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Asymptotic Stability for Some Types of Nonlinear Fractional Order Differential-Algebraic Control Systems

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Abstract
The aim of this paper is to study the asymptotically stable solution of nonlinear single and multi fractional differential-algebraic control systems, involving feedback control inputs, by an effective approach that depends on necessary and sufficient conditions.

Keywords: Asymptotically Stable, Feedback Control, Differential-Algebraic Equation , Fractional Order , Gronwall inequality, Mult-Fractional Order.

الاستقرارية التامة لبعض الانواع من انظمة السيطرة التفاضلية – الجبرية غير الخطية ذات الرتب الكسرية

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الخلاصه
الهدف في هذا البحث هو دراسة االاستقرارية التامة لحل انظمة السيطرة غير الخطية التفاضلية- الجبرية ذات مشتقة واحدة او متعدد المشتقات الكسرية التي تحوي مدخلات سيطرة ذات تغذية تراجعية من خلال اسلوب فعال يعتمد على شروط ضرورية وكافية.

1. Introduction

The nonlinear fractional order differential-algebraic control systems appear in a variety of theories and applications. The theory of fractional descriptor ordinary and fractional partial differential equations, with different types of derivatives, have recently been addressed by several researchers for different problems. It is well known that descriptor systems or differential-algebraic systems are the major research fields of the control theory. During the past two decades, differential-algebraic systems attracted much attention due to the comprehensive applications in economics singular systems, not only those containing differential or difference equations as normal systems but also algebraic equations. Thus, their description is considered as being more general. Their class of systems has been widely studied, not only because of theoretical interest but also because of its extensive applications in areas such as robotics and power systems. The necessary and sufficient conditions for the solvability, positivity, and asymptotic stability and stabilization of the fractional descriptor linear systems were established [1-7, 8, 9]. Earlier works [10, 11] studied the partial eigenvalue assignment for stabilization of descriptor fractional discrete-time linear systems or by derivative state feedback. In other investigations [12, 10], the stabilization problem of singular fractional-order systems with

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fractional commensurate fractional order, via static output feedback, was studied . The stability problem of descriptor second-order systems was also considered [13]. Lyapunov equations for stability of second-order systems were established by using Lyapunov method. The robust admissibility problem in singular fractional-order continuous time systems was also studied with a static output feedback controller that is designed for the uncertain closed-loop system to be admissible [14]. Other articles studied the robust stability and stabilization of uncertain fractional-order differential-algebraic nonlinear systems [15, 13].

Our intersect in this paper is to study the asymptotic stability of nonlinear fractional order differential- algebraic control systems, involving feedback input controls. Also, we aim to study single-fractional (15-16) and multi-fractional (21-22) order differential – algebraic control equations .

The following definitions and results are needed later on.

Definition (1.1), [16]

Let f be a function such that $f:[0,\infty)\to R$. The α fractional order Caputo derivative is defined as $({}_0^cD_t^\alpha)f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-s)^{n-\alpha-1}f^{(n)}(s)ds$, $n-1<\alpha<n$, where Γ denotes the gamma function.

Lemma (1.1), [17]

The degree polynomial $a_n\lambda^n+a_{n-1}\lambda^{n-1}+\cdots+a_0=0$ is asymptotically stable if it holds the following condition: $|arg\lambda_i|>\alpha\frac{\pi}{2}$ for all zeros $\lambda_i, i=1,\dots,n$.

Definition (1.1), [18]

The following fractional order system

$({}_0^cD_t^\alpha)x(t)=Ax(t)$

$x(0)=x_0,\ 0<\alpha<1,$

where x=$(x_1,x_2,\dots,x_n)^T$ and $x_0=(x_{10},x_{20},\dots,x_{n0})^T$, $A\epsilon R^{n\times n}$ is

a) stable if for any x_0 there exists $\epsilon>0$ such that $\|x\|\le\epsilon$ for $t\ge 0$.

b) asymptotically stable if $\lim_{t\to\infty}\|x\|=0$.

Lemma (1.2), [18]

Consider the linear fractional control system

$({}_0^cD_t^\alpha)x(t)=Ax(t)+Bu(t)$

$x(0)=x_0,\ 0<\alpha\le 1$, where x=$(x_1,x_2,\dots,x_n)^T$ and $x_0=(x_{10},x_{20},\dots,x_{n0})^T$,

$u\in L([0,\infty],R^m), A\epsilon R^{n\times n}B\epsilon R^{n\times m}$. The system is stable if and only if $|arg\lambda_i|>\alpha\frac{\pi}{2}$, for all zeros $\lambda_i,\ i=1,\dots,n$.

