DOI: 10.24996/ijs.2021.62.4.24

ISSN: 0067-2904

On Closed Quasi Principally Injective Acts over Monoids

Shaymaa Amer Abdul-Kareem[1]*, Ahmed Amer Abdulkareem2, Yusra Amer Abdul-Kareem[3]

[1]Department of Mathematics, College of Basic Education, Mustansiriyah University, Baghdad, Iraq

[2]Ministry of trade, Department of Follow-up and Monitoring Electronic

[3]Ministry of trade, Department of Foreign and Economic Relations

Received: 8/6/2020 Accepted: 5/9/2020

Abstract

The concept of closed quasi principally injective acts over monoids is introduced ,which signifies a generalization for the quasi principally injective as well as for the closed quasi injective acts. Characterization of this concept is intended to show the behavior of a closed quasi principally injective property. At the same time, some properties of closed quasi principally injective acts are examined in terms of their endomorphism monoid. Also, the characterization of a closed self-principally injective monoid is given in terms of its annihilator. The relationship between the following concepts is also studied; closed quasi principally injective acts over monoids, Hopfian, co Hopfian, and directly finite property. Ultimately, based on the results obtained, the conditions on subacts to inherit a closed quasi principally injective property were shown. Part of this paper was dedicated to studying the relationship between the classes of closed quasi principally injective acts with some generalizations of injectivity. Conclusions and future remarks of this work are given.

Keywords: Quasi Principally Injective Acts, Closed Quasi Principally Injective Acts, Closed Self Principally Injective Monoid, Continuous Acts, Extending Acts, Generalization of Quasi Principally injective acts. **AMS Subject Classification:** 20M30, 20M99, 08B30.

حول الانظمة شبه الرئيسية الاغمارية المغلقة على اشباه الزمر الاحادية

شيماء عامر عبد الكريم[1]*، احمد عامر عبد الكريم[2] ، يسرى عامر عبد الكريم[3]

[1]قسم الرياضيات،كلية التربية الاساسية، الجامعة المستنصرية،بغداد،عراق

[2]وزارة التجارة، دائرة تسجيل الشركات، قسم المتابعة والرصد الالكتروني

[3]وزارة التجارة، دائرة العلاقات الاقتصادية الخارجية، قسم الدراسات والترجمة

*Email: Shaymma_amer.edbs@uomustansiriyah.edu.iq

الخلاصة

تم تقديم مفهوم الأنظمة شبه الرئيسية الاغمارية المغلقة على اشباه الزمرالاحادية وهو يمثل اعمام الى الانظمة شبه الرئيسية الاغمارية بالاضافة الى انه يمثل ايضا اعمام الى الانظمة شبه الاغمارية المغلقة. الغرض من خصائص هذا المفهوم هو إظهار سلوك الخاصية شبه الرئيسية الاغمارية المغلقة. في الوقت نفسه ، تم فحص بعض خصائص الانظمة شبه الرئيسية الاغمارية المغلقة بدلالة تشاكلها الداخلي. أيضا ، يتم وصف خصائص شبه الاغمارية الذاتية المغلقة بدلالة annihilator. تم دراسة العلاقة بين المفاهيم التالية Hopfian ، co-Hopfian، directly finite property. في النهاية ، استنادا إلى النتائج التي حصلنا عليها ، تم عرض شروط على الانظمة الجزئية لوراثة خاصية الانظمة شبه الرئيسية الاغمارية المغلقة. تم تخصيص جزء من هذه الورقة لدراسة العلاقة بين الانظمة شبه الرئيسية الاغمارية المغلقة مع بعض اعمامات الانظمة الاغمارية. تم تقديم الاستنتاجات وعمل مستقبلي لهذا العمل.

1. Introduction

For any given mathematical structure on a set, the collection of structure-preserving maps on the set to itself is an example of an abstract algebraic “object”, referred to as a semigroup. Thereby, semigroups pervade mathematics. On the other hand, given an abstractly defined semigroup, when can it be represented as a semigroup of maps of a mathematical structure? The answer is represented by actions. In mathematics, an action of a semigroup on a set is an operation that associates each element of the semigroup with a transformation on the set. It is familiar that, from an algebraic perspective, an action for the semigroup is a generalization of the notion of group action in group theory, and a major special case is a monoid action or act, in which the semigroup is a monoid and the identity element of the monoid acts as the identity transformation of a set. It is recognized that the theory of monoids and acts is a generalization of the theory of rings and modules, which has a number of direct applications in theoretical computer science, theory of differential equations and functional analysis, etc. [1].

Throughout this work, every right S-act M is a unitary S-act (contains identity element), with zero element Θ represented by M_S, and S is a monoid with zero elements 0. Let M_S refers to a right S-act with zero where it is a non-empty set with a function f: $M\times S\longrightarrow M$, $(m, s) \longmapsto ms$ such that the following properties hold: **(1)** $m\cdot 1= m$ **(2)** $m (st) = (ms) t$, for all $m\in M$ and $s, t\in S$, where 1 denotes the identity element of S. For other basic definitions, theorems, lemmas, corollaries, results and notations for S-acts, annihilators, homomorphism, endomorphism, monomorphism, epimorphism, isomorphism …etc. we refer to [2, 3 and 4].

