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The Dynamics of a Food Web System: Role of a Prey Refuge Depending on Both Species

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Abstract
This paper aims to study the role of a prey refuge that depends on both prey and predator species on the dynamics of a food web model. It is assumed that the food transfer among the web levels occurs according to Lotka-Volterra functional response. The solution properties, such as existence, uniqueness, and uniform boundedness, are discussed. The local, as well as the global, stabilities of the solution of the system are investigated. The persistence of the system is studied with the assistance of average Lyapunov function. The local bifurcation conditions that may occur near the equilibrium points are established. Finally, numerical simulation is used to confirm our obtained results. It is observed that the system has only one type of attractors that is a stable point, while periodic dynamics do not exist even on the boundary planes.

Keywords: Food web, prey refuge, Stability, bifurcation, persistence.

ديناميكية نظام الشبكة الغذائية: دور ملجأ الفريسة المعتمد على كلا الجنسين

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الخلاصة
يهدف هذا البحث الى دراسة دور ملجأ الفريسة المعتمد على كلا الجنسين (الفريسة و المفترس) على ديناميكية نموذج الشبكة الغذائية . تم أعتماد دالة لوتكا فولتيرا لوصف انتقال التغذية بين مستويات الشبكة. وقد تم مناقشة خواص الحل (وجود , وحدانية , القيد المنتظم) .كما ان الاستقرارية المحلية و الشاملة لحل النظام تم مناقشتها ايضا. كذلك فأن اصرار النظام درس بمساعدة دالة معدل ليبانوف. شروط التفرع المحلي المحتمل الحدوث بالقرب من نقطاط التوازن تم اعدادها.واخيرا المحاكاة العددية استخدمت لتأكيد النتائج التي تم الحصول عليها, لاحظنا ان النظام له نوع واحد من الجواذب وهو نقطة مستقرة بحيث لا يوجد ديناميكية الدورية حتى في المستويات الحدودية.

1. Introduction
Food webs are established in ecosystems and are significant for all living organisms. Energy needs for all activities of animals are provided through food consumption. Furthermore, food webs represent the pathways by which energy flows from a level to another. They are fundamental and inescapable in any attempt to describe how the real world life in the environment is organized or complexes of species interact. The three types of food web samples are the bases of large measure ecosystems. Therefore, to understand the dynamical behavior of an ecosystem, it is substantial to realize the dynamics of the three kinds of food web models. Hence, attention has been given to the

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dynamics with 3-species food chains and food web systems [1-8]. Later on, many researchers investigated the dynamical behavior of food webs with various types of functional responses and including many biological factors. Agarwal and Kumarb [9] proposed and studied the alternative resource effects for top predators using Holling type III model. They showed that an alternative resource has the ability to prohibit top predator annihilation. Xiao *et al*. [10] proposed and investigated a predator–prey model that associates time delay with a Holling type II and Allee impact in prey. They observed that if the Allee impact is high or the birth rate is low, then both species of the system are fading and hence the stability of the system is affected.

In fact, the refuge mechanism has a role in promoting the growth rate of prey and reducing both predator growth and the products lurking behavior of prey [11]. Therefore, the existence of prey refuges has substantial effects on the cohabitation of predators and their prey. There are three types of refuges; the first type is the supply of continual spatial protection for a small branch of the prey population, while the second type is the provision of temporary spatial preservation and, finally, the third type is the supply of a temporary refuge in numbers, which means the reduction of the venture of predation by rising the abundance of vulnerable prey [12]. The problem of prey-predator interactions under a prey refuge has been studied by some authors. Mukherjee [11] investigated a resource according to Holling type I and II functional responses and then showed the effect of constant prey refuge. Das *et al*. [13] described constant prey refuge and harvesting to both predator and prey. Ghosh *et al*. [14] studied the effects of extra food for predator with prey refuge. Santra *et al*. [15] investigated the dynamical actions with Crowley–Martin functional response-associated prey refuge. Finally, Molla *et al*. [16] suggested a model for Holling type-II prey– predator interactions in a prey refuge.

In this paper, a Lotka-Volttera food web model with a prey refuge that consists of prey and predator is proposed and studied. The remaining sections are organized as follows. In section (2), the mathematical model with symbols of system is introduced. Section (3) deals with the analysis of local stability. In section (4), global stability analysis is investigated. Section (5) describes the persistence of the model. In section (6), the local bifurcation of the model is studied. The numerical simulation is given in section (7), while section (8) includes some discussion of the obtained results.

2. The model formulation

In this section, a mathematical modeling approach was used to study the role of a non-constant prey refuge in the dynamical behavior of a three-species food web system. It is assumed that the food web is consisting of prey at a lower level of density at time T, given by $X(T)$. The prey grows logistically in the absence of the predators and has a non-constant refuge property that is dependent on the density of their predator. The middle predator, denoted for density at time T by $Y(T)$, feeds on prey at the lower level and is preyed on by the predator at the higher level. It has another limited source of food in the absence of their preferred food. The top predator, denoted for density at time T by $Z(T)$, at the higher level, feeds on both the prey at the lower level and the middle predator at the second level. Accordingly, the dynamics of such a food web system can be described by using mathematical modeling with differential equations.

$$\begin{aligned}
\frac{dX}{dT} &= rX\left(1-\frac{X}{K_1}\right)-\alpha_1\left(X-X_{R_1}\right)Y-\alpha_2\left(X-X_{R_2}\right)Z,\\
\frac{dY}{dT} &= sY\left(1-\frac{Y}{K_2}\right)+e_1\alpha_1\left(X-X_{R_1}\right)Y-\alpha_3YZ-d_1Y, \qquad (1)\\
\frac{dZ}{dT} &= e_2\alpha_2\left(X-X_{R_2}\right)Z+e_3\alpha_3YZ-d_2Z,
\end{aligned}$$

with $X_{R_1}=cXY$ and $X_{R_2}=cXZ$, the model (1) becomes

$$\begin{aligned}
\frac{dX}{dT} &= rX\left(1-\frac{X}{K_1}\right)-\alpha_1X(1-cY)Y-\alpha_2X(1-cZ)Z,\\
\frac{dY}{dT} &= sY\left(1-\frac{Y}{K_2}\right)+e_1\alpha_1X(1-cY)Y-\alpha_3YZ-d_1Y, \qquad (2)\\
\frac{dZ}{dT} &= e_2\alpha_2X(1-cZ)Z+e_3\alpha_3YZ-d_2Z,
\end{aligned}$$

with $X(0)\geq 0, Y(0)\geq 0, Z(0)\geq 0$. The description of the parameters for the system (2) is shown in Table-1.

Table 1-Description of parameters in system (2)

Parameter	**Description**
$r,\ s$	Intrinsic growth rates for the prey and middle predator, respectively
K_1, K_2	Carrying capacity of the environment for the prey and middle predator, respectively
$1-cY, 1-cZ$	Predator dependent refuge rates, where c denotes the coefficient of prey refuge in (0, 1] with $Y \le \frac{1}{c}$ and $Z \le \frac{1}{c}$, respectively.
$\alpha_1, \alpha_2, \alpha_3$	attack rates of middle and top predator, respectively
e_1, e_2, e_3	Conversion rates
d_1, d_2	Natural death rates of middle and top predators, respectively

It is assumed that all the above parameters are positive constants and hence the domain of system (2) will be given by $\mathbb{R}^3_+ = \{ (X,Y,Z) \in \mathbb{R}^3 : X \ge 0, Y \ge 0, Z \ge 0\}$. Now to simplify the model analysis, the following parameters and dimensionless variables are used in system (2).

$$t = rT, x = \frac{X}{K_1}, y = cY, z = cZ,$$
$$w_1 = \frac{\alpha_1}{rc}, w_2 = \frac{\alpha_2}{rc}, w_3 = \frac{s}{r}, w_4 = cK_2, w_5 = \frac{e_1\alpha_1 K_1}{r}, \tag{3}$$
$$w_6 = \frac{\alpha_3}{rc}, w_7 = \frac{d_1}{r}, w_8 = \frac{e_2\alpha_2 K_1}{r}, w_9 = \frac{e_3\alpha_3}{rc}, w_{10} = \frac{d_1}{r}.$$

According to Eq. (3), the dimensionless system corresponding to system (2) will be

$$\frac{dx}{dt} = x(1-x) - w_1 x(1-y)y - w_2 x(1-z)z = xf_1,$$
$$\frac{dy}{dt} = w_3 y\left(1 - \frac{y}{w_4}\right) + w_5 x(1-y)y - w_6 yz - w_7 y = yf_2, \tag{4}$$
$$\frac{dz}{dt} = w_8 x(1-z)z + w_9 yz - w_{10} z = zf_3,$$

where $x(0) \ge 0, y(0) \ge 0, z(0) \ge 0$.The functions on the right hand side of system (4) are continuous and have continuous partial derivatives. Hence, they are Lipschitzian functions [17]. Therefore, the solution of system (1) exists and is unique. In addition, the following theorem shows the boundedness of the solution in $\mathbb{R}^3_+$.

Theorem 1. The region $\mathcal{L}_1 = \{(x,y,z) \in \mathbb{R}^3_+ : x \le 1; 0 < x(t) + y(t) + z(t) \le \frac{1}{\mu_1}(2 + \frac{w_3 w_4}{4})\}$ is a uniformly bounded region that contains all the solutions of system (4).

Proof. From system (4), we have that

$$\frac{dx}{dt} \le x(1-x).$$

By solving the inequality, it is simple to verify that $x(t) \le 1$ as $t \to \infty$. Let $W(t) = x(t) + y(t) + z(t)$, then after some algebraic manipulation we have

$$\frac{dW}{dt} \le 2x + \frac{w_3 w_4}{4} - x - w_7 y - w_{10} z.$$

Then $\frac{dW}{dt} + \mu_1 W \le 2 + \frac{w_3 w_4}{4}$, where $\mu_1 = \min\{1, w_7, w_{10}\}$. By using Gronwall inequality [18], it is easy to obtain that $0 < W(t) \le \frac{1}{\mu_1}(2 + \frac{w_3 w_4}{4})$ as $t \to \infty$. Hence, all solutions are uniformly bounded in $\mathcal{L}_1$.

3. Local Stability Analysis

There are seven equilibrium points of system (4). Their existence conditions and local stability analyses are established.