2. Nonlinear Fractional Order Differential-Algebraic Control Systems

The following two types of nonlinear fractional order differential-algebraic control systems are presented.

2.1. Single- Fractional Order Differential -Algebraic Control Equations

Consider the nonlinear single fractional order differential -algebraic control system:

$$({}_0^cD_t^\alpha)x_1(t)=\sum_{i=1}^3 a_{1i}x_i(t)+\sum_{j=1}^2 b_{1j}\,u_j(t)+f_1(x_1,x_2,x_3)\mathrm{g}_1({}_0^cD_t^\beta x_1,{}_0^cD_t^\beta x_2,{}_0^cD_t^\beta x_3) \quad (1)$$

$$({}_0^cD_t^\alpha)x_2(t)=\sum_{i=1}^3 a_{2i}x_i(t)+\sum_{j=1}^2 b_{2j}\,u_j(t)+f_2(x_1,x_2,x_3)\,\mathrm{g}_2({}_0^cD_t^\gamma x_1,{}_0^cD_t^\gamma x_2,{}_0^cD_t^\gamma x_3) \quad (2)$$

$$x_3(t)=\sum_{i=1}^3 a_{3i}x_i(t)+\sum_{j=1}^2 b_{3,j}\,u_j(t)+f_3(x_1,x_2,x_3)\,\mathrm{g}_3({}_0^cD_t^\delta x_1,{}_0^cD_t^\delta x_2,{}_0^cD_t^\delta x_3) \quad (3)$$

where $a_{1i},a_{2i},a_{3i},b_1,b_2,$and b_3 are constants, $x_i\in R$ are state vectors, i=1…3, $0\le\alpha,\beta,\gamma,\delta\le 1$ $u_j(t)\in R, j$ =1,2 are control input functions, and f_i, g_i , i=1…3 are varying nonlinear time values .

The linear system of (1-3) is:

$$({}_0^cD_t^\alpha)x_1(t)=\sum_{i=1}^3 a_{1i}x_i(t)+\sum_{j=1}^2 b_{1,j}\,u_j(t) \quad (4)$$

$$({}_0^cD_t^\alpha)x_2(t)=\sum_{i=1}^3 a_{2i}x_i(t)+\sum_{j=1}^2 b_{2,j}\,u_j(t) \quad (5)$$

$$x_3(t)=\sum_{i=1}^3 a_{3i}x_i(t)+\sum_{i=1}^2 b_{3,i}\,u_i(t) \quad (6)$$

From (6), it is given that :

$$x_3(t)=\frac{a_{31}x_1(t)}{1-a_{33}}+\frac{a_{32}x_2(t)}{1-a_{33}}+\frac{\sum_{i=1}^2 b_{3,i}u_i(t)}{1-a_{33}}$$

Hence,

$$({}_0^cD_t^\alpha)x_1(t)=a_{11}x_1(t)+a_{12}x_2(t)+a_{13}\left[\frac{a_{31}x_1(t)}{1-a_{33}}+\frac{a_{32}x_2(t)}{1-a_{33}}+\frac{\sum_{i=1}^2 b_{3,i}u_i(t)}{1-a_{33}}\right]+\sum_{j=1}^2 b_{1j}\,u_j(t)$$

$({}_0^cD_t^\alpha)x_2(t) = a_{21}x_1(t) + a_{22}x_2(t) + a_{23}\left[\frac{a_{31}x_1(t)}{1-a_{33}} + \frac{a_{32}x_2(t)}{1-a_{33}} + \frac{\sum_{i=1}^2 b_{3,i}u_i(t)}{1-a_{33}}\right] + \sum_{j=1}^2 b_{2j}\, u_j(t)$

then,

$$({}_0^cD_t^\alpha)x_1(t) = \left[a_{11} + \frac{a_{13}a_{31}}{1-a_{33}}\right]x_1(t) + [a_{12} + \frac{a_{13}a_{32}}{1-a_{33}}]x_2(t) + \left[a_{13}\frac{\sum_{i=1}^2 b_{3,i}u_i(t)}{1-a_{33}}\right] + \sum_{i=1}^2 b_{1i}\, u_i(t) \quad (7)$$

$$({}_0^cD_t^\alpha)x_2(t) = \left[a_{21} + \frac{a_{23}a_{31}}{1-a_{33}}\right]x_1(t) + [a_{22} + \frac{a_{23}a_{32}}{1-a_{33}}]x_2(t) + \left[a_{23}\frac{\sum_{i=1}^2 b_{3,i}u_i(t)}{1-a_{33}}\right] + \sum_{i=1}^2 b_{2,i}\, u_i(t) \quad (8)$$