It is possible to find an S-act in different names such as S-acts, S-sets, S-operands, S-polygons, transition acts, and S-automata [2].We will freely make use of the standard notations, terminologies as well as results of [1, 5, 6, 7, 8 and 9].

Let A_S and M_S be two S-acts. A_S is referred to as an M-injective in case of an S-monomorphism α: $N\rightarrow M_S$ where N is a subact of M_S and every S-homomorphism β: $N\longrightarrow A_S$, can be extended to an S-homomorphism σ: $M_S\longrightarrow A_S$ [10].

An S-act A_S is an injective if it is an M-injective for all S-acts M_S. An S-act A_S is quasi injective if and only if it is an A-injective. Quasi injective S-acts were studied by Lopez and Luedeman [11]. In [1], the author developed the concept presented by Lopez to C-quasi injective act. An S-act N_S is called closed M-injective (for short C-M-injective) if for any homomorphism from a closed subact of S-act M_S to N_S can be extended to a homomorphism from M_S to N_S [1]. An S-act N_S is referred to as a C-quasi injective if N_S is C-N-injective. In a similar way, a monoid S is called the right C-self-injective if it is C-S-injective. Besides, the author continues to develop and generalize the concept of the quasi injective act introduced by Lopez to quasi principally injective act. An S-act N_S is called M-principally injective if every S-homomorphism from M-cyclic subact of S-act M_S into N_S can be extended to an S-homomorphism from M_S into N_S (for short N_S is M-P-injective) [8]. Equally, an S-act M_S is referred to as quasi-principally injective if it is M-P-injective [8]. In [12], Al-Bahrani and Rahman introduced a generalization of Rickart Modules to y-closed Rickart Modules. Because the S-act theory is a generalization of module theory, we introduced the generalization of quasi principally injective acts over monoids to closed quasi principally injective acts over monoids.

This paper aims to introduce and study the concept of closed quasi principally injective acts by examining their structure and properties. The importance of this concept is attributed to two points: Firstly, it represents a generalization of closed quasi injective acts and, secondly, it signifies a generalization of quasi principally injective acts. In addition, we characterized the behavior of the property that is considered under well-known constructions such as the product, coproduct, and direct sum. This article is divided into three sections. Section two is devoted to introduce and investigate a new kind of generalization of quasi principally injective S-acts, namely closed quasi principally injective act over monoids. Certain classes of subacts which inherit the property of closed quasi principally injective acts were considered. Also, the characterizations of this new class of S-acts were investigated. An example was given to demonstrate closed quasi principally injective acts over monoids. Some known results on closed quasi principally injective for general modules were generalized to S-acts.

For future work, one can consider the subact that is closed and finitely M-generated.

2. Results

Definition2.1: [8] Let M_S and N_S be two S-acts. An S-act N_S is called M-principally injective if every S-homomorphism of M-cyclic subact of M_S into N_S can be extended to an S-homomorphism from M_S into N_S (if this is the case, we write N_S as M-P-injective).

Definition2.2: [1] Let M_S and N_S be two S-acts, N_S is called closed M-injective (for short C-M - injective) if any homomorphism of a closed subact of M_S to N_S can be extended to homomorphism from M_S to N_S. An S-act N_S is called closed quasi injective if N_S is C-N-injective. A monoid S is called right closed self-injective if it is C-S-injective.

Definition2.3: An S-act N_S is called closed M-principally injective (for short, C-M-P-injective) if every S-homomorphism of closed M-cyclic subact of M_S to N_S extends to S-homomorphism from M_S to N_S. Meanwhile, an S-act M_S is called closed quasi principally injective (for short, C-QP-injective) if it is closed M-principally injective. Similarly, a monoid S is called closed self principally injective monoid (for short, C-self-P-injective) in case that S_S is closed quasi principally injective.

Remarks and Example2.4

(1) Recall that an S-act M_S is called quasi-principally injective if it is M-P-injective, that is every S-homomorphism from M-cyclic subact of M_S to M_S can be extended to S-endomorphism of M_S. Accordingly, we mention that M_S is QP-injective [8]. For this reason, every QP-injective is C-QP-injective, but the converse is not true in general. For example, Z with usual multiplication monoid as Z-act is C-QP-injective which cannot be called quasi principally injective act.

(2) Obviously, definition2.3 is up to isomorphism. This means that every object may be replaced by an isomorphic object.

Recall that α is an S-homomorphism if it is a mapping (i.e.α: $A_S \longrightarrow B_S$) from S-act A_S into S-act B_S such that for any $a \in A_S$ and $s \in S$, $\alpha(as) = \alpha(a)s$. The usual meanings of monomorphism, epimorphism, and isomorphism are also satisfied [7]. Besides, an S-homomorphism f: $M_S \longrightarrow M_S$ is called an endomorphism of M_S, where M_S is S-act.

Definition 2.5: An endomorphism f∈End(M) is called a closed homomorphism if f(M) is a closed subact of M_S.

The following theorem illustrates the characterization of C-quasi principally injective act (for definition of annihilators see definition (1.1.27) in [7]).

