. (**2001**). Modules whose maximal submodules have supplementes. *Comm. Algebra* 29(6): 2389-2405 .
8. **8.** Orhan , N. , Tutuncu , D. K. and Tribak , R. **2007**. On Hollow- lifting modules , *Taiwanese J. Math*, 545-568.

9. Talibi ,Y., Amoozegar ,T. **2009.** Fully invariant τ_M –lifting modules , *Albanian. J. of Mathematics* , 49-53 .

10. Birkenmeier , G. F., Muller, B. J., Rizvi, S. T. **2002**. Modules in which every fully invariant submodules is essential in a direct summand , *Comm. Algabra* , 30,1395-1415 .

11. $\ddot{O}$zcan, A .C., Harmanci, A. and Smith , P. F. **2006**. Duo modules , *Glasgow Math. J .Trust*., 48, 533-545.

12. Talibi , Y. Amoozegar , T. **2008**. Strongly FI- lifting modules , *International electronic J. Algebra* 3, 75-82 .