For more simplicity, let $\tilde{a}_{11} = a_{11} + \frac{a_{13}a_{31}}{1-a_{33}}$, $\tilde{a}_{12} = a_{12} + \frac{a_{13}a_{32}}{1-a_{33}}$, $\tilde{a}_{21} = a_{21} + \frac{a_{23}a_{31}}{1-a_{33}}$, $\tilde{a}_{22} = a_{22} + \frac{a_{23}a_{32}}{1-a_{33}}$, $\tilde{a}_{31} = \frac{a_{31}}{1-a_{33}}$, $\tilde{a}_{32} = \frac{a_{32}}{1-a_{33}}$, $\tilde{b}_{11} = \frac{a_{13}b_{3,1}}{1-a_{33}} + b_{11}$, $\tilde{b}_{12} = \frac{a_{13}b_{3,2}}{1-a_{33}} + b_{12}$, $\tilde{b}_{21} = \frac{a_{23}b_{31}}{1-a_{33}} + b_{21}$, $\tilde{b}_{22} = \frac{a_{23}b_{32}}{1-a_{33}} + b_{22}$, $\tilde{b}_{31} = \frac{b_{31}}{1-a_{33}}\sum_{i=0}^T a_{33}^i$, $\tilde{b}_{32} = \frac{b_{32}}{1-a_{33}}\sum_{i=0}^T a_{33}^i$

Equations (7) and (8) become ,

$$({}_0^cD_t^\alpha)x_1(t) = \tilde{a}_{11}x_1(t) + \tilde{a}_{12}x_2(t) + \sum_{j=1}^2 \tilde{b}_{1j}\, u_j(t) \quad (9)$$

$$({}_0^cD_t^\alpha)x_2(t) = \tilde{a}_{21}x_1(t) + \tilde{a}_{22}x_2(t) + \sum_{j=1}^2 \tilde{b}_{2j}\, u_j(t) \quad (10)$$

$$x_3(t) = \tilde{a}_{31}x_1(t) + \tilde{a}_{32}x_2(t) + \sum_{j=1}^2 \tilde{b}_{3j}\, u_j(t) \quad (11)$$

Now, we consider the following related linear feedback control system

$$({}_0^cD_t^{\alpha_1})x_{1\rho}(t) + ({}_0^cD_t^{\alpha_2})x_{1\rho}(t) = \tilde{a}_{11\rho}x_{1\rho}(t) + \tilde{a}_{12\rho}x_{2\rho}(t) + \tilde{b}_{11\rho}x_3(t) + \tilde{b}_{12\rho}x_3(t) \quad (12)$$

$$({}_0^cD_t^{\alpha_1})x_{2\rho}(t) + ({}_0^cD_t^{\alpha_2})x_{2\rho}(t) = \tilde{a}_{21\rho}x_{1\rho}(t) + \tilde{a}_{22\rho}x_{2\rho}(t) + \tilde{b}_{21\rho}x_3(t) + \tilde{b}_{22\rho}x_3(t) \quad (13)$$

$$u_1(t) = K_\rho x_{1\rho}(t) u_2(t) = K_\rho x_{2\rho}(t) \quad (14)$$

where $x_{1\rho}, x_{2\rho} \in R$, $\tilde{a}_{11\rho}, \tilde{a}_{12\rho}, \tilde{a}_{21\rho}, \tilde{a}_{22\rho}$, and K_ρ are constants.