Theorem 2.6: Let M_S be an S-act and T=End (M). Then the following conditions are equivalent:

(1) M_S is C-M-P-injective.

(2) $\ell_T(\ker\alpha) = T\alpha$ For every closed homomorphism $\alpha \in T$.

(3) If f: $\alpha(M) \longrightarrow M_S$ is a homomorphism, then $f\alpha \in T\alpha$ for closed homomorphism $\alpha \in T$.

(4) $\mathrm{Ker}\alpha \subseteq \mathrm{Ker}\beta$ implies that $T\beta \subseteq T\alpha$ for any $\alpha, \beta \in T$, where α is a closed homomorphism.

(5) $\ell_T[(\beta(M) \times \beta(M)) \cap \mathrm{Ker}\alpha] = \ell_T(\beta(M) \times \beta(M)) \cup T\alpha$, for each $\alpha, \beta \in T$ where ($\alpha\beta$) is a closed homomorphism and T is right cancellative.

Proof: (1→2) Let $\beta \in T\alpha$ be a closed homomorphism. For any $\sigma \in T$, we have then that $\beta = \sigma\alpha$, so Ker (α) ⊆ Ker ($\sigma\alpha$). For each s, t∈S with ms = mt, we have α(ms) = α(mt) and then $\sigma\alpha$(ms) = $\sigma\alpha$(mt). This implies that β(ms) = β(mt), so $\beta \in \ell_T(\mathrm{Ker}\sigma\alpha) \subseteq \ell_T(\mathrm{Ker}\alpha)$. Conversely, let $\beta \in \ell_T(\mathrm{Ker}\alpha)$, then define$\sigma$: $\alpha(M_S) \to M_S$ by $\sigma(\alpha(m)) = \beta(m)$ for some m∈M_S. It is clear that σ is a well-defined and S-homomorphism with Kerα ⊆ Kerβ. In fact, if α(ms) =α(mt), then $\sigma(\alpha(ms))$ = $\sigma(\alpha(mt))$.Thus, β(ms)=β(mt), which means that σ is well-defined. From this, it is easy to see that σ is S-homomorphism. Let $\alpha^{/}$: $M_S \longrightarrow \alpha(M_S)$, and $\beta^{/}$: $M_S \longrightarrow \beta(M_S)$ be S-epimorphisms induced by α and β, respectively. Let i_1: $\alpha(M_S) \longrightarrow M_S$, i_2: $\beta(M_S) \longrightarrow M_S$ be the inclusion maps.Since $\alpha^{/}$ is S-epimorphism, so there is an S-homomorphism φ:$\alpha(M_S) \longrightarrow \beta(M_S)$ such that $\varphi\alpha^{/}=\beta^{/}$, ∀m∈$M_S$. Let $\alpha(m) \in \alpha(M_S)$. Since $\alpha^{/}$ is epimorphism, so there exists $x \in M_S$ such that $\alpha^{/}(x)=\alpha(m)$, so $\varphi(\alpha(m))=\beta(x)$, where $\alpha^{/}(x)$ =α(m) and $\beta^{/}(x)=\beta(x)$. Now, φ is well-defined. If $\alpha(m_1)=\alpha(m_2)$ with $\alpha^{/}(x_1)$ =$\alpha(m_1)$ and $\alpha^{/}(x_2)=\alpha(m_2)$, then, ($x_1$, x_2)∈Ker($\alpha^{/}$)=Ker (α)⊆Ker(β)=Ker($\beta^{/}$). So, $\beta^{/}(x_1)$ = $\beta^{/}(x_2)$. Since M_S is C-M-P-injective, so there exists an S-homomorphismσ: $M_S \longrightarrow M_S$ such that σi_1 = $i_2\varphi$, then $\sigma i_1\alpha^{/}=i_2\varphi\alpha^{/}$.This implies that $\sigma i_1\alpha^{/}=i_2\beta^{/}$,then$\beta = \sigma\alpha$.Therefore,$\beta \in T\alpha$ and $\ell_T(\mathrm{Ker}\alpha) \subseteq T\alpha$. Then, we have $T\alpha = \ell_T(\mathrm{Ker}\alpha)$.

(2→3) Let f:$\alpha(M) \longrightarrow M_S$ be a homomorphism, where α is a closed homomorphism. Since Kerα ⊆ Kerfα,then$\ell_T(\mathrm{Kerf}\alpha) \subseteq \ell_T(\mathrm{Ker}\alpha)$. By (2), we have $\ell_T(\mathrm{Kerf}\alpha) \subseteq \ell_T(\mathrm{Ker}\alpha)=T\alpha$ and so fα ∈ Tα.

(3→4) Let α,β and σ ∈ T, where α is a closed homomorphism. Suppose that Kerα⊆Kerβ⊆Ker$\sigma\beta$. Then, by the homomorphism theorem 4.21 in ([2], P.53), there exists unique homomorphism f:α (M) →M_S such that $\sigma\beta$ = fα. By using (3), fα∈Tα, then $\sigma\beta$ ∈ Tα. Therefore, we get Tβ⊆Tα.