- The vanishing equilibrium point that is denoted by $E_0 = (0,0,0)$.
- The first axial equilibrium point (FAEP) that is denoted by $E_1 = (\tilde{\tilde{x}}, 0, 0) = (1,0,0)$ and always exists.
- The second axial equilibrium point (SAEP) that is denoted by $E_2 = (0, \hat{y}, 0)$, where

$\hat{y} = \frac{w_4(w_3 - w_7)}{w_3}$. (5a)

Clearly, the SAEP exists uniquely on y −axis if and only if it satisfies the following condition

$w_3 > w_7$. (5b)

- The prey-free equilibrium point (PFEP) that is denoted by $E_3 = (0, \breve{y}, \breve{z})$, where

$\breve{y} = \frac{w_{10}}{w_9}$, $\breve{z} = \frac{w_3 w_4 w_9 - w_3 w_{10} - w_4 w_7 w_9}{w_4 w_6 w_9}$. (6a)

Obviously, the PFEP exists uniquely in the yz −plane if and only if the following condition holds

$w_3 w_4 w_9 > w_3 w_{10} + w_4 w_7 w_9$. (6b)

- The middle predator-free equilibrium point (MPFEP) that is denoted by $E_4 = (\bar{x}, 0, \bar{z})$, where

$\bar{x} = 1 - w_2 \bar{z} + w_2 \bar{z}^2$, (7a)

with $\bar{z} > 0$ is a root of the following equation

$\gamma_1 \bar{z}^3 + \gamma_2 \bar{z}^2 + \gamma_3 \bar{z} + \gamma_4 = 0$, (7b)

with $\gamma_1 = -w_2 w_8 < 0$, $\gamma_2 = 2 w_2 w_8 > 0$, $\gamma_3 = -w_8(1 + w_2) < 0$ and $\gamma_4 = w_8 - w_{10}$. Therefore, with the aid of the Descartes rule of signs, Eq. (7b) has either one or three positive roots but there are no negative roots, provided that the following condition holds

$w_8 > w_{10}$. (7c)

By using Wolfram Mathematica 11.3, it is observed that Eq. (7b) has only one positive root denoted by $\bar{z}$ and two other complex conjugate roots. Accordingly, the MPFEP remains uniquely in the xz −plane, which stipulates that, in the extension to condition (7c), the following condition holds too

$1 - w_2 \bar{z} + w_2 \bar{z}^2 > 0$. (7d)

- The top predator-free equilibrium point (TPFEP) that is denoted by $E_5 = (\bar{\bar{x}}, \bar{\bar{y}}, 0)$, where

$\bar{\bar{x}} = 1 - w_1 \bar{\bar{y}} + w_1 \bar{\bar{y}}^2$, (8a)

with $\bar{\bar{y}} > 0$ is the root of equation

$\alpha_1 \bar{\bar{y}}^3 + \alpha_2 \bar{\bar{y}}^2 + \alpha_3 \bar{\bar{y}} + \alpha_4 = 0$, (8b)

with $\alpha_1 = -w_1 w_5 < 0$, $\alpha_2 = 2 w_1 w_5 > 0$, $\alpha_3 = -\left(\frac{w_3}{w_4} + w_1 w_5 + w_5\right) < 0$ and $\alpha_4 = w_3 - w_7 + w_5$. Again, by utilizing the Descartes rule of signs, Eq. (8b) has either one or three positive roots but there is no negative roots, provided the following condition holds

$w_3 + w_5 > w_7$. (8c)

Similarly, Wolfram Mathematica 11.3 is used to compute the roots of Eq. (8b). It was found that Eq. (8b) has only one positive root denoted by $\bar{\bar{y}}$ and two other complex conjugate roots. Therefore, the TPFEP remains uniquely in the xy −plane, which stipulates that, in the extension to condition (8c), the following condition holds too

$1 - w_1 \bar{\bar{y}} + w_1 \bar{\bar{y}}^2 > 0$. (8d)

- The positive equilibrium point (PEP) that is denoted by $E_6 = (x^*, y^*, z^*)$, where

$y^* = \frac{1}{w_9}(w_{10} - w_8 x^* + w_8 x^* z^*)$, (9a)

where (x^*, z^*) is a positive intersection point for the following two isoclines

$$g_1(x,z) = 1 - x - \frac{w_1 w_{10}}{w_9} + \frac{w_1 w_8}{w_9} x - \frac{w_1 w_8}{w_9} xz + w_1 \left(\frac{w_{10}}{w_9}\right)^2 - 2\frac{w_1 w_8 w_{10}}{{w_9}^2} x + 2\frac{w_1 w_8 w_{10}}{{w_9}^2} xz + w_1 \left(\frac{w_8}{w_9}\right)^2 x^2 (1 - z)^2 - w_2 z + w_2 z^2 = 0, \quad (9b)$$

$$g_2(x,z) = w_3 - \frac{w_3 w_{10}}{w_4 w_9} + \frac{w_3 w_8}{w_4 w_9} x - \frac{w_3 w_8}{w_4 w_9} xz + w_5 x - \frac{w_5 w_{10}}{w_9} x + \frac{w_5 w_8}{w_9} x^2 - \frac{w_5 w_8}{w_9} x^2 z - w_6 z - w_7 = 0. \quad (9c)$$

Then, as $z \to 0$, we obtain that

$$g_1(x) = w_1 \left(\frac{w_8}{w_9}\right)^2 x^2 - \left[1 - \frac{w_1 w_8}{w_9} + 2\frac{w_1 w_8 w_{10}}{{w_9}^2}\right] x + 1 - \frac{w_1 w_{10}}{w_9} + w_1 \left(\frac{w_{10}}{w_9}\right)^2 = 0, \quad (9d)$$

$$g_2(x) = \frac{w_5 w_8}{w_9} x^2 + \left[w_5 + \frac{w_3 w_8}{w_4 w_9} - \frac{w_5 w_{10}}{w_9}\right] x + w_3 - \frac{w_3 w_{10}}{w_4 w_9} + w_7 = 0. \quad (9e)$$

Clearly, each of Eq. (9d) and Eq. (9e) has a unique positive root on the x −axis, denoted by x_1 and x_2, respectively, provided that the next conditions hold

$$1 + w_1 \left(\frac{w_{10}}{w_9}\right)^2 < \frac{w_1 w_{10}}{w_9},$$
$$w_3 + w_7 < \frac{w_3 w_{10}}{w_4 w_9}. \tag{9f}$$

Therefore, the two isoclines given by equations (9b) and (9c) have a unique positive intersection point (x^*, z^*) in the interior of xz −plane and then the PEP exists uniquely, provided that

$$x_1 < x_2, \tag{10a}$$
$$\frac{dz}{dx} = -\left(\frac{\partial g_1(x,z)}{\partial x} \div \frac{\partial g_1(x,z)}{\partial z}\right) > 0, \tag{10b}$$
$$\frac{dz}{dx} = -\left(\frac{\partial g_2(x,z)}{\partial x} \div \frac{\partial g_2(x,z)}{\partial z}\right) < 0, \tag{10c}$$
$$w_{10} + w_8 x^* z^* > w_8 x^*. \tag{10d}$$

Now, to establish the local stability, the Jacobian matrix, that is denoted by $J(x, y, z)$ of the system (4) at the (x, y, z), is determined by

$$J(x,y,z) = \begin{bmatrix} -x + f_1 & -w_1 x + 2w_1 xy & -w_2 x + 2w_2 xz \\ w_5 y - w_5 y^2 & -\frac{w_3}{w_4} y - w_5 xy + f_2 & -w_6 y \\ w_8 z - w_8 z^2 & w_9 z & -w_8 xz + f_3 \end{bmatrix}. \tag{11}$$

It is clear that system (4) has the Jacobian at E_0, specified by

$$J(E_0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & w_3 - w_7 & 0 \\ 0 & 0 & -w_{10} \end{bmatrix}. \tag{12a}$$

The eigenvalues of $J(E_0)$ are

$$\lambda_{0x} = 1 > 0, \lambda_{0y} = w_3 - w_7, \lambda_{0z} = -w_{10}. \tag{12b}$$

Therefore, E_0 is a saddle point.
The Jacobian matrix at FAEP is

$$J(E_1) = \begin{bmatrix} -1 & -w_1 & -w_2 \\ 0 & w_3 + w_5 - w_7 & 0 \\ 0 & 0 & w_8 - w_{10} \end{bmatrix}. \tag{13a}$$

Therefore, the eigenvalues of $J(E_1)$ are given by

$$\lambda_{1x} = -1 < 0, \lambda_{1y} = w_3 + w_5 - w_7, \lambda_{1z} = w_8 - w_{10}. \tag{13b}$$

Clearly, all the above eigenvalues are negative and E_1 is locally asymptotically stable in $\mathbb{R}^3_+$ if the next conditions are satisfied

$$w_3 + w_5 < w_7, \tag{13c}$$
$$w_8 < w_{10}. \tag{13d}$$

The Jacobian matrix at SAEP is

$$J(E_2) = \begin{bmatrix} 1 - w_1\hat{y} + w_1\hat{y}^2 & 0 & 0 \\ w_5\hat{y} - w_5\hat{y}^2 & -\frac{w_3}{w_4}\hat{y} & -w_6\hat{y} \\ 0 & 0 & w_9\hat{y} - w_{10} \end{bmatrix}. \tag{14a}$$

Therefore, the eigenvalues can be written as

$$\lambda_{2x} = 1 - w_1\hat{y} + w_1\hat{y}^2, \lambda_{2y} = -\frac{w_3}{w_4}\hat{y}, \lambda_{2z} = w_9\hat{y} - w_{10}. \tag{14b}$$

The eigenvalues of $J(E_2)$ will be negative and then E_2 is locally asymptotically stable if the next conditions are held

$$1 + w_1\hat{y}^2 < w_1\hat{y}, \tag{14c}$$
$$w_9\hat{y} < w_{10}. \tag{14d}$$

The Jacobian matrix at PFEP is

$$J(E_3) = \begin{bmatrix} 1 - w_1\breve{y} + w_1\breve{y}^2 - w_2(1 - \breve{z})\breve{z} & 0 & 0 \\ w_5\breve{y} - w_5\breve{y}^2 & -\frac{w_3}{w_4}\breve{y} & -w_6\breve{y} \\ w_8\breve{z} - w_8\breve{z}^2 & w_9\breve{z} & 0 \end{bmatrix} = (C_{ij}). \tag{15a}$$

The characteristic equation of $(15a)$ is as follows

$$(C_{11} - \lambda)\left(\lambda^2 + \frac{w_3}{w_4}\breve{y}\lambda + w_6 w_9 \breve{y}\breve{z}\right) = 0. \tag{15b}$$

Therefore, the eigenvalues of $J(E_3)$ are

$$\lambda_{3x} = -w_1\breve{y} + w_1\breve{y}^2 - w_2\breve{z} + w_2\breve{z}^2 + 1, \tag{15c}$$

$$\lambda_{3y} = -\frac{w_3}{2w_4}\breve{y} + \frac{\sqrt{\left(\frac{w_3}{w_4}\breve{y}\right)^2 - 4w_6 w_9 \breve{y}\breve{z}}}{2}, \tag{15d}$$

$$\lambda_{3z} = -\frac{w_3}{2w_4}\breve{y} - \frac{\sqrt{\left(\frac{w_3}{w_4}\breve{y}\right)^2 - 4w_6 w_9 \breve{y}\breve{z}}}{2}. \tag{15e}$$

Hence, if the next condition holds, E_3 will be locally asymptotically stable

$$1 < w_1\breve{y} - w_1\breve{y}^2 + w_2\breve{z} - w_2\breve{z}^2. \tag{15f}$$

The Jacobian matrix at MPFEP is presented as follows

$$J(E_4) = \begin{bmatrix} -\bar{x} & -w_1\bar{x} & -w_2\bar{x} + 2w_2\bar{x}\bar{z} \\ 0 & w_3 + w_5\bar{x} - w_6\bar{z} - w_7 & 0 \\ w_8\bar{z} - w_8\bar{z}^2 & w_9\bar{z} & -w_8\bar{x}\bar{z} \end{bmatrix} = (M_{ij}). \tag{16a}$$

The characteristic of (16a) is shown as

$$(M_{22} - \lambda)(\lambda^2 - T_4\lambda + D_4) = 0, \tag{16b}$$

where $T_4 = -\bar{x}(1 + w_8\bar{z}) < 0$ and $D_4 = w_8\bar{x}^2\bar{z} + w_2 w_8\bar{x}\bar{z}(1 - 2\bar{z})(1 - \bar{z})$.