For the nonlinear multi-fractional order differential -algebraic control system in equations (10-11) with equations (12-13), we obtain:

$$\begin{bmatrix} ({}_0^cD_t^\alpha)x_1(t) \\ ({}_0^cD_t^\alpha)x_2(t) \\ ({}_0^cD_t^\alpha)x_{1,\rho}(t) \\ ({}_0^cD_t^\alpha)x_{2,\rho}(t) \end{bmatrix} = \begin{bmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \tilde{b}_{11}K_\rho & \tilde{b}_{12}K_\rho \\ \tilde{a}_{21} & \tilde{a}_{22} & \tilde{b}_{2,1}K_\rho & \tilde{b}_{2,2}K_\rho \\ \tilde{a}_{31}(\tilde{b}_{11\rho} + \tilde{b}_{12\rho}) & \tilde{a}_{32}(\tilde{b}_{11\rho} + \tilde{b}_{12\rho}) & \sum_{j=1}^2 \tilde{b}_{1j\rho}\tilde{b}_{31}K_\rho + \tilde{a}_{11\rho} & \sum_{j=1}^2 \tilde{b}_{1j\rho}\tilde{b}_{32}K_\rho + \tilde{a}_{12\rho} \\ \tilde{a}_{31}(\tilde{b}_{21\rho} + \tilde{b}_{22\rho}) & \tilde{a}_{32}(\tilde{b}_{21\rho} + \tilde{b}_{22\rho}) & \sum_{j=1}^2 \tilde{b}_{2j\rho}\tilde{b}_{31}K_\rho + \tilde{a}_{21\rho} & \sum_{j=1}^2 \tilde{b}_{2j\rho}\tilde{b}_{32}K_\rho + \tilde{a}_{22\rho} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_{1,\rho}(t) \\ x_{2,\rho}(t) \end{bmatrix}$$

$$+ \begin{bmatrix} f_1(x_1, x_2, x_3)g_1\,({}_0^cD_t^\beta x_1, {}_0^cD_t^\beta x_2, {}_0^cD_t^\beta x_3) \\ f_2(x_1, x_2, x_3)g_2\,({}_0^cD_t^\beta x_1, {}_0^cD_t^\beta x_2, {}_0^cD_t^\beta x_3) \\ \tilde{b}_{1,2,\rho}f_3(x_1, x_2, x_3)g_3\,({}_0^cD_t^\beta x_1, {}_0^cD_t^\beta x_2, {}_0^cD_t^\beta x_3) \\ \tilde{b}_{2,2,\rho}f_3(x_1, x_2, x_3)g_3\,({}_0^cD_t^\beta x_1, {}_0^cD_t^\beta x_2, {}_0^cD_t^\beta x_3) \end{bmatrix} \quad (15)$$

$$x_3(t) = \begin{bmatrix} \tilde{a}_{31} & \tilde{a}_{32} & \tilde{b}_{3,1}K_\rho & \tilde{b}_{3,2}K_\rho \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_{1,\rho}(t) \\ x_{2,\rho}(t) \end{bmatrix} + f_3(x_1, x_2, x_3)g_3\,({}_0^cD_t^\beta x_1, {}_0^cD_t^\beta x_2, {}_0^cD_t^\beta x_3) \quad (16)$$

Thus,

$$({}_0^cD_t^\alpha)x(t) = Ax + F(x)G({}_0^cD_t^\alpha x) \quad (17)$$

$$x_3(t) = Bx + f_3(x)g_3({}_0^cD_t^\alpha x) \quad (18)$$

where

$$A = \begin{bmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \tilde{b}_{11}K_\rho & \tilde{b}_{12}K_\rho \\ \tilde{a}_{21} & \tilde{a}_{22} & \tilde{b}_{2,1}K_\rho & \tilde{b}_{2,2}K_\rho \\ \tilde{a}_{31}(\tilde{b}_{11\rho} + \tilde{b}_{12\rho}) & \tilde{a}_{32}(\tilde{b}_{11\rho} + \tilde{b}_{12\rho}) & \sum_{j=1}^2 \tilde{b}_{1j\rho}\tilde{b}_{31}K_\rho + \tilde{a}_{11\rho} & \sum_{j=1}^2 \tilde{b}_{1j\rho}\tilde{b}_{32}K_\rho + \tilde{a}_{12\rho} \\ \tilde{a}_{31}(\tilde{b}_{21\rho} + \tilde{b}_{22\rho}) & \tilde{a}_{32}(\tilde{b}_{21\rho} + \tilde{b}_{22\rho}) & \sum_{j=1}^2 \tilde{b}_{2j\rho}\tilde{b}_{31}K_\rho + \tilde{a}_{21\rho} & \sum_{j=1}^2 \tilde{b}_{2j\rho}\tilde{b}_{32}K_\rho + \tilde{a}_{22\rho} \end{bmatrix}$$