(4→5) Let $\sigma \in \ell_T[\beta(M_S) \times \beta(M_S) \cap \mathrm{Ker}\alpha]$ and $\alpha\beta$ is a closed homomorphism. We claim that Ker$\alpha\beta$ ⊆Ker$\sigma\beta$. For this, let (m_1, m_2)∈Ker$\alpha\beta$, so $\alpha\beta(m_1)$ =$\alpha\beta(m_2)$. This implies that ($\beta(m_1)$, $\beta(m_2)$)∈ $[\beta(M_s) \times \beta(M_s) \cap \mathrm{Ker}\alpha]$, then $\sigma\beta(m_1)=\sigma\beta(m_2)$. Thus ($m_1$, m_2)∈ker$\sigma\beta$. By (4), we have T$\sigma\beta$ ⊆T$\alpha\beta$ and $\sigma\beta$=u$\alpha\beta$ for some u∈T. Therefore, this means that there is u∈T such that $\sigma\beta = u\alpha\beta$ for each $\alpha, \beta \in T$. Since T is cancellative, so $\sigma = u\alpha$. Thus, $\sigma \in \ell_T[\beta(M_s) \times \beta(M_s)$. This means that $\sigma \in \ell_T(\beta(M_s) \times \beta(M_s)) \cup T\alpha$ and $\ell_T[\beta(M_s) \times \beta(M_s) \cap \mathrm{Ker}\alpha] \subseteq \ell_T(\beta(M_s) \times \beta(M_s)) \cup T\alpha$.Conversely, let$\sigma \in \ell_T(\beta(M_s) \times \beta(M_s)) \cup T\alpha$, hence this means that $\sigma \in \ell_T(\beta(M_s) \times \beta(M_s))$ or $\sigma = u\alpha$ for some u∈T. If $\sigma \in \ell_T[\beta(M_s) \times \beta(M_s)]$, this means that $\sigma\beta(m_1)=\sigma\beta(m_2)$, ∀ m_1,m_2∈M_S. Now, for each m_1 and m_2∈M_S, we have (m_1, m_2) ∈ $[\mathrm{Ker}\alpha \cap \beta(M_s) \times \beta(M_s)]$, which implies that $\alpha(m_1)=\alpha(m_2)$ and $\beta(m_1)=\beta(m_2)$. Since u is well-defined, so $u\alpha(m_1)=u\alpha(m_2)$. If $\sigma = u\alpha$, then this implies that $\sigma(m_1)=\sigma(m_2)$. Thus, $\sigma \in \ell_T(\beta(M_s) \times \beta(M_s) \cap \mathrm{Ker}\alpha)$. If $\sigma \in \ell_T[\beta(M_s) \times \beta(M_s)$, then $\sigma\beta(m_1)=\sigma\beta(m_2)$. Hence,$\sigma \in \ell_T(\beta(M_s) \times \beta(M_s) \cap \mathrm{Ker}\alpha)$ and $\ell_T(\beta(M_s) \times \beta(M_s)) \cup T\alpha \subseteq \ell_T[\beta(M_s) \times \beta(M_s) \cap \mathrm{Ker}\alpha]$.

(5⟶1) By taking $\beta = I_M$, we identified the map of M_S in (5) and obtained that $\ell_T(\ker\alpha) = T\alpha$ for every closed homomorphism $\alpha \in T$. Now, let N be a closed M-cyclic subact of S-act M_S, so N=α (M) for some closed homomorphism $\alpha \in T$. Assume that i_1:α (M)⟶M_S is the inclusion map and α_1: M_S⟶ α(M) is a homomorphism induced by closed homomorphismα. Let φ be S-homomorphism from N into M_S. It is clear that $\varphi\alpha_1$ is S-endomorphism of M_S. Since Kerα=Ker$\alpha_1 \subseteq$ Ker$\varphi\alpha_1$, whence for each (x, y)∈Kerα implies α(x)=α(y), and since φ is well-defined, so φ(α(x))= φ(α(y)). Thus, we obtain (x, y)∈Kerφ.This implies that $\ell_T(\ker\varphi\alpha) \subseteq \ell_T(\ker\alpha)$. Because $\ell_T(\ker\alpha) = T\alpha$, then we have $T\varphi\alpha \subseteq T\alpha$. Thus, $\varphi\alpha \in T\alpha$ and then there exists $\sigma \in T$ such that $\varphi\alpha = \sigma\alpha$. Therefore M_S is a C-M-P-injective act.

We can define a closed ideal as follows: an ideal of a monoid S is called a closed ideal if there is no proper essential extension (i.e. no proper ⋂-large) inside S.

The next corollary characterizes the closed self principally injective monoids.

By Definition 2.3, a monoid S is called closed self principally injective monoid (for short, C-self-P-injective) in case that S_S is closed quasi principally injective. Thus, the proof of the next corollary is clear by Theorem 2.6 and, hence, it is omitted.

Corollary 2.7: The following conditions are equivalent for a monoid S:

(1) S is C-self-P-injective.

(2) $\ell_S(\gamma_S(a)) = Sa$, ∀a∈ S, and closed right ideal aS.

(3) If f: aS⟶S is S-homomorphism, then f (a) ∈Sa for all a∈S and closed right ideal aS.