Thus, the eigenvalues of $J(E_4)$ are given by

$$\lambda_{4y} = w_3 + w_5\bar{x} - w_6\bar{z} - w_7, \tag{16c}$$

$$\lambda_{4x} = \frac{T_4}{2} + \frac{1}{2}\sqrt{T_4^{\,2} - 4D_4}, \tag{16d}$$

$$\lambda_{4z} = \frac{T_4}{2} - \frac{1}{2}\sqrt{T_4^{\,2} - 4D_4}. \tag{16e}$$

Accordingly, all the above eigenvalues have real parts of less than zero and then E_4 is locally asymptotically stable if the next conditions are satisfied

$$w_3 + w_5\bar{x} < w_6\bar{z} + w_7, \tag{16f}$$

$$\bar{x} + w_2 + 2w_2\bar{z}^2 > 3w_2\bar{z}. \tag{16g}$$

The Jacobian matrix at TPFEP is written as follows

$$J(E_5) = \begin{bmatrix} -\bar{\bar{x}} & -w_1\bar{\bar{x}} + 2w_1\bar{\bar{x}}\bar{\bar{y}} & -w_2\bar{\bar{x}} \\ w_5\bar{\bar{y}} - w_5\bar{\bar{y}}^2 & -\frac{w_3}{w_4}\bar{\bar{y}} - w_5\bar{\bar{x}}\bar{\bar{y}} & -w_6\bar{\bar{y}} \\ 0 & 0 & w_8\bar{\bar{x}} + w_9\bar{\bar{y}} - w_{10} \end{bmatrix} = (Q_{ij}). \tag{17a}$$

Therefore, the characteristic equation of $J(E_5)$ is given by

$$(Q_{33} - \lambda)(\lambda^2 - T_5\lambda + D_5) = 0, \tag{17b}$$

where $T_5 = -\left[\bar{\bar{x}} + \frac{w_3}{w_4}\bar{\bar{y}} - w_5\bar{\bar{x}}\bar{\bar{y}}\right] < 0$ and $D_5 = \frac{w_3}{w_4}\bar{\bar{x}}\bar{\bar{y}} + w_5\bar{\bar{x}}^2\bar{\bar{y}} + w_1 w_5\bar{\bar{x}}\bar{\bar{y}}(1 - 2\bar{\bar{y}})(1 - \bar{\bar{y}})$.

Hence, the eigenvalues of $J(E_5)$ are

$$\lambda_{5z} = w_8\bar{\bar{x}} + w_9\bar{\bar{y}} - w_{10}, \tag{17c}$$

$$\lambda_{5x} = \frac{T_5}{2} + \frac{1}{2}\sqrt{T_5^{\,2} - 4D_5}, \tag{17d}$$

$$\lambda_{5y} = \frac{T_5}{2} - \frac{1}{2}\sqrt{T_5^{\,2} - 4D_5}. \tag{17e}$$

Consequently, all the above eigenvalues have real parts of less than zero and then E_5 is locally asymptotically stable if the next conditions are satisfied

$$w_8\bar{\bar{x}} + w_9\bar{\bar{y}} < w_{10}, \tag{17f}$$

$$\frac{w_3}{w_4} + w_5\bar{\bar{x}} + w_1 w_5 + 2w_1 w_5\bar{\bar{y}}^2 > 3w_1 w_5\bar{\bar{y}}. \tag{17g}$$

Next, the local stability conditions of the PEP are determined.

Theorem 2: Suppose that the PEP of the system (4) exists, then it is a locally asymptotically stable with the following sufficient conditions are satisfied

$$y^* < \frac{1}{2}, \tag{18a}$$

$$z^* < \frac{1}{2}, \tag{18b}$$

$$w_9 > w_1 w_8(1 - 2y^*)(1 - z^*), \tag{18c}$$

$$w_2 w_5 w_9(1 - y^*)(1 - 2z^*) < 2w_8 x^*\left(\frac{w_3}{w_4} + w_5 x^*\right) + w_1 w_6 w_8(1 - 2y^*)(1 - z^*). \tag{18d}$$

Proof: The Jacobian matrix at the PEP is as follows

$$J(E_6)=\begin{bmatrix} -x^* & -w_1x^*+2w_1x^*y^* & -w_2x^*+2w_2x^*z^* \\ w_5y^*-w_5{y^*}^2 & -\frac{w_3}{w_4}y^*-w_5x^*y^* & -w_6y^* \\ w_8z^*-w_8{z^*}^2 & w_9z^* & -w_8x^*z^* \end{bmatrix}=(L_{ij}). \quad (19a)$$

The characteristic equation of (19a) is

$$\lambda^3+\delta_1\lambda^2+\delta_2\lambda+\delta_3=0, \quad (19b)$$

where

$$\delta_1=-(L_{11}++L_{22}+L_{33}),$$
$$\delta_2=L_{11}L_{22}-L_{12}L_{21}+L_{22}L_{33}-L_{23}L_{32}+L_{11}L_{33}-L_{13}L_{31},$$
$$\delta_3=-L_{33}(L_{11}L_{22}-L_{12}L_{21})-L_{12}L_{23}L_{31}-L_{13}L_{21}L_{32}+L_{11}L_{23}L_{32}+L_{13}L_{22}L_{31},$$

and

$$\begin{aligned}\Delta=\delta_1\,\delta_2-\delta_3=&-(L_{11}+L_{22})[L_{11}L_{22}-L_{12}L_{21}]\\ &-(L_{11}+L_{33})[L_{11}L_{33}-L_{13}L_{31}]\\ &-(L_{22}+L_{33})[L_{22}L_{33}-L_{23}L_{32}]\\ &-2L_{11}L_{22}L_{33}+L_{12}L_{23}L_{31}+L_{13}L_{21}L_{32}.\end{aligned}$$

By using Routh-Hurwitz criterion [18], all the roots of equation (19b) have real parts of less than zero and then $E_6=(x^*,y^*,z^*)$ is locally asymptotically stable if δ_1, δ_3 and Δ are positive. It is found by computation that all Routh-Hurwitz constraints are satisfied under the sufficient conditions given by (18a) – (18d).

4. Global *s*tability analysis

In this section, the global dynamics of system (4) is investigated using Lyapnuov functions, as demonstrated in the next theorems [18].

Theorem 3: Suppose that the FAEP is locally asymptotically stable. Then E_1 is a globally asymptotically stable with the following sufficient conditions are satisfied

$$\tilde{\tilde{x}}\le\min\{\frac{w_{10}}{w_8},\frac{w_7-w_3}{w_5}\}, \quad (20a)$$

$$\frac{w_2w_9}{w_8}<\frac{w_1w_6}{w_5}. \quad (20b)$$

Proof: Define Ω_1 as a real valued function that is given by

$$\Omega_1=q_1\left(x-\tilde{\tilde{x}}-\tilde{\tilde{x}}\ln\frac{x}{\tilde{\tilde{x}}}\right)+q_2y+q_3z,$$

where the constants q_1,q_2,q_3 are the greater than zero constants that are to be determine. Obviously, Ω_1 is a positive definite function that is defined for all $x>0, y\ge0$ and $z\ge0$. Then, $\frac{d\Omega_1}{dt}$ can be written as follows

$$\begin{aligned}\frac{d\Omega_1}{dt}=&-q_1(x-\bar{\bar{x}})^2-(w_1q_1-w_5q_2)xy-(-w_1q_1\bar{\bar{x}}-q_2w_3+w_7q_2)y\\ &-(w_5q_2-w_1q_1)xy^2-\left(w_1q_1\bar{\bar{x}}+\frac{q_2w_3}{w_4}\right)y^2\\ &-(-w_2q_1+w_8q_3)xz^2-w_2q_1\bar{\bar{x}}z^2\\ &-(q_1w_2-q_3w_8)xz-(q_3w_{10}-q_1w_2\bar{\bar{x}})z-(q_2w_6-q_3w_9)yz.\end{aligned}$$

Now, by choosing the positive constants as $q_1=1, q_2=\frac{w_1}{w_5}, q_3=\frac{w_2}{w_8}$, then we get that

$$\begin{aligned}\frac{d\Omega_1}{dt}\le&-(x-\bar{\bar{x}})^2-w_1\left(-\bar{\bar{x}}+\frac{w_7-w_3}{w_5}\right)y\\ &-w_2\left(\frac{w_{10}}{w_8}-\bar{\bar{x}}\right)z-\left(\frac{w_1w_6}{w_5}-\frac{w_2w_9}{w_8}\right)yz.\end{aligned}$$

Then, by using the above sufficient conditions (20a)-(20b), the derivative $\frac{d\Omega_1}{dt}$ is a negative definite function. Hence, E_1 is a globally asymptotically stable.

Theorem 4: Suppose that the SAEP is locally asymptotically stable. Then E_2 has a basin of attraction that satisfies the following sufficient conditions

$$w_1\hat{y}y<x, \quad (21a)$$

$$\frac{w_2w_9}{w_8}<\frac{w_1w_6}{w_5}<\frac{w_2w_{10}}{w_8\hat{y}}. \quad (21b)$$

Proof: Consider the following real valued function

$$\Omega_2=\wp_1x+\wp_2\left(y-\hat{y}-\hat{y}\ln\frac{y}{\hat{y}}\right)+\wp_3z,$$

where the constants $\wp_1, \wp_2, \wp_3$ are greater than zero and must be determined. Obviously, Ω_2 is a positive definite function that is defined for all $x \geq 0, y > 0$ and $z \geq 0$. Then $\frac{d\Omega_2}{dt}$ can be written as follows

$$\frac{d\Omega_2}{dt} = -(\wp_2 w_5 \hat{y} - \wp_1)x - \wp_1 x^2 - (\wp_1 w_1 - \wp_2 w_5 - \wp_2 w_5 \hat{y})xy$$
$$-(\wp_2 w_5 - \wp_1 w_1)xy^2 - (\wp_1 w_2 - \wp_3 w_8)xz - (w_8 \wp_3 - \wp_1 w_2)xz^2$$
$$-\frac{w_3 \wp_2}{w_4}(y - \hat{y})^2 - (\wp_2 w_6 - \wp_3 w_9)yz - (\wp_3 w_{10} - \wp_2 w_6 \hat{y})z.$$

Then, by choosing the positive constants as $\wp_1 = 1, \wp_2 = \frac{w_1}{w_5}, \wp_3 = \frac{w_2}{w_8}$, then by substituting these constants, we obtain

$$\frac{d\Omega_2}{dt} = -(w_1 \hat{y} - 1)x - x^2 + w_1 \hat{y} xy - \frac{w_1 w_3}{w_4 w_5}(y - \hat{y})^2$$
$$-\left(\frac{w_1 w_6}{w_5} - \frac{w_2 w_9}{w_8}\right) yz - \left(\frac{w_2 w_{10}}{w_8} - \frac{w_1 w_6}{w_5} \hat{y}\right) z.$$

Using the above sufficient conditions (21a)-(21b) along with the local stability conditions, the derivative $\frac{d\Omega_2}{dt}$ is a negative definite function. Hence, E_2 has a basin of attraction that satisfies the given sufficient conditions.

Theorem 5: Assume that the PFEP of system (4) is locally asymptotically stable. Then E_3 has a basin of attraction that satisfies the following sufficient conditions

$$\frac{w_8}{w_2} < \frac{w_5 w_9}{w_6} \breve{y} + w_8 \breve{z}, \quad (22a)$$
$$w_2 \breve{z} z < x, \quad (22b)$$
$$\frac{w_5 w_9}{w_6} \breve{y} < \left(\frac{w_1 w_8}{w_2} - \frac{w_5 w_9}{w_6}\right)(1 - y). \quad (22c)$$

Proof: Define the following real valued function

$$\Omega_3 = \mathscr{b}_1 x + \mathscr{b}_2 \left(y - \breve{y} - \breve{y} \ln \frac{y}{\breve{y}}\right) + \mathscr{b}_3 \left(z - \breve{z} - \breve{z} \ln \frac{z}{\breve{z}}\right)$$

where the constants $\mathscr{b}_1, \mathscr{b}_2, \mathscr{b}_3$ are the greater than zero constants that are to be determine. Obviously, Ω_3 is a positive definite function that is defined for all $x \geq 0, y > 0$ and $z > 0$. Then $\frac{d\Omega_3}{dt}$ can be written as follows

$$\frac{d\Omega_3}{dt} = -(\mathscr{b}_2 w_5 \breve{y} + \mathscr{b}_3 w_8 \breve{z} - \mathscr{b}_1)x - \mathscr{b}_1 x^2 - (\mathscr{b}_2 w_5 - \mathscr{b}_1 w_1)xy^2$$
$$-(\mathscr{b}_1 w_2 - \mathscr{b}_3 w_8 - \mathscr{b}_3 w_8 \breve{z})xz - (w_8 \mathscr{b}_3 - \mathscr{b}_1 w_2)xz^2$$
$$-\frac{w_3 \mathscr{b}_2}{w_4}(y - \breve{y})^2 - (\mathscr{b}_1 w_1 - \mathscr{b}_2 w_5 - w_5 \mathscr{b}_2 \breve{y})xy$$
$$-(\mathscr{b}_2 w_6 - \mathscr{b}_3 w_9)(y - \breve{y})(z - \breve{z}).$$

Now, by choosing the positive constants as $\mathscr{b}_1 = \frac{w_8}{w_2}, \mathscr{b}_2 = \frac{w_9}{w_6}, \mathscr{b}_3 = 1$, we have

$$\frac{d\Omega_3}{dt} = -\left(\frac{w_5 w_9}{w_6} \breve{y} + w_8 \breve{z} - \frac{w_8}{w_2}\right) x - \frac{w_8}{w_2}(x - w_2 \breve{z} z)x - \frac{w_3 w_9}{w_4 w_6}(y - \breve{y})^2$$
$$-\left(\left(\frac{w_1 w_8}{w_2} - \frac{w_5 w_9}{w_6}\right)(1 - y) - \frac{w_5 w_9}{w_6} \breve{y}\right) xy.$$

Then, by using the above sufficient conditions (22a)-(22c), the derivative $\frac{d\Omega_3}{dt}$ is a negative definite function. Hence, E_3 has a basin of attraction that satisfies the given sufficient conditions.