$B = \begin{bmatrix}\tilde{a}_{31} & \tilde{a}_{32} & \tilde{b}_{31}K_\rho & \tilde{b}_{32}K_\rho\end{bmatrix}$, $F(x)G({}_0^cD_t^\alpha x) = \begin{bmatrix} f_1(x_1,\ x_2,x_3)g_1\ ({}_0^cD_t^\beta x_1, {}_0^cD_t^\beta x_2, {}_0^cD_t^\beta x_3) \\ f_2(x_1,x_2,x_3)g_2\ ({}_0^cD_t^\beta x_1, {}_0^cD_t^\beta x_2, {}_0^cD_t^\beta x_3) \\ \tilde{b}_{1,2,\rho} f_3(x_1,x_2,x_3)g_3\ ({}_0^cD_t^\beta x_1, {}_0^cD_t^\beta x_2, {}_0^cD_t^\beta x_3) \\ \tilde{b}_{2,2,\rho} f_3(x_1,x_2,x_3)g_3\ ({}_0^cD_t^\beta x_1, {}_0^cD_t^\beta x_2, {}_0^cD_t^\beta x_3)\end{bmatrix}$

$f_3(x)g_3({}_0^cD_t^\alpha x) = f_3(x_1,x_2,x_3)g_3\ ({}_0^cD_t^\beta x_1, {}_0^cD_t^\beta x_2, {}_0^cD_t^\beta x_3)$.

Lemma (2.1.1), [19]

The Mittage-Leffler function $E_{\alpha,\beta}(At^\alpha)$ satisfies the following

i. $E_{\alpha,1}(At^\alpha) \leq K_{E_{\alpha,1}} \|e^{At^\alpha}\|$, $\alpha > 1$.

ii. $E_{\alpha,\alpha}(At^\alpha) \leq K_{E_{\alpha,\alpha}} \|e^{At^\alpha}\|$, $\alpha > 1$,

where $A \in R^{nxn}$, $K_{E_{\alpha,1}} K_{E_{\alpha,\alpha}}$ are finite real constants such that $K_{E_{\alpha,1}} > 1$, $K_{E_{\alpha,\alpha}} > 1$.

Lemma (2.1.2), [19]

Let $\alpha > 0$ v(t) be a nonnegative function that is locally integrable on [0,T), let a(t) be a nonnegative,nondecreasing continuous function that is defined on [0,T), and let a(t) <M . Suppose that z(t) is nonnegative and locally integrable on [0,T) with $z(t) \leq v(t)+a(t)\int_0^t (t-\tau)^{\alpha-1} z(\tau)d\tau$. If $v(t)$ is a non-decreasing function on [0,T), then we have

$z(t) \leq v(t)\, E_\alpha(\Gamma(\alpha)a(t)t^\alpha)$.

Theorem (2.1.3)

Suppose that the following nonlinear fractional order differential-algebraic control system (17-18) with feedback control (14) satisfies the following conditions:

1. Re(eig (A))<0 and –max Re(eig (A)) >$\Gamma(\alpha)$
2. $\|F(x)G({}_0^cD_t^\alpha x)\| = \|f_3(x)\| = o(\|x\|)$ as $\|x\| \to 0$.

where $F(x) = \begin{bmatrix} f_1(x_1,\ x_2,x_3)g_1\ ({}_0^cD_t^\beta x_1, {}_0^cD_t^\beta x_2, {}_0^cD_t^\beta x_3) \\ f_2(x_1,x_2,x_3)g_2\ ({}_0^cD_t^\beta x_1, {}_0^cD_t^\beta x_2, {}_0^cD_t^\beta x_3) \\ \tilde{b}_{2,2,\rho} f_3(x_1,x_2,x_3)g_3\ ({}_0^cD_t^\beta x_1, {}_0^cD_t^\beta x_2, {}_0^cD_t^\beta x_3) \\ \tilde{b}_{1,2,\rho} f_3(x_1,x_2,x_3)g_3\ ({}_0^cD_t^\beta x_1, {}_0^cD_t^\beta x_2, {}_0^cD_t^\beta x_3)\end{bmatrix}$ and K_{g_i}, i=1,..3 are the nonnegative continuous function, and μ_{g_i} are the continuous non-decreasing positive functions , such that $\|g_i(t, {}_0^cD_t^\alpha x)\| \leq K_{g_i}(t)\ \mu_{g_i}(\|{}_0^cD_t^\alpha x\|)$, i=1,..3.