(4) $\gamma_S(b) \subseteq \gamma_S(a)$, which implies that Sa⊆Sb for any a, b∈ S, where bS is closed right ideal.

(5) $\ell_S\left(bS \cap \left(\gamma_S(a) \times \gamma_S(a)\right)\right) = \ell_S(bS \times bS) \cup Sa$,∀a, b∈ S.

In the subsequent theorem, we study some general properties of C-M-P-injective acts and C-self-P-injective monoids.

Theorem2.8: Let M_S be C-M-P-injective S-act and $\alpha, \beta \in$ T=End(M), where α is a closed homomorphism. Then, the following statements hold:

(1) If $f: \alpha(M) \longrightarrow \beta(M)$ is a monomorphism (otherwise, an epimorphism), then there exists a T-epimorphism (otherwise, a T- monomorphism)$\sigma: T\beta \longrightarrow T\alpha$.

(2) Ifα(M) ≅ β(M), then $T\alpha \cong T\beta$.

Proof: (1) Let $f: \alpha(M) \longrightarrow \beta(M)$ be an S-monomorphism, where M_S is an S-act. Let i_1 (otherwise, i_2) be the inclusion maps of α(M) (otherwise, β(M)) into M_S. Then fα(m) =β(m) for all m∈M_S. Since M_S is C-M-P-injective and α(M) is closed M-cyclic subact of M_S, so the S-homomorphism $i_2$of can be extended to the S-homomorphism $\bar{f}: M_S \longrightarrow M_S$, such that $\bar{f} \circ i_1 = i_2 \circ f$. This means that $\bar{f}\, i_1\alpha(m) = i_2 f\alpha(m)$ for all m∈M_S. Therefore, β(m)=$\bar{f}$α(m) for all m∈M_S. Define $\sigma: T\beta \longrightarrow T\alpha$ by σ(λ β) =λ $\bar{f}$α, $\lambda\beta \in T\beta$. If $\lambda_1\beta = \lambda_2\beta$ for m∈M_S, then $\bar{f}$α(m) = ($\bar{f} \circ i_1$) (α(m)) = (i_2∘ f) (α(m)) = f(α(m)), and hence λ $\bar{f}$ α(m)=λf(α(m)). Thus, σ is well-defined. It is clear that σ is T-homomorphism; in fact, let $\lambda\ \beta \in T\beta$ and g∈ T, then $\sigma(g(\lambda\beta)) = \sigma((g\,\lambda)\beta) = g\lambda\bar{f}\alpha$= g (λ$\bar{f}$ α) =gσ(λβ). We claim that Ker ($\bar{f}$α)⊆Kerα. Let (x_1, x_2)∈Ker ($\bar{f}$α), which implies that $\bar{f}$α(x_1)=$\bar{f}$α(x_2). This implies that f (α(x_1)) =f (α(x_2)).Since f is monomorphism, so α(x_1) =α(x_2). Thus, (x_1, x_2)∈Kerα. By theorem3.4(4), we have $T\alpha \subseteq T\bar{f}\alpha$, so there exists $\lambda \in T$ such that $\alpha = \lambda\bar{f}\alpha$, then $\alpha = \lambda\ \bar{f}\alpha = \sigma(\lambda\ \ \beta) \in \sigma(T\beta)$. This implies that $T\alpha = \sigma(T\beta)$.Then,σ is T-epimorphism.

For the second part (i.e. If $f: \alpha(M) \longrightarrow \beta(M)$ is epimorphism, then there exists a T- monomorphism $\sigma: T\beta \longrightarrow T\alpha$), as in above. Let $f: \alpha(M) \longrightarrow \beta(M)$, and by assumption, f is S-epimorphism. Since M_S is C-M-P-injective, so $i_2$of can be extended to $\bar{f}: M_S \longrightarrow M_S$ such that $\bar{f}$oi_1 =$i_2$of, where i_1 and i_2 are the inclusion map of $\alpha(M)$ into M_S and $\beta(M)$ into M_S, respectively. Define $\sigma: T\beta \longrightarrow T\alpha$ by $\sigma(\lambda\beta) = \lambda\bar{f}\alpha$, for $\lambda \in T$. As in the first part of the proof, σ is well-defined, then $\lambda_1\bar{f}\alpha = \lambda_2\bar{f}\alpha$. Since $\bar{f}\alpha(M) = \bar{f}oi_1(\alpha(M)) = i_2of(\alpha(M)) = f\alpha(M) = \beta(M)$, then $\lambda\bar{f}\alpha(M) = \lambda\beta(M)$, hence $\lambda_1\beta(M) = \lambda_1\bar{f}\alpha(M) = \lambda_2\bar{f}\alpha(M) = \lambda_2\beta(M)$, then $\lambda_1\beta = \lambda_2\beta$. Hence σ is T-monomorphism.

(2) By using (1).

Before describing the next corollary, we need the following definition:

Definition 2.9: [2, P.20] Let S be a semigroup. A nonempty subset K of S is called left ideal of S if $SK \subseteq K$, a right ideal of S if $KS \subseteq K$, and an ideal of S if $SK \subseteq K$ and $KS \subseteq K$.