Theorem 6: Suppose that the MPFEP is locally asymptotically stable. Then E_4 is a globally asymptotically stable, provided that

$$\left[\left(\frac{w_2 w_5}{w_1} - \frac{w_6 w_8}{w_9}\right)(1 - z) - \frac{w_2 w_5}{w_1} \bar{z}\right]^2 < 4\left(\frac{w_5}{w_1}\right)\left(\frac{w_6 w_8}{w_9} \bar{x}\right). \quad (23)$$

Proof: Define the following real valued function

$$\Omega_4 = \mathfrak{h}_1 \left(x - \bar{x} - \bar{x} \ln \frac{x}{\bar{x}}\right) + \mathfrak{h}_2 y + \mathfrak{h}_3 \left(z - \bar{z} - \bar{z} \ln \frac{z}{\bar{z}}\right),$$

where the constants $\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3$ are greater than zero and must be determined. Obviously, Ω_4 is a positive definite function that is defined for all $x > 0, y \geq 0$ and $z > 0$. Therefore, $\frac{d\Omega_4}{dt}$ can be written as follows

$$\frac{d\Omega_4}{dt} = -\mathfrak{h}_1(x-\bar{x})^2 - [\mathfrak{h}_1 w_2(1-(z+\bar{z}) - \mathfrak{h}_3 w_8(1-z)](x-\bar{x})(z-\bar{z})$$
$$-\mathfrak{h}_3 w_8 \bar{x}(z-\bar{z})^2 - [\mathfrak{h}_1 w_1 - \mathfrak{h}_2 w_5]xy$$
$$-[\mathfrak{h}_2(w_7-w_3) + \mathfrak{h}_3 w_9 \bar{z} - \mathfrak{h}_1 w_1 \bar{x}]y - [\mathfrak{h}_2 w_5 - \mathfrak{h}_1 w_1]xy^2$$
$$-\left[\mathfrak{h}_1 w_1 \bar{x} + \mathfrak{h}_2 \frac{w_3}{w_4}\right] y^2 - [\mathfrak{h}_2 w_6 - \mathfrak{h}_3 w_9]yz.$$

Now, by choosing the positive constants as $\mathfrak{h}_1 = \frac{w_5}{w_1}, \mathfrak{h}_2 = 1, \mathfrak{h}_3 = \frac{w_6}{w_9}$, and substituting these constants, we obtain

$$\frac{d\Omega_4}{dt} < -\frac{w_5}{w_1}(x-\bar{x})^2 - \left[\left(\frac{w_2 w_5}{w_1} - \frac{w_6 w_8}{w_9}\right)(1-z) - \frac{w_2 w_5}{w_1}\bar{z}\right](x-\bar{x})(z-\bar{z})$$
$$-\frac{w_6 w_8}{w_9}\bar{x}(z-\bar{z})^2 - [(w_7-w_3) + w_6\bar{z} - w_5\bar{x}]y.$$

Accordingly, by using the condition (23) along with the local stability condition (16f), the derivative $\frac{d\Omega_4}{dt}$ becomes

$$\frac{d\Omega_4}{dt} < -\left[\sqrt{\frac{w_5}{w_1}}(x-\bar{x}) + \sqrt{\frac{w_6 w_8}{w_9}\bar{x}}(z-\bar{z})\right]^2 - [(w_7-w_3) + w_6\bar{z} - w_5\bar{x}]y.$$

Clearly, $\frac{d\Omega_4}{dt}$ is negative definite and hence E_4 is a globally asymptotically stable.

Theorem 7: Assume that the TPFEP of system (4) is locally asymptotically stable. Then E_5 is globally asymptotically stable, provided that

$$\left[\frac{w_1 w_8}{w_2} - \frac{w_1 w_8}{w_2}(y+\bar{\bar{y}}) - \frac{w_5 w_9}{w_6}(1-\bar{\bar{y}})\right]^2 < 4\left(\frac{w_8}{w_2}\right)\left(\frac{w_9}{w_6}\left[\frac{w_3}{w_4} + w_5 x\right]\right). \quad (24)$$

Proof: Define the following real valued function

$$\Omega_5 = \mathfrak{Y}_1\left(x - \bar{\bar{x}} - \bar{\bar{x}}\ln\frac{x}{\bar{\bar{x}}}\right) + \mathfrak{Y}_2\left(y - \bar{\bar{y}} - \bar{\bar{y}}\ln\frac{y}{\bar{\bar{y}}}\right) + \mathfrak{Y}_3 z$$

where the constants $\mathfrak{Y}_1, \mathfrak{Y}_1, \mathfrak{Y}_1$ are greater than zero and must be determined. Obviously, Ω_5 is a positive definite function, which is defined for all $x > 0, y > 0$ and $z \geq 0$. Therefore, $\frac{d\Omega_5}{dt}$ can be written as follows

$$\frac{d\Omega_5}{dt} = -\mathfrak{Y}_1(x-\bar{\bar{x}})^2 - [\mathfrak{Y}_1 w_1 - \mathfrak{Y}_1 w_1(y+\bar{\bar{y}}) - \mathfrak{Y}_2 w_5(1-\bar{\bar{y}})](x-\bar{\bar{x}})(y-\bar{\bar{y}})$$
$$-\mathfrak{Y}_2\left[\frac{w_3}{w_4} + w_5 x\right](y-\bar{\bar{y}})^2 - [\mathfrak{Y}_1 w_2 - \mathfrak{Y}_3 w_8]xz$$
$$-[\mathfrak{Y}_3 w_{10} - \mathfrak{Y}_1 w_2\bar{\bar{x}} - \mathfrak{Y}_2\bar{\bar{y}}w_6]z - [\mathfrak{Y}_3 w_8 - \mathfrak{Y}_1 w_2]xz^2$$
$$-\mathfrak{Y}_1 w_2 xz^2 - [\mathfrak{Y}_2 w_6 - \mathfrak{Y}_3 w_9]yz.$$

By choosing the positive constants as $\mathfrak{Y}_1 = \frac{w_8}{w_2}, \mathfrak{Y}_2 = \frac{w_9}{w_6}, \mathfrak{Y}_3 = 1$, and substituting these constants, we get

$$\frac{d\Omega_5}{dt} < -\frac{w_8}{w_2}(x-\bar{\bar{x}})^2 - \left[\frac{w_1 w_8}{w_2} - \frac{w_1 w_8}{w_2}(y+\bar{\bar{y}}) - \frac{w_5 w_9}{w_6}(1-\bar{\bar{y}})\right](x-\bar{\bar{x}})(y-\bar{\bar{y}})$$
$$-\frac{w_9}{w_6}\left[\frac{w_3}{w_4} + w_5 x\right](y-\bar{\bar{y}})^2 - [w_{10} - w_8\bar{\bar{x}} - w_9\bar{\bar{y}}]z.$$

Therefore, by using the condition (24) along with the local stability condition (17f), the derivative $\frac{d\Omega_5}{dt}$ becomes

$$\frac{d\Omega_5}{dt} < -\left[\sqrt{\frac{w_8}{w_2}}(x-\bar{\bar{x}}) + \sqrt{\frac{w_9}{w_6}\left[\frac{w_3}{w_4} + w_5 x\right]}(y-\bar{\bar{y}})\right]^2 - [w_{10} - w_8\bar{\bar{x}} - w_9\bar{\bar{y}}]z.$$

Clearly, $\frac{d\Omega_5}{dt}$ is negative definite and hence E_5 is globally asymptotically stable.

Theorem 8: Suppose that the PEP is locally asymptotically stable. Then E_6 is globally asymptotically stable if the following sufficient conditions hold

$$[w_1 - w_1(y+y^*) - w_5 + w_5 y]^2 < \left[\frac{w_3}{w_4} + w_5 x^*\right], \quad (25a)$$

$$[w_2 - w_2(z+z^*) - w_8 + w_8 z^*]^2 < w_8 x, \quad (25b)$$

$$[w_6 - w_9]^2 < \left[\frac{w_3}{w_4} + w_5 x^*\right][w_8 x]. \quad (25c)$$

Proof: Define the following real valued function

$\Omega_6 = x - x^* - x^* \ln \frac{x}{x^*} + y - y^* - y^* \ln \frac{y}{y^*} + z - z^* - z^* \ln \frac{z}{z^*}$.

Clearly, Ω_6 is a positive definite function that is defined for all $x > 0, y > 0$ and $z > 0$. Therefore, $\frac{d\Omega_6}{dt}$ can be written as follows

$$\frac{d\Omega_6}{dt} = -(x - x^*)^2 - [w_1 - w_1(y + y^*) - w_5 + w_5 y](x - x^*)(y - y^*)$$
$$-\left[\frac{w_3}{w_4} + w_5 x^*\right](y - y^*)^2 - [w_2 - w_2(z + z^*) - w_8 + w_8 z^*](x - x^*)(z - z^*)$$
$$-w_8 x(z - z^*)^2 - [w_6 - w_9](y - y^*)(z - z^*).$$

Therefore, by using the conditions (25a)-(25c), the derivative $\frac{d\Omega_6}{dt}$ becomes

$$\frac{d\Omega_6}{dt} < -\frac{1}{2}\left[(x - x^*) + \sqrt{\frac{w_3}{w_4} + w_5 x^*}(y - y^*)\right]^2$$
$$-\frac{1}{2}\left[(x - x^*) + \sqrt{w_8 x}(z - z^*)\right]^2$$
$$-\frac{1}{2}\left[\sqrt{\frac{w_3}{w_4} + w_5 x^*}(y - y^*) + \sqrt{w_8 x}(z - z^*)\right]^2.$$

Obviously, $\frac{d\Omega_6}{dt}$ is negative definite and E_6 is globally asymptotically stable.

5. Persistence

In this section the persistence of system (4) is investigated using the Gard technique [19]. First, we need to check the existence of periodic dynamics in the boundary planes. Obviously, the system (4) has three possible subsystems which are obtained in the absence of x, y and z and can be written, respectively, as

$$\begin{aligned}\frac{dy}{dt} &= y\left[w_3\left(1 - \frac{y}{w_4}\right) - w_6 z - w_7\right] = g_1(y, z),\\ \frac{dz}{dt} &= z(w_9 y - w_{10}) = g_2(y, z).\end{aligned} \tag{26}$$

$$\begin{aligned}\frac{dx}{dt} &= x[(1 - x) - w_2(1 - z)z] = g_3(x, z),\\ \frac{dz}{dt} &= z[w_8 x(1 - z) - w_{10}] = g_4(x, z).\end{aligned} \tag{27}$$

$$\begin{aligned}\frac{dx}{dt} &= x\big((1 - x) - w_1(1 - y)y\big) = g_5(x, y),\\ \frac{dy}{dt} &= y(w_3\left(1 - \frac{y}{w_4}\right) + w_5 x(1 - y) - w_7 = g_6(x, y).\end{aligned} \tag{28}$$

Clearly, subsystems (26), (27) and (28) fall in the interior of yz −plane, xz −plane and xy −plane, respectively. Now, define the function $H_1(y, z) = \frac{1}{yz}$, which is a C^1 function in interior of $\mathbb{R}^2_+$ of the yz −plane. Then, determine the following quantity

$$\Delta(y, z) = \frac{\partial(g_1 H_1)}{\partial y} + \frac{\partial(g_2 H_1)}{\partial z} = -\frac{w_3}{w_4}.\frac{1}{z} \neq 0.$$

According to the Dulic-Bendixon criterion [20], the quantity $\Delta(y, z)$ does not have a changed sign and it is not identically zero, then there is no closed curve in the interior of $\mathbb{R}^2_+$ of the yz −plane. Further, since the subsystem (26) is bounded and has a unique positive equilibrium point in the interior of $\mathbb{R}^2_+$ of the yz −plane, that is given by $(\breve{y}, \breve{z})$, then, according to Poincare-Bendixon theorem [20], it is globally asymptotically stable, whenever it exists. and locally asymptotically stable.