Then, the system (15) is a locally asymptotically stable.

Proof

By taking the Laplace transformation to (17), we get

$s^\alpha X(s) - s^{\alpha-1}x_0 = \tilde{A}X(s) + \mathcal{L}\left\{F(x)G({}_0^cD_t^\beta)\right\}$, thus

$X(s) = \left(Is^\alpha - \tilde{A}\right)^{-1}\left(-s^{\alpha-1}x_0 + \mathcal{L}\left\{F(x)G({}_0^cD_t^\beta x)\right\}\right)$

(19)

By taking the Laplace inverse transformation to (19), we obtain

x(t)=$E_{\alpha,1}(\tilde{A}t^\alpha)x_0 + \int_0^t (t-\tau)^{\alpha-1}\, E_{\alpha,\alpha}(\tilde{A}(t-\tau)^\alpha)\ F(x(\tau))G({}_0^cD_t^\beta x(\tau)d\,\tau$

From lemma (2.1.1), we have

$\|x(t)\| = L_1\|e^{\tilde{A}t^\alpha}\|\|x_0\| + L_3\int_0^t (t-\tau)^{\alpha-1}\,\|e^{\tilde{A}(t-\tau)^\alpha}\|\,\left\|F(x(\tau))G({}_0^cD_t^\beta x(\tau)\right\|\,d\,\tau$

From condition (1), the matrix $\tilde{A}$ is stable and there is a constant $L_4 > 0$ such that $\|e^{\tilde{A}t^\alpha}\| \leq L_4 e^{-\omega t^\alpha}$, hence,

$\|x(t)\| \leq L_1L_4e^{-\omega t^\alpha}\|x_0\| + L_3L_4\int_0^t (t-\tau)^{\alpha-1}\, e^{-\omega(t-\tau)^\alpha}\,\left\|F(x(\tau))G({}_0^cD_t^\beta x(\tau)\right\|\,d\,\tau$

From condition (2), we get

$\left\|F\big(x(\tau)\big)G({}_0^cD_t^\beta x(\tau)\right\| =$

$\left\|f_1(x_1,\ x_2,x_3)g_1\ \left({}_0^cD_t^\beta x_1, {}_0^cD_t^\beta x_2, {}_0^cD_t^\beta x_3\right)\right\| + \left\|f_2(x_1,x_2,x_3)g_2\ \left({}_0^cD_t^\beta x_1, {}_0^cD_t^\beta x_2, {}_0^cD_t^\beta x_3\right)\right\| +$