Recall that an ideal of a monoid S is called a closed ideal if there is no proper essential extension (i.e. no proper $\cap$-large) inside S.

In a similar way, we can define a closed right (otherwise, left) ideal of a monoid S if there is no proper essential extension right (otherwise, left) ideal (i.e. no proper $\cap$-large right (otherwise, left) ideal) inside S.

Corollary 2.10: Let S be C-self-P-injective monoid. Then, for any s, $t \in S$, and closed right ideal bS, the following statements hold:

(1) If $f: bS \longrightarrow aS$ is monomorphism (otherwise, an epimorphism), then there exists an epimorphism (otherwise, a monomorphism) $\sigma: Sa \longrightarrow Sb$.

(2) If $bS \cong aS$, then $Sb \cong Sa$.

The next proposition explains the concepts of Co-Hopfian and the directly finite that coincide under C- quasi principal injectivity condition.

Proposition 2.11: Every C- quasi principally injective act and directly finite is co-Hopfian.

Proof: Similar to the proof of proposition 2.17 in [1], by replacing M_S being C-M-injective act by being C-QP-injective.

The following proposition shows that the concepts of Hopfian and co-Hopfian are coincided in-terms of C-QP-injective property.

Proposition 2.12: Let M_S be C-QP-injective act. M_S is Hopfian act if and only if M_S is co-Hopfian.

Proof:$\Rightarrow$) As every Hopfian is directly finite (For this, if for any $\alpha, \beta \in End(M_S)$ and $\alpha o\beta = I$), where I is the identity endomorphism, then this means that α is surjective. Since M_S is Hopfian, then α is an isomorphism and β is the inverse of α. Thus $\beta o\alpha = I$, which implies that M_S is a directly finite act, so by Proposition 2.11, M_S is co-Hopfian.

$\Leftarrow$) Let f be surjective endomorphism of M_S, then the inclusion map $i: f(M) \rightarrow M_S$ is isomorphism (since M_S is co-Hopfian). Thus $foi = I_{f(M)}$. By proposition 2.11, M_S is directly finite, so $iof = I_M$ (since $f(M) \cong M_S$). Thus f is injective and then it is isomorphism. Therefore, M_S is Hopfian.

Recall that A right S-act B_S is a retract of a right S-act, as if and only if there exists a subact W of A_S and epimorphism $f: A_S \longrightarrow W$, such that $B_S \cong W$ and $f(x) = x$ for every $x \in W$ ([2],P.84).

Proposition2.13: Let M_S be S-act and N be closed M-cyclic subact of M_S.If N is C-M-P-injective, then N is a retract subact of M_S.

Proof: Let i_N be the inclusion map of closed M-cyclic subact N of S-act M_S.Since N is C-M-P-injective, then there exists an S-homomorphism $g: M_S \longrightarrow N$ such that $g \circ i_N = I_N$, hence i_N has left inverse and i(N) is a retract subact of M_S, but N= i(N),so N is a retract subact of M_S.

By replacing the property of M_S from C-quasi injective act to C-M-P-injective act in proposition 2.5 in [1], we can proof the following Proposition:

Proposition 2.14: Let M_S be a C-M-P-injective act. Then every fully invariant closed subact of M_S is C-quasi principally injective.

Proposition 2.15: Every retract subact of C-M-P-injective is C-M-P-injective.

Proof: Assume that N is C-M-P-injective S-act and A is a retract subact of N. Let X be closed M-cyclic subact of S-act M_S and f be S-homomorphism from X into A. Since N is C-M-P-injective act, so there exists S-homomorphism g from M into N_S such that $g\circ i_X= j_A\circ f$, where i_X is the inclusion map of X into M_S and j_A is the injection map of A into N_S. Put $h=\pi_A o g$, where π_A is the projection map of N_S onto A, then $h\circ i_X = \pi_A\circ g\circ i_X = \pi_A\circ j_A\circ f = f$ and A is C-M-P-injective act.

Proposition 2.16: Let M_S and N_S are two S-acts. If N_S is C-M-P-injective act, and B_S is a closed M-cyclic subact of M_S, then N_S is C-B-P-injective act.

Proof: Let X be closed B-cyclic subact of B. Since B is closed M-cyclic subact of M_S, so by lemma2.4 in [13], X is closed M-cyclic. Let f be S-homomorphism from X into N_S. Since N_S is C-M-P-injective act, so there exists S-homomorphism g from M_S into N_S such that $g\circ i_B\circ i_X = f$, where i_X, i_B be the inclusion map of X into B and B into M_S, respectively. Put $h=g\circ i_B$, then $h\circ i_X = g\circ i_B\circ i_X = f$. Thus N_S is C-B-P-injective act.

Corollary 2.17: Let M_S and N_S be two S-acts. Then, N_S is C-M-P-injective act if and only if N_S is C-X-P-injective act for every closed M-cyclic subact X of M_S.

Proof: Suppose that N_S is C-M-P-injective act, then by proposition 2.16, we have N_S is C-X-P-injective for every closed M-cyclic subact X of M_S. The converse is clear.