Similarly, by using the C^1 functions in the interior of $\mathbb{R}^2_+$ of the xz and xy planes, which are defined as $H_2(x, z) = \frac{1}{xz}$ and $H_3(x, y) = \frac{1}{xy}$, respectively, it is easy to verify that there is no periodic dynamics in the interior of $\mathbb{R}^2_+$ of their planes, and the positive equilibrium point of each subsystem $(\bar{x}, \bar{z})$ and $(\bar{\bar{x}}, \bar{\bar{y}})$ is globally asymptotically stable, whenever it exists, and locally asymptotically stable.

Consequently, the necessary and sufficient conditions for the uniform persistence of system (4) are derived in the following theorem.

Theorem 9: The system (4) is uniformly persistent if the following conditions hold

$(w_7 < w_3 + w_5)$ or $(w_{10} < w_8)$, (29a)

$(w_1(1 - \hat{y})\hat{y} < 1)$ or $(w_{10} < w_9 \hat{y})$, (29b)

$$(1-\breve{y})\breve{y}+w_1(1-\breve{z})\breve{z}<1, \quad (29c)$$
$$w_6\bar{z}+w_7<w_3+w_5\bar{x}, \quad (29d)$$
$$w_{10}<w_8\bar{\bar{x}}+w_9\bar{\bar{y}}. \quad (29e)$$

Proof: Define $\sigma(x,y,z)=x^{\rho_1}y^{\rho_2}z^{\rho_3}$ as an average Lyapunov function with positive constant ρ_i; $i=1,2,3$.

Obviously, $\sigma(x,y,z)$ is a C^1 positive definite function, and if $x\to 0$ or $y\to 0$ or $z\to 0$, then $\sigma(x,y,z)\to 0$. Consequently, we get that

$$Þ(x,y,z)=\frac{\sigma'(x,y,z)}{\sigma(x,y,z)}=\rho_1[(1-x)-w_1(1-y)y-w_2(1-z)z]$$
$$+\rho_2\left[w_3\left(1-\frac{y}{w_4}\right)+w_5x(1-y)-w_6z-w_7\right]$$
$$+\rho_3[w_8x(1-z)+w_9y-w_{10}].$$

Now, according to the Gard technique, the proof is followed and the system uniformly persists if we can prove that $Þ(x,y,z)>0$ at each of the boundary equilibrium points. Since

$$Þ(E_0)=\rho_1+\rho_2(w_3-w_7)-\rho_3(w_{10}).$$

Clearly, $Þ(E_0)>0$ if we chose the positive constants, so that ρ_1 is sufficiently large with respect to others.

$$Þ(E_1)=\rho_2(w_3+w_5-w_7)+\rho_3(w_8-w_{10}).$$

Clearly, $Þ(E_1)>0$ if any one of the conditions given by equation (29a) holds with suitable selection of positive constants ρ_2 and ρ_3.

$$Þ(E_2)=\rho_1(1-w_1(1-\hat{y})\hat{y})+\rho_2(w_9\hat{y}-w_{10}).$$

Again, $Þ(E_2)>0$ if any one of the conditions given by equation (29b) holds with suitable selection of positive constants ρ_1 and ρ_3.

$$Þ(E_3)=\rho_1[1-w_1(1-\breve{y})\breve{y}-w_2(1-\breve{z})\breve{z}].$$

Obviously, $Þ(E_3)>0$ under condition (29c).

$$Þ(E_4)=\rho_2[w_3+w_5\bar{x}-w_6\bar{z}-w_7].$$

Obviously, $Þ(E_4)>0$ under condition (29d).

$$Þ(E_5)=\rho_3[w_8\bar{\bar{x}}+w_9\bar{\bar{y}}-w_{10}].$$

Clearly, $Þ(E_5)>0$ under the condition (29e).

Hence, all the requirements of Gard technique are satisfied.

6. Bifurcation analysis

In this section, the appearance of the local bifurcation in system (4) is investigated. In a dynamical system, a bifurcation is carried out when a small smooth variation that is made to the parameter of a model gives a sudden or topical variation in its action. In general, the local stability properties of equilibrium point, periodic orbits, or other invariant sets, change at a bifurcation. Sotomayor's theorem [17] will be applied to study the occurrence of local bifurcation. We can write system (4) in the vector form as

$$\frac{dX}{dt}=F(X), \quad (30)$$

where $X=(x,y,z)^T$ and $F(X)=(xf_1,yf_2,zf_3)^T$.

Now, it is easy to verify that for any vector $V=(v_1,v_2,v_3)^T$, the second derivative of the system (30) can be written as

$$D^2F(X,\beta)(V,V)=(c_{i1})_{3\times1}, \quad (31a)$$

where

$$c_{11}=-2{v_1}^2-2w_1v_1v_2+4w_1yv_1v_2-2w_2v_1v_3$$
$$+4w_2zv_1v_3+2w_1x{v_2}^2+2w_2x{v_3}^2,$$
$$c_{21}=-4w_5yv_1v_2+2w_5v_1v_2-\frac{2w_3}{w_4}{v_2}^2-2w_5x{v_2}^2-2w_6v_2v_3,$$
$$c_{31}=-4w_8zv_1v_3+2w_8v_1v_3+2w_9v_2v_3-2w_8x{v_3}^2,$$

with β is any bifurcation parameter. Moreover, the third derivative of the system (30) is given by

$$D^3F(X,\beta)(V,V,V)=(d_{i1})_{3\times1}, \quad (31b)$$

where

$$d_{11}=6w_1v_1{v_2}^2+6w_2v_1{v_3}^2;\ d_{21}=-6w_5v_1{v_2}^2;\ d_{31}=-6w_8v_1{v_3}^2.$$

Theorem 10. Assume that condition (13c) holds with the parameter $w_8\equiv w_8^*=w_{10}$, then the system (4) near FAEP undergoes a transcritical bifurcation provided that

$w_2 \neq 1.$ (32)

However, it has a pitchfork bifurcation otherwise.

Proof. Clearly, under the condition (13c) with $w_8 \equiv w_8^*$, the Jacobian matrix $J(E_1)$ that is given by equation (13a) has the following eigenvalues: ${\lambda_{11}}^* = -1$, ${\lambda_{12}}^* = w_3 + w_5 - w_7 < 0$ and ${\lambda_{13}}^* = 0$. So, the FAEP is a non-hyperbolic point and the Jacobian matrix can be written as

$$J_1 = J(E_1, w_8^*) = \begin{bmatrix} -1 & -w_1 & -w_2 \\ 0 & w_3 + w_5 - w_7 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Let $V_1 = (v_{11}, v_{12}, v_{13})^T$ be the eigenvectors of J_1 corresponding to ${\lambda_{13}}^* = 0$. Then, we will get

$$V_1 = (-w_2 v_{13}, 0, v_{13})^T,$$

where v_{13} is any nonzero real number. In the same way, $\Phi_1 = (\Phi_{11}, \Phi_{12}, \Phi_{13})^T$ is the eigenvectors associated with the zero eigenvalue ${\lambda_{13}}^* = 0$ of ${J_1}^T$. Then we obtain that

$\Phi_1 = (0,0,\Phi_{13})^T$,

where $\Phi_{13} \neq 0 \in \mathbb{R}$. Furthermore, it is simple to establish that

$$F_{W_8}(X, w_8) = (0,0, xz - xz^2)^T.$$

Hence, $F_{W_8}(E_1, w_8^*) = (0,0,0)^T$, which gives ${\Phi_1}^T F_{W_8}(E_1, w_8^*) = 0$.

Thus, according to Sotomayor theorem, the saddle node bifurcation cannot occur. Moreover, since

$${\Phi_1}^T \left[DF_{W_8}(E_1, w_8^*) V_1 \right] = v_{13}\Phi_{13} \neq 0,$$

here DF_{W_8}represents the derivative of F_{W_8} with respect to X. Now, by substituting E_1, w_8^* and V_1 in Eq. (31a), we get that

${\Phi_1}^T [D^2 F(E_1, w_8^*)(V_1, V_1)] = 2w_8^*(w_2 - 1){v_{13}}^2 \Phi_{13}$.

Clearly, ${\Phi_1}^T [D^2 F(E_1, w_8^*)(V_1, V_1)] \neq 0$ under the condition (32) and then system (4) undergoes a transcritical bifurcation when $w_8 \equiv w_8^*$.

Otherwise, we have ${\Phi_1}^T [D^2 F(E_1, w_8^*)(V_1, V_1)] = 0$, when $w_2 = 1$. Now, by substituting E_1, w_8^* and V_1 in Eq. (31b), we get that

$${\Phi_1}^T [D^3 F(E_1, w_8^*)(V_1, V_1, V_1)] = 6w_8^* {v_{13}}^3 \Phi_{13} \neq 0.$$

Hence, pitchfork bifurcation takes place.

Theorem 11. Suppose that condition (14c) holds with the parameter $w_{10} \equiv {w_{10}}^* = w_9 \hat{y}$, then the system (4) near SAEP undergoes a transcritical bifurcation.

Proof. Clearly, under the condition (14c) with $w_{10} \equiv {w_{10}}^*$, the Jacobian matrix $J(E_2)$ that is given by equation (14a) has the following eigenvalues ${\lambda_{21}}^* = 1 - w_1\hat{y} + w_1\hat{y}^2 < 0$, ${\lambda_{22}}^* = -\frac{w_3}{w_4}\hat{y}$ and ${\lambda_{23}}^* = 0$. So, the SAEP is a non-hyperbolic point and the Jacobian matrix can be written as

$$J_2 = J(E_2, w_{10}^*) = \begin{bmatrix} 1 - w_1\hat{y} + w_1\hat{y}^2 & 0 & 0 \\ w_5\hat{y} - w_5\hat{y}^2 & -\frac{w_3}{w_4}\hat{y} & -w_6\hat{y} \\ 0 & 0 & 0 \end{bmatrix}.$$

Similarly, as in theorem (10), direct computation gives that

$V_2 = \left(0, -\frac{w_4 w_6}{w_3} v_{23}, v_{23}\right)^T$; $v_{23} \neq 0 \in \mathbb{R}$ as an eigenvector corresponding to ${\lambda_{23}}^* = 0$ of J_2.

$\Phi_2 = (0,0,\Phi_{23})^T$; $\Phi_{23} \neq 0 \in \mathbb{R}$ as an eigenvector corresponding to ${\lambda_{23}}^* = 0$ of ${J_2}^T$.

${\Phi_2}^T F_{W_{10}}(E_1, w_{10}^*) = 0$.

${\Phi_2}^T [DF(E_2, w_{10}^*) V_2] = -v_{23}\Phi_{23} \neq 0$.

${\Phi_2}^T [D^2 F(E_2, w_{10}^*)(V_2, V_2)] = -2{v_{23}}^2 \Phi_{23} \left(\frac{w_4 w_6 w_9}{w_3} + w_8 x \right) \neq 0$.