$\left\| \tilde{b}_{2,2,\rho} f_3(x_1, x_2, x_3) g_3 \left({}_0^cD_t^{\beta} x_1, {}_0^cD_t^{\beta} x_2, {}_0^cD_t^{\beta} x_3 \right) \right\| +$
$\left\| \tilde{b}_{1,2,\rho} f_3(x_1, x_2, x_3) g_3 \left({}_0^cD_t^{\beta} x_1, {}_0^cD_t^{\beta} x_2, {}_0^cD_t^{\beta} x_3 \right) \right\|$
$\leq \|f_1(x_1, x_2, x_3)\| K_{g_1}(t)\, \mu_{g_1}(\|{}_0^cD_t^{\alpha} x\|) + \|f_2(x_1, x_2, x_3)\| K_{g_2}(t) \mu_{g_2}(\|{}_0^cD_t^{\alpha} x\|)$
$+ |\tilde{b}_{2,2,\rho}| \|f_3(x_1, x_2, x_3)\| K_{g_3}(t)\, \mu_{g_3}(\|{}_0^cD_t^{\alpha} x\|) + |\tilde{b}_{1,2,\rho}| \|f_3(x_1, x_2, x_3)\| K_{g_3}(t)\, \mu_{g_3}(\|{}_0^cD_t^{\alpha} x\|)$.
Since $\|F(x)G({}_0^cD_t^{\alpha} x)\| = o(\|x\|)$ as $\|x\| \to 0$, thus $\lim_{\|x\| \to 0} \frac{\|F(x)G({}_0^cD_t^{\alpha} x)\|}{\|x\|} = 0$, which implies that $\frac{\|F(x)G({}_0^cD_t^{\alpha} x)\|}{\|x\|} < \frac{1}{L_3 L_4}$, therefore
$\left\| F(x(\tau)) G({}_0^cD_t^{\beta} x(\tau) \right\| \leq \frac{1}{L_3 L_4} \left[K_{g_1}(t) \mu_{g_1}(\|{}_0^cD_t^{\alpha} x\|) + \|f_2(x_1, x_2, x_3)\| K_{g_2}(t) \mu_{g_2}(\|{}_0^cD_t^{\alpha} x\|) + \right.$
$\left. \left(|\tilde{b}_{2,2,\rho}| + |\tilde{b}_{1,2,\rho}| \right) K_{g_3}(t) \mu_{g_3}(\|{}_0^cD_t^{\alpha} x\|) \right] \|x\|$.
Now,we set that $L_5 = \left[K_{g_1}(t) \mu_{g_1}(\|{}_0^cD_t^{\alpha} x\|) + \|f_2(x_1, x_2, x_3)\| K_{g_2}(t) \mu_{g_2}(\|{}_0^cD_t^{\alpha} x\|) \right]$
$+ \left(|\tilde{b}_{2,2,\rho}| + |\tilde{b}_{1,2,\rho}| \right) K_{g_3}(t) \mu_{g_3}(\|{}_0^cD_t^{\alpha} x\|)]$
We have
$\|x(t)\| \leq L_1 L_4 e^{-\omega t^{\alpha}} \|x_0\| + L_5 \int_0^t (t-\tau)^{\alpha-1} e^{-\omega(t-\tau)^{\alpha}} \|x(\tau)\| d\tau$, thus
$e^{\omega t} \|x(t)\| \leq L_1 L_4 e^{-\omega t^{\alpha}} \|x_0\| + L_5 \int_0^t (t-\tau)^{\alpha-1} e^{\omega \tau} \|x(\tau)\|$
By using Gronwall inequality and lemma(2.1.2), we have
$e^{\omega t} \|x(t)\| \leq L_1 L_4 \|x_0\| E_{\alpha,1}(L_5 \Gamma(\alpha) t^{\alpha}) \leq L_1 L_4 \|x_0\| e^{L_5 \Gamma(\alpha) t^{\alpha}}$, then
$\|x(t)\| \leq c_1 L_1 L_4 \|x_0\| e^{-(\omega - L_5 \Gamma(\alpha)) t^{\alpha}}$
As $\longrightarrow \infty$, $\|x(t)\| \longrightarrow 0$ for $\omega > L_5 \Gamma(\alpha)^{1/\alpha}$.
From condition (2) , we have that $x \to 0$ as $t \to \infty$, $x_3(t) = Bx + f_3(x) g_3({}_0^cD_t^{\alpha} x)$. We get $\|x_3(t)\| \leq \|Bx\| + \|f_3(x)\| \left\| g_3\left(t, {}_0^cD_t^{\alpha}(x)\right) \right\|$. Now, since $\|f_3(x)\| = o(\|x\|)$ as $\|x\| \to 0$, this implies that $\lim_{\|x\| \to 0} \frac{\|f_3(x)\|}{\|x\|} = 0$, so that $\frac{\|f_3(x)\|}{\|x\|} < \delta_1$, and since $\left\| g_3\left(t, {}_0^cD_t^{\alpha}(x)\right) \right\| \leq K_{g_3}(t)$ $\mu_{g_3}(\|{}_0^cD_t^{\alpha} x\|)$, therefore
$\|x_3(t)\| \leq (\|B\| + \delta_1 K_{g_3}(t)\, \mu_{g_3}(\|{}_0^cD_t^{\alpha} x\|)) \|x\|$.
Hence, $\|x_3(t)\| \to 0$, which implies that system (17-18) is asymptotically stable.

Example (2.1.2)

The following single-fractional order differential-algebraic equations (19-20) with Caputo derivative are asymptotically stable, as follows

$$\begin{bmatrix} \left({}_0^cD_t^{1/2}\right) x_1(t) \\ \left({}_0^cD_t^{1/2}\right) x_2(t) \\ \left({}_0^cD_t^{1/2}\right) x_{1,\rho}(t) \\ \left({}_0^cD_t^{1/2}\right) x_{2,\rho}(t) \end{bmatrix} =$$

$$\begin{bmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \tilde{b}_{11} K_\