Proposition 2.18: Let M_S be an S-act and $\{N_i \mid i\in I\}$ be a family of S-acts. Then $\prod_{i\in I} N_i$ is C-M-P-injective act if and only if N_i is C-M-P-injective act for every $i\in I$.

Proof:$\Longrightarrow$) Assume that $N_S = \prod_{i\in I} N_i$ is C-M-P-injective. Let X be closed M-cyclic subact of S-act M_S and f be S-homomorphism from X to N_i. Since N_S is C-M-P-injective act then there exists S-homomorphism g: $M_S \longrightarrow N_S$ such that $g\circ i_X = j_i\circ f$, where i_X is the inclusion map of X into M_S and j_i is the injection map of N_i into N_S. Define h: $M_S\longrightarrow N_i$ by $h=\pi_i \circ g$, where π_i is the projection map of N_S onto N_i. Then $h\circ i_X=\pi_i \circ g\circ i_X = \pi_i \circ j_i\circ f = f$. That is, for all $x\in X$, $h(x) = h(i_X(x)) = \pi_i(g(x)) = \pi_i(g(i_X(x))) = \pi_i(j_i(f(x))) = (\pi_i \circ j_i)(f(x)) = f(x)$.

$\Longleftarrow$) Assume that N_i is C-M-P-injective act for each $i\in I$, where M_S is S-act. Let X be closed M-cyclic subact of M_S and f be S-homomorphism from X to $N_S = \prod_{i\in I} N_i$. Since N_i is C-M-P-injective act, then there exists S-homomorphism $\beta_i: M_S\longrightarrow N_i$, such that $\beta_i \circ i_X=\pi_i \circ f$, so there exists S-homomorphism β: $M_S\longrightarrow N_S$ such that $\beta=j_i \circ \beta_i$. We claim that $\beta \circ i_X = f$. Since $\beta \circ i_X = j_i \circ \beta_i \circ i_X = j_i \circ \pi_i \circ f = f$, so we obtain $f = \beta \circ i$. Therefore, N_S is C-M-P-injective.

Corollary 2.19: Let M_S and N_i be S-acts, where $i\in I$ and I is a finite index set. Then, for every i, N_i is C-M-P-injective if and only if $\oplus_{i=1}^{n} N_i$ is C-M-P-injective.

The next theorem gives the relationship between injective and C-N-P-injective acts:

Theorem 2.20: The following statements are equivalent for S-act M_S:

(1) M_S is injective act,

(2) M_S is C-N-P-injective act for every S-act N.

Proof: ($1\Longrightarrow 2$) It is obvious.

($2\Longrightarrow 1$) Assume that M_S is C-N-P-injective act and E(M) is injective envelope of M_S. By corollary 2.19, $M_S\oplus E$ (M) is C-N-P-injective. Put $N_S= M_S \oplus E$ (M). Thus, $M_S \oplus E$ (M) is C-M$\oplus$E-P-injective. By proposition 2.15, M_S is C- M$\oplus$E-P-injective act. Consider the inclusion map $i:M_S\longrightarrow E(M)$ and the injection maps $j_1:E(M) \longrightarrow M_S\oplus E(M)$, $j_2:M_S\longrightarrow M_S\oplus E(M)$, and I_M: $M_S\longrightarrow M_S$ are the identity maps of M_S. Letπ_M: $M_S\oplus E$ (M) $\longrightarrow M_S$ be the projection map such that $\pi_M \circ j_2 = I_M$. Now, $M_S\oplus E$ (M) is C-quasi injective, so this implies that there exists S-homomorphism g: $M_S\oplus E$ (M) $\longrightarrow M_S\oplus E$ (M) such that $g\circ j_1 \circ i=j_2 \circ I_M$, then $\pi_M \circ g\circ j_1 \circ i = \pi_M \circ j_2 \circ I_M$. Thus $I_M=\pi_M \circ g\circ j_1 \circ i$. Put $f= \pi_M \circ g\circ j_1$ and then $I_M = f \circ i$. Therefore, M_S is a retract of E (M) and then it is injective.

Definition 2.21: An S-act M_S satisfies the CM-property if every closed subact of M_S is an M-cyclic subact of M_S.

The following proposition provides a relationship among the extending act, C-M-injective, and C-M-P-injective:

Proposition 2.22: The following statements are equivalent for S-act M_S:

(1) M_S is an extending act,

(2) Every S-act is C-M-injective,

(3) Every S-act is C-M-P-injective and M_S satisfies CM-property.

Proof: (1⟹2) It is obvious.

(2⟹3) Let N be a closed subact of S-act M_S. By using(2), N is C-M-injective act. Thus, by proposition 2.7, N is a retract subact of M_S and hence every retract subact is M-cyclic, by remarks and examples 2.3(2) in [8]. Thus, M_S satisfies the CM-property. The other part is obvious.

(3⟹1) Let N be any closed subact of S-act M_S. Since M_S satisfies the CM-property, so N is M-cyclic. By using (3), N is C-M-P-injective act. By proposition 2.13, N is a retract subact of M_S. Thus M_S is extending act.

Theorem 2.23: The following statements are equivalent for the projective act M_S:

(1) Every homomorphic image of any C-M-P-injective act is C-M-P-injective.