Hence, system (4) undergoes a transcritical bifurcation when $w_{10} \equiv {w_{10}}^*$.

Theorem 12. Assume that $w_1 \equiv {w_1}^* = \frac{1 - w_2(1-\breve{z})\breve{z}}{(1-\breve{y})\breve{y}}$, then system (4) near PFEP undergoes a transcritical bifurcation, provided that

$1 + w_1\gamma_1(1 - 2\breve{y}) + w_2\gamma_2(1 - 2\breve{z}) \neq 0,$ (33)

where γ_1 and γ_2 are given in the proof. However, it has a pitchfork bifurcation otherwise.

Proof. Clearly, when $w_1 \equiv {w_1}^*$, the Jacobian matrix $J(E_3)$ that is given by equation (15a) has the following eigenvalues

${\lambda_{31}}^* = 0$, ${\lambda_{32}}^* = -\frac{w_3}{2w_4}\breve{y} + \frac{\sqrt{\left(\frac{w_3}{w_4}\breve{y}\right)^2 - 4w_6w_9\breve{y}\breve{z}}}{2}$ and ${\lambda_{33}}^* = -\frac{w_3}{2w_4}\breve{y} - \frac{\sqrt{\left(\frac{w_3}{w_4}\breve{y}\right)^2 - 4w_6w_9\breve{y}\breve{z}}}{2}$.

So, the PFEP is a non-hyperbolic point and the Jacobian matrix can be written as

$$J_3 = J(E_3, w_1^*) = \begin{bmatrix} 0 & 0 & 0 \\ w_5\breve{y} - w_5\breve{y}^2 & -\frac{w_3}{w_4}\breve{y} & -w_6\breve{y} \\ w_8\breve{z} - w_8\breve{z}^2 & w_9\breve{z} & 0 \end{bmatrix}.$$

Direct computation gives that

$V_3 = (v_{31}, \gamma_1 v_{31}, \gamma_2 v_{31})^T$; $v_{31} \neq 0 \in \mathbb{R}$ as an eigenvector corresponding to ${\lambda_{31}}^* = 0$ of J_3, where $\gamma_1 = -\frac{w_8(1-\breve{z})}{w_9} < 0$ and $\gamma_2 = \frac{w_3w_8(1-\breve{z}) + w_4w_5w_9(1-\breve{y})}{w_4w_6w_9} > 0$.

$\Phi_3 = (\Phi_{31}, 0, 0)^T$; $\Phi_{31} \neq 0 \in \mathbb{R}$ as an eigenvector corresponding to ${\lambda_{31}}^* = 0$ of ${J_3}^T$.

${\Phi_3}^T F_{W_1}(E_3, w_1^*) = 0$.

${\Phi_3}^T \left[DF_{W_1}(E_3, w_1^*)V_3\right] = -\breve{y}(1-\breve{y})v_{31}\Phi_{31} \neq 0$.

${\Phi_3}^T [D^2F(E_3, w_1^*)(V_3, V_3)] = -2{v_{31}}^2\Phi_{31}[1 + w_1\gamma_1(1-2\breve{y}) + w_2\gamma_2(1-2\breve{z})]$.

Clearly, ${\Phi_3}^T [D^2F(E_3, w_1^*)(V_3, V_3)] \neq 0$ under the condition (33) and then system (4) undergoes a transcritical bifurcation when $w_1 \equiv w_1^*$.

Otherwise, we have ${\Phi_3}^T [D^2F(E_3, w_1^*)(V_3, V_3)] = 0$, in case of violating the condition (33). Now, by substituting E_3, w_1^* and V_3 in Eq. (31b), we get that

$${\Phi_3}^T [D^3F(E_3, w_1^*)(V_3, V_3, V_3)] = 6(w_1^*{\gamma_1}^2 + w_2{\gamma_2}^2){v_{13}}^3\Phi_{13} \neq 0.$$

Hence, pitchfork bifurcation takes place.

Theorem 13. Suppose that condition (16g) holds with the parameter $w_7 \equiv w_7^* = w_3 + w_5\bar{x} - w_6\bar{z}$, then the system (4) near MPFEP undergoes a transcritical bifurcation, provided that

$$w_5\gamma_3 - \frac{w_3}{w_4} - w_5\bar{x} - w_6\gamma_4 \neq 0, \qquad (34)$$

where γ_3 and γ_4 are given in the proof. However, it has a pitchfork bifurcation otherwise.

Proof. Note that, when $w_7 \equiv {w_7}^*$, the Jacobian matrix $J(E_4)$ that is given by equation (16a) has the following eigenvalues

${\lambda_{41}}^* = \frac{T_4}{2} + \frac{1}{2}\sqrt{{T_4}^2 - 4D_4}$, ${\lambda_{42}}^* = 0$ and ${\lambda_{43}}^* = \frac{T_4}{2} - \frac{1}{2}\sqrt{{T_4}^2 - 4D_4}$,

where $T_4 < 0$ and $D_4 > 0$ are given in equation (16b). So, the MPFEP is a non-hyperbolic point and the Jacobian matrix can be written as

$$J_4 = J(E_4, {w_7}^*) = \begin{bmatrix} -\bar{x} & -w_1\bar{x} & -w_2\bar{x} + 2w_2\bar{x}\bar{z} \\ 0 & 0 & 0 \\ w_8\bar{z} - w_8\bar{z}^2 & w_9\bar{z} & -w_8\bar{x}\bar{z} \end{bmatrix}.$$

Direct computation gives

$V_4 = (\gamma_3 v_{42}, v_{42}, \gamma_4 v_{42})^T$; $v_{42} \neq 0 \in \mathbb{R}$ as an eigenvector corresponding to ${\lambda_{42}}^* = 0$ of J_4, where $\gamma_3 = \frac{\bar{x}\bar{z}[w_9 - w_1w_8(1-\bar{z})]}{D_4}$ and $\gamma_4 = -\frac{\bar{x}\bar{z}[w_2w_9(1-2\bar{z}) + w_1w_8\bar{x}]}{D_4}$.

$\Phi_4 = (0, \Phi_{42}, 0)^T$; $\Phi_{31} \neq 0 \in \mathbb{R}$ as an eigenvector corresponding to ${\lambda_{42}}^* = 0$ of ${J_4}^T$.

${\Phi_4}^T F_{W_7}(E_4, {w_7}^*) = 0$.

${\Phi_4}^T \left[DF_{W_4}(E_4, {w_7}^*)V_4\right] = -v_{42}\Phi_{42} \neq 0$.

${\Phi_4}^T [D^2F(E_4, {w_7}^*)(V_4, V_4)] = 2{v_{42}}^2\Phi_{42}\left[w_5\gamma_3 - \frac{w_3}{w_4} - w_5\bar{x} - w_6\gamma_4\right]$.

Clearly, ${\Phi_4}^T [D^2F(E_4, {w_7}^*)(V_4, V_4)] \neq 0$ under the condition (34) and then system (4) undergoes a transcritical bifurcation when $w_7 \equiv {w_7}^*$.

Otherwise, we have ${\Phi_4}^T [D^2F(E_4, {w_7}^*)(V_4, V_4)] = 0$, in case of violating the condition (34). Now, by substituting E_4, ${w_7}^*$ and V_4 in Eq. (31b), we get that

$${\Phi_4}^T [D^3F(E_4, {w_7}^*)(V_4, V_4, V_4)] = -6w_5\gamma_3{v_{42}}^3\Phi_{42} \neq 0.$$

Hence, pitchfork bifurcation takes place.

Theorem 14. Suppose that condition (17g) holds with the parameter $w_{10} \equiv {w_{10}}^* = w_8\bar{\bar{x}} + w_9\bar{\bar{y}}$, then the system (4) near TPFEP undergoes a transcritical bifurcation, provided that

$$w_8\gamma_5 + w_9\gamma_6 - w_8x \neq 0, \qquad (35)$$

where γ_5 and γ_6 are given in the proof. However, it has a pitchfork bifurcation otherwise.

Proof. Now, when $w_{10} \equiv w_{10}^*$, the Jacobian matrix $J(E_5)$ that given is by equation (17a) has the following eigenvalues

$\lambda_{51}^* = \frac{T_5}{2} + \frac{1}{2}\sqrt{T_5^2 - 4D_5}$, $\lambda_{52}^* = \frac{T_5}{2} - \frac{1}{2}\sqrt{T_5^2 - 4D_5}$ and $\lambda_{53}^* = 0$,

where $T_5 < 0$ and $D_5 > 0$ are given in equation (17b). So the TPFEP is a non-hyperbolic point and the Jacobian matrix can be written as

$$J_5 = J(E_5, w_{10}^*) = \begin{bmatrix} -\bar{\bar{x}} & -w_1\bar{\bar{x}} + 2w_1\bar{\bar{x}}\bar{\bar{y}} & -w_2\bar{\bar{x}} \\ w_5\bar{\bar{y}} - w_5\bar{\bar{y}}^2 & -\frac{w_3}{w_4}\bar{\bar{y}} - w_5\bar{\bar{x}}\bar{\bar{y}} & -w_6\bar{\bar{y}} \\ 0 & 0 & 0 \end{bmatrix}.$$

Again, as in above theorems, by direct computation we obtain

$V_5 = (\gamma_5 v_{51}, \gamma_6 v_{52}, v_{53})^T$; $v_{53} \neq 0 \in \mathbb{R}$ as an eigenvector corresponding to $\lambda_{53}^* = 0$ of J_5, where $\gamma_5 = \frac{\bar{\bar{x}}\bar{\bar{y}}\left[w_1 w_6 (1-2\bar{\bar{y}}) - w_2(\frac{w_3}{w_4} + w_5\bar{\bar{x}})\right]}{D_5}$ and $\gamma_6 = -\frac{\bar{\bar{x}}\bar{\bar{y}}[w_2 w_5(1-\bar{\bar{y}}) + w_6]}{D_5} < 0$.

$\Phi_5 = (0,0,\Phi_{53})^T$; $\Phi_{53} \neq 0 \in \mathbb{R}$ as an eigenvector corresponding to $\lambda_{53}^* = 0$ of J_5^T.

$\Phi_5^T F_{W_{10}}(E_5, w_{10}^*) = 0$.

$\Phi_5^T \left[DF_{W_{10}}(E_5, w_{10}^*) V_5\right] = -v_{53}\Phi_{53} \neq 0$.

$\Phi_5^T [D^2 F(E_5, w_{10}^*)(V_5, V_5)] = 2v_{53}^2 \Phi_{53}[w_8\gamma_5 + w_9\gamma_6 - w_8 x]$.

Clearly, $\Phi_5^T [D^2 F(E_5, w_{10}^*)(V_5, V_5)] \neq 0$ under the condition (35) and then system (4) undergoes a transcritical bifurcation when $w_{10} \equiv w_{10}^*$.

Otherwise, we have $\Phi_5^T [D^2 F(E_5, w_{10}^*)(V_5, V_5)] = 0$, in case of violating the condition (35). Now, by substituting E_5, w_{10}^* and V_5 in equation (31b), we get that

$$\Phi_5^T [D^3 F(E_5, w_{10}^*)(V_5, V_5, V_5)] = -6w_8\gamma_5 v_{53}^3 \Phi_{53} \neq 0.$$

Hence, pitchfork bifurcation takes place.

Theorem 15. Suppose that the conditions (18a) - (18b) with the condition

$$w_9 < w_1 w_8 (1 - 2y^*)(1 - z^*), \qquad (36)$$

are satisfied, then as the parameter

$w_6 \equiv w_6^* = \frac{L_{33}(L_{11}L_{22} - L_{12}L_{21}) + L_{13}(L_{21}L_{32} - L_{22}L_{31})}{y^*(L_{12}L_{31} - L_{11}L_{32})}$,

where $L_{ij}; i,j = 1,2,3$ are given in equation (19a), then system (4) near PEP undergoes a saddle node bifurcation, provided that the following condition holds

$$\gamma_9\Gamma_1 + \gamma_{10}\Gamma_2 + \Gamma_3 \neq 0, \qquad (37)$$

where all symbols are specified in the proof.