(2) Every homomorphic image of any C-M-injective act is C-M-P-injective.

(3) Every homomorphic image of any M-injective act is C-M-P-injective.

(4) Every homomorphic image of any injective act is C-M-P-injective.

(5) Every closed M-cyclic subact of M_S is projective.

Proof: (1⟹2), (2⟹3) and (3⟹4) are obvious.

(4 ⟹5) Let A be closed M-cyclic subact of M_S and f be S-epimorphism from S-act N_S onto S-act B_S. Let g be S-homomorphism from A into B_S. Since every act can be embedded into an injective act, by corollary 1.6 in ([2], P.186), thus N_S embedded into E and i_N is the inclusion map of N into E. Let π: $E \longrightarrow E/\rho$ is the canonical projection map such that ρ = Kerf. Define ℓ: $B_S \longrightarrow E/\rho$ by $\ell(b) = [b]_\rho$, for all $b \in B_S$, where b=f (n) and $n \in N_S$. It is clear that ℓ is well-defined and an S-homomorphism. By using (4), E/ρ is C-M-P-injective, so $\ell \circ g$ extends to S-homomorphism g^* from M into E/ρ, such that $g^* \circ i_A = \ell \circ g$.Since M_S is projective, so g^* can be lifted to S-homomorphism h from M_S into E, such that $\pi \circ h = g^*$.Since E is injective by assumption, then h represents the extension of the S-homomorphism α from A into E. This means that $h \circ i_A = \alpha$. Let h^*: $A \to N_S$ is defined by $h^*(a) = \alpha(a)$, for all $a \in A$. Now, $\ell \circ g = g^* \circ i_A = \pi \circ h \circ i_A = \pi \circ \alpha = \pi \circ i_N \circ h^* = \ell \circ f \circ h^*$. Thus, $f \circ h^* = g$ (since ℓ is monomorphism) and A is projective act.

(5 ⟹1) Let N_S be C-M-P-injective act and f: $N \longrightarrow W_S$ be S-epimorphism. Let A be a closed M-cyclic subact of S-act M_S and g be any S-homomorphism from A into W_S. Now, since A is projective by using (5), so g can be lifted to S-homomorphism h from A into N_S. Since N_S is C-M-P-injective act, so h extends to S-homomorphism h^* from M_S into N_S (this means that $h = h^* \circ i_A$, where i_A is the inclusion map of A into M_S). Put $g^* = f \circ h^*$. Now, $g^* \circ i_A = f \circ h^* \circ i_A = f \circ h = g$. Thus, $g^* \circ i_A = g$ and W_S is C-M-P-injective act.

Acknowledgements

The first author would like to thank Mustansiriyah University (www.uomustansiriyah.edu.iq) Baghdad-Iraq for its support in the present work.

Conflicts of Interest: The authors declare no conflicts of interest in this paper.

References

1. Shaymaa, A. A. Kareem and Ahmed, A. A. Kareem **2019**. About the Closed Quasi Injective S-Acts Over Monoids, *Pure and Applied Mathematics Journal. Special Issue: Algebra with Its Applications*, **8**(5): 88-92. doi: 10.11648/j.pamj.20190805.12
2. Kilp, M., Knauer, U. and Mikhalev, A.V. **2000**. *Monoids acts and categories with applications to wreath products and graphs*, Walter de Gruyter. Berlin. New York.
3. Yan, T. **2011**. Generalized injective S-acts on a monoid, *Advances in mathematics*, **40**(4): 421-432.
4. Jupil, K. **2008**. PI-S-systems , *Journal of Chungcheong Mathematical Society*, **21**(4): 591-599.
5. Shaymaa, A. **2018**. *about the generalizations in acts over monoids*, LAP LAMBERT Academic Publishing, Germany.
6. Shaymaa, A.**2018**.*On Finitely Generated in S-acts over monoids*, Noor Publishing, Germany.
7. Shaymaa, A. **2015**. Generalizations of quasi injective S-acts over monoids, PhD. thesis, College of Science, University of Al-Mustansiriyah, Baghdad, Iraq.
8. Abbas, M.S. and Shaymaa, A. **2015**. Quasi principally injective acts over monoids, *Journal of advances in mathematics*, **10**(5): 3493-3502.
9. Abdul Kareem, S.A. **2020**. Dual of Extending Acts. *Iraqi Journal of Science*, 64-71. https://doi.org/10.24996/ijs.2020.SI.1.9
10. Ahsan, J. **1987**. Monoids characterized by their quasi injective S-acts, *Semigroup forum*, **36**(3): 285-292.
11. Lopez, A. M., Jr. and Luedeman, J. K. **1979**. Quasi-injective S-acts and their S-endomorphism Semigroup, *Czechoslovak Math.J.* , **29**(104): 97-104.
12. Al-Bahrani B. H., Rahman M. Q. **2020**. On y-closed Rickart Modules, *Iraqi Journal of Science*, **61**(10): 2681-2686 .DOI: 10.24996/ijs.2020.61.10.25
13. Shaymaa, A. **2017**. Extending and P-extending S-act over monoids, *International Journal of Advanced Scientific and Technical Research*, **2**(7): 171-178.