Proof. Obviously, the Jacobian matrix at the PEP with $w_6 \equiv w_6^*$ can be written as

$$J_6 = J(E_6, w_6^*) = \begin{bmatrix} -x^* & -w_1 x^*(1 - 2y^*) & -w_2 x^*(1 - 2z^*) \\ w_5 y^*(1 - y^*) & -y^*\left(\frac{w_3}{w_4} + w_5 x^*\right) & -w_6^* y^* \\ w_8 z^*(1 - z^*) & w_9 z^* & -w_8 x^* z^* \end{bmatrix} = \left(L_{ij}\right)_{3\times 3}.$$

Clearly, the elements of J_6 coincide with elements of $J(E_6)$ that are given in equation (19a), except for the element L_{23}, which is written in term of w_6^* and denoted by $L_{23}^* = -w_6^* y^*$.

Straightforward computation shows that the determinant of J_6 that is denoted by δ_3 in equation (19b) equals to zero ($\delta_3 = 0$) when $w_6 \equiv w_6^*$, which is positive under condition (36). Therefore, the characteristic equation that is given by equation (19b) has zero root with the other two negative real part roots, denoted by:

$$\lambda_{61}^* = 0, \lambda_{62}^* = -\frac{\delta_1}{2} + \frac{1}{2}\sqrt{\delta_1^2 - 4\delta_2} \text{ and } \lambda_{63}^* = -\frac{\delta_1}{2} - \frac{1}{2}\sqrt{\delta_1^2 - 4\delta_2},$$

where δ_1 and δ_2 are positive due conditions (18a) and (18b), and given by equation (19b).

Therefore, the PEP is a non-hyperbolic equilibrium point. Now, to test the possibility of the occurrence of saddle node bifurcation, the following quantities are determined.

$V_6 = (\gamma_7 v_{63}, \gamma_8 v_{63}, v_{63})^T$; $v_{63} \neq 0 \in \mathbb{R}$ as an eigenvector corresponding to $\lambda_{61}^* = 0$ of J_6, where $\gamma_7 = \frac{L_{12}L_{23}^* - L_{22}L_{13}}{L_{11}L_{22} - L_{12}L_{21}}$ and $\gamma_8 = \frac{L_{13}L_{21} - L_{11}L_{23}^*}{L_{11}L_{22} - L_{12}L_{21}} < 0$.

$\Phi_6 = (\gamma_9\Phi_{63}, \gamma_{10}\Phi_{63}, \Phi_{63})^T$; $\Phi_{63} \neq 0 \in \mathbb{R}$ as an eigenvector corresponding to $\lambda_{61}{}^* = 0$ of $J_5{}^T$, where $\gamma_9 = \frac{L_{21}L_{32}-L_{22}L_{31}}{L_{11}L_{22}-L_{12}L_{21}} > 0$ and $\gamma_{10} = \frac{L_{12}L_{31}-L_{11}L_{32}}{L_{11}L_{22}-L_{12}L_{21}}$.

$\Phi_6{}^T F_{w_6}(E_6, w_6{}^*) = -\gamma_{10}y^*z^*\Phi_{63} \neq 0.$

$$\Phi_6{}^T[D^2F(E_6, w_6{}^*)(V_6, V_6)] = -2v_{63}{}^2\Phi_{63}[\gamma_9\Gamma_1 + \gamma_{10}\Gamma_2 + \Gamma_3],$$

where $\Gamma_1 = \gamma_7{}^2 + w_1\gamma_7\gamma_8(1-2y^*) + w_2\gamma_7(1-2z^*) - w_1x^*\gamma_8{}^2 - w_2x^*$,

$\Gamma_2 = -w_5\gamma_7\gamma_8(1-2y^*) + \left(\frac{w_3}{w_4} + w_5x^*\right)\gamma_8{}^2 + w_6{}^*\gamma_8$,

$\Gamma_3 = -w_8\gamma_7(1-2z^*) - w_9\gamma_8 + w_8x^*$.

Accordingly, $\Phi_6{}^T[D^2F(E_6, w_6{}^*)(V_6, V_6)] \neq 0$ under the condition (37), hence the saddle node takes place and the proof is completed.

7. Numerical Simulation

In this section, a numerical simulation of the solution of system (4) is performed to authenticate the obtained analytical results and understand the influence of the change in the parameters on the dynamical action of the system. The following hypothetical set of parameters is used in the numerical simulation, which is applied by using predictor-corrector four steps method along with the sixth orders Runge-Kutta method [21] for solving system (4). Then, MATLAB version 6.0.0.88 was used to plot the figures.

$$\begin{aligned} &w_1 = 0.5,\ w_2 = 0.5,\ w_3 = 1,\ w_4 = 1,\ w_5 = 0.2, \\ &w_6 = 0.5, w_7 = 0.1, w_8 = 0.2, w_9 = 0.2, w_{10} = 0.2. \end{aligned} \tag{38}$$

It is observed that, for the data set given by equation (38) with different initial sets of values, the system (4) approaches asymptotically to a PEP, as shown in (Figure-1).

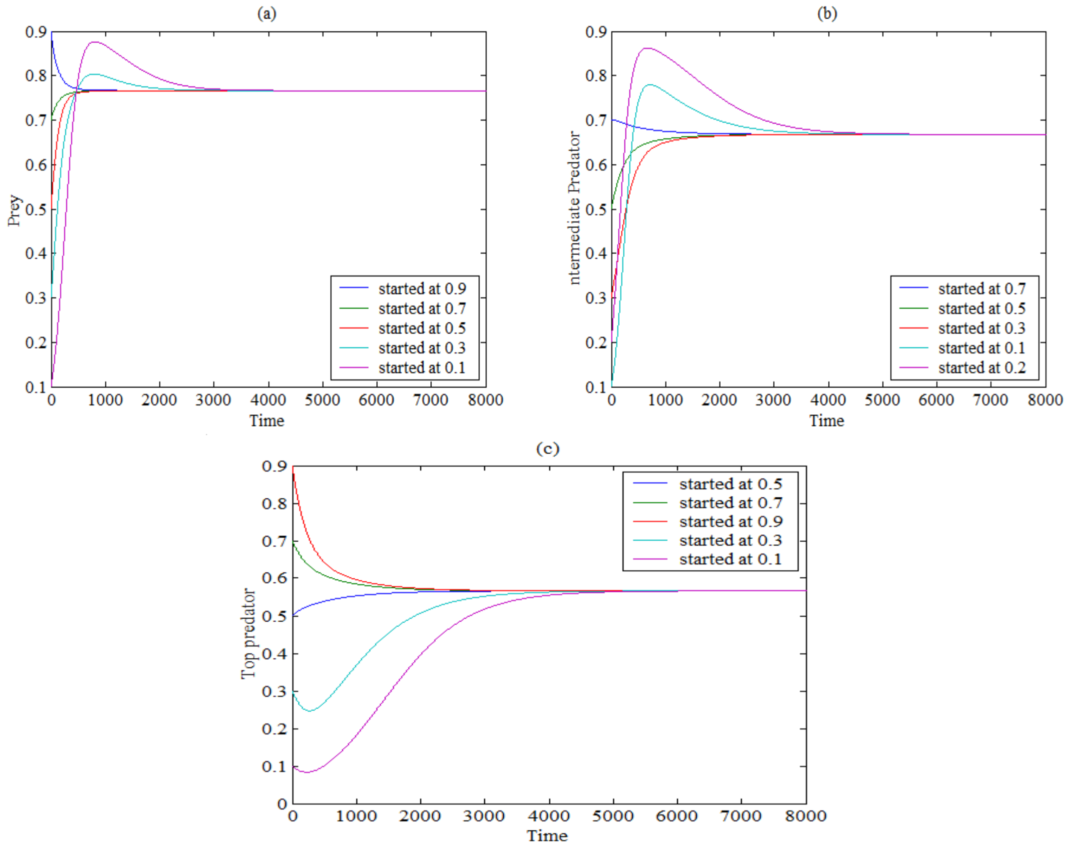

Figure 1- The time series of the trajectories of system (4) using data set (38) with different initial points, which approach asymptotically to $E_6 = (0.76, 0.66, 0.56)$. (a) Trajectories of x. (b) Trajectories of y. (c) Trajectories of z.

Accordingly, system (4) is globally asymptotically stable at PEP. Now, for the data set (38), with decreasing the value of the parameter w_8 up to $w_8 = 0.01$, it is noticed that system (4) approaches asymptotically to the TPFEP, as shown in (Figure-2). Otherwise, the system still persists at the PEP.

Similar observation was obtained for varying w_9.

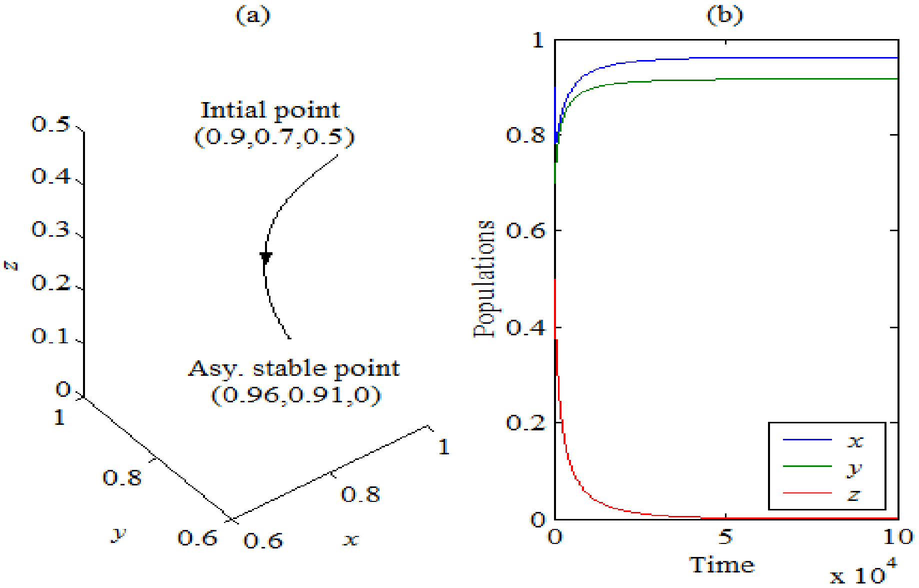

Figure 2-The solution of system (4) approaches asymptotically to TPFEP that is given by $E_5 = (0.96,0.91,0)$ using the data set (38) with $w_8 = 0.01$. (a) The trajectory of system (4). (b) Time series of the trajectory given by (a).

Now, increasing the parameter w_{10}, so that $w_{10} \geq 0.38$, with other parameters as in set (38), leads to an extinction in the top predator too, and the solution approaches asymptotically to the TPFEP as shown typically by (Figure-3). Otherwise, it is still persistent at PEP.

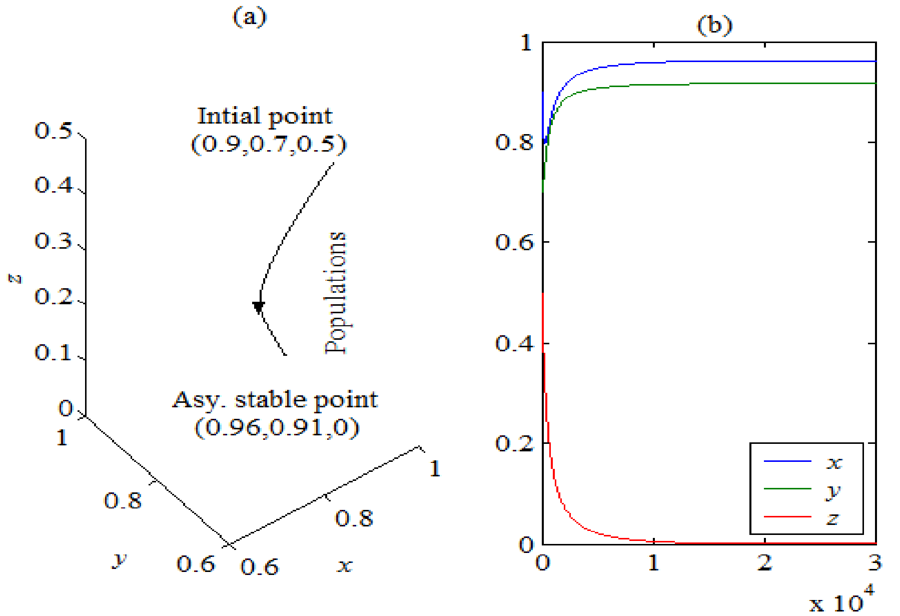

Figure 3-The solution of system (4) approaches asymptotically to TPFEP given by $E_5 = (0.96,0.91,0)$ using the data set (38) with $w_{10} = 0.4$. (a) The trajectory of system (4). (b) Time series of the trajectory given by (a).

Further investigation of the dynamical behavior of system (4) using data set (38) is performed with varying one parameter each time to understand their effects on the solution and persistence of the system. It is observed that all the parameters have a quantitative change on the solution of system (4). This is due to the existence of more than on source of food in each level, which makes the extinction of any species difficult through using only one parameter. Therefore, in the following, we will investigate the system under the effects of varying of multi parameters simultaneously.

For the data set (38) with $w_1 = 1.25$ and $w_{10} = 0.21$, it is noticed that the solution of system (4) approaches asymptotically to SAEP, $E_1 = (1,0,0)$, as shown in (Figure-4).

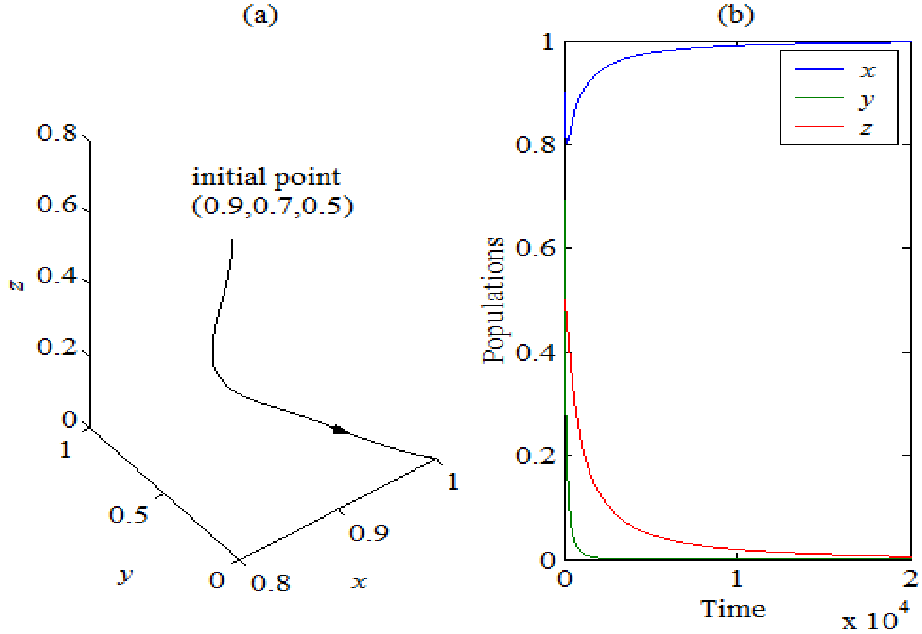

Figure 4-The solution of system (4) approaches asymptotically to FAEP given by $E_1 = (1,0,0)$ using the data set (38) with $w_1 = 1.25, w_{10} = 0.21$. (a) The trajectory of system (4). (b) Time series of the trajectory given by (a).

Note that it is simple to prove that the conditions (13c) and (13d) are held. Moreover, for the data set (38) with $w_1 = 4.5$ and $w_{10} = 0.5$, it is noticed that the solution of system (4) approaches asymptotically to SAEP, that is given by $E_2 = (0,0.5,0)$, as shown in (Figure-5).

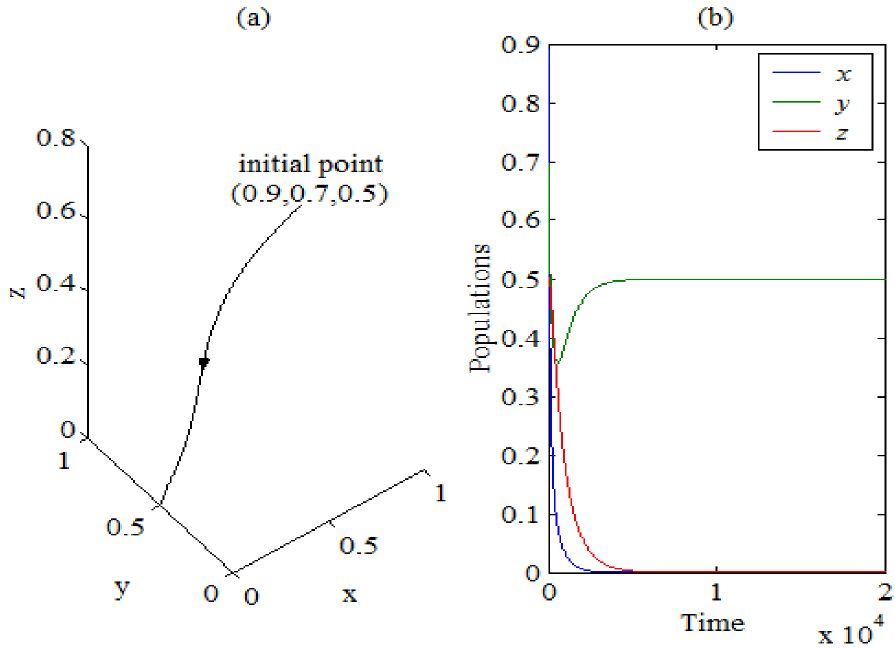

Figure 5-The solution of system (4) approaches asymptotically to SAEP given by $E_2 = (0,0.5,0)$ using the data set (38) with $w_1 = 4.5, w_7 = 0.5$. (a) The trajectory of system (4). (b) Time series of the trajectory given by (a).

Clearly, the data used in (Figure-5) satisfy the conditions (14c) and (14d). Again, for the data set (38) with $w_2 = 6.5$ and $w_4 = 2$, the system approaches asymptotically to the PFEP, that is given by $E_3 = (0,1,0.8)$, as shown in (Figure-6).

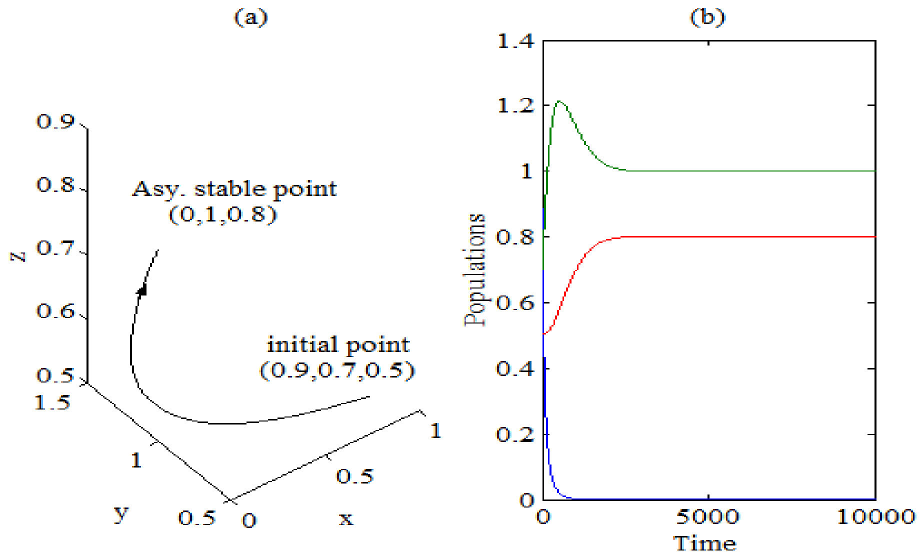

Figure 6-The solution of system (4) approaches asymptotically to PFEP given by $E_3 = (0,1,0.8)$ using the data set (38) with $w_2 = 6.5, w_4 = 2$. (a) The trajectory of system (4). (b) Time series of the trajectory given by (a).

Direct computation shows that the data used in (Figure- 6) satisfy the condition (15f). Finally, the solution of system (4) approaches asymptotically to the MPFEP, that is given by $E_4 = (0.87,0,0.54)$, using the data set (38) with $w_3 = 0.5, w_8 = 0.5$ and $w_7 = 0.5$, as shown in (Figure-7).

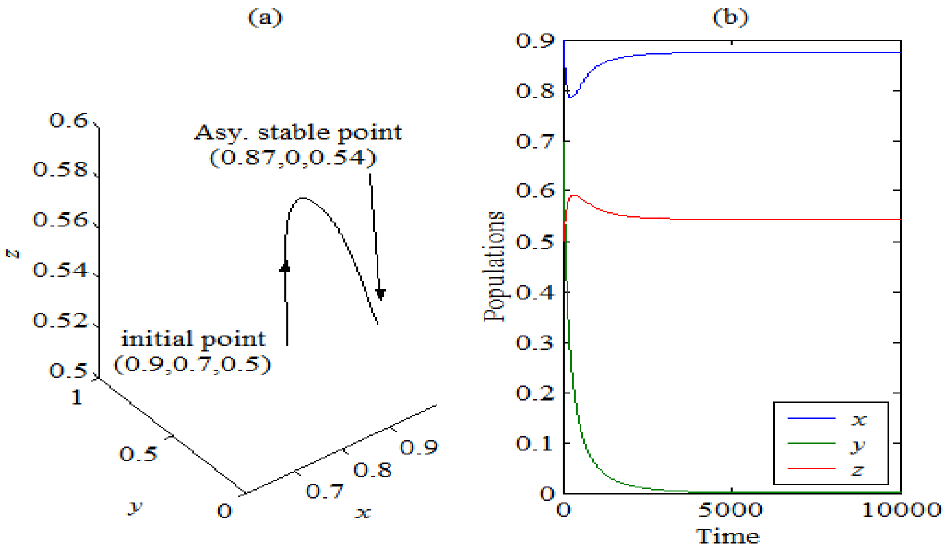

Figure 7-The solution of system (4) approaches asymptotically to MPFEP given by $E_4 = (0.87,0,0.54)$ using the data set (38) with $w_3 = 0.5, w_7 = 0.5$, $w_8 = 0.5$. (a) The trajectory of system (4). (b) Time series of the trajectory given by (a).

Again, the data used in (Figure-7) satisfied the conditions (16f) and (16g).

1. Discussion

In this paper, a food web model incorporating a prey refuge that depends on both prey and predator species is proposed and studied. The food is consumed according to Lotka-Volterra functional response. Moreover, the intermediate predator grows logistically by the addition of favorite food at the lower level. The top predator behaves as a generalist predator and preys upon both species in the lower level and second level. All the properties of the solution of system (4) are investigated. It is observed that the system has seven nonnegative equilibrium points. The local stability conditions for each point are constructed. The global dynamics, whenever exists, is investigated too. The dynamics of all possible subsystems is also studied and it is observed that there is no periodic dynamics in the boundary planes. On the other hand, the persistence conditions of the food web system are established too. It is observed that the system has a wide range of persistence due to the existence of the prey refuge that is depending on both prey and predator species as well as the multisource of food for each species. The probability of occurrence of local bifurcation around the non-hyperbolic equilibrium point is also discussed. It is observed that the system has multi-types of bifurcations which may occur near the equilibrium points. On the other hand, numerical simulation is used to confirm our obtained results. It is observed that the system has only one type of attractors that is a stable point, while periodic dynamics does not exist even in the boundary planes. This indicates that the existence of the prey refuge that is depending on both prey and predator species is a stabilizing factor on the dynamics of food web model and extends the range of the parameters at which the system persists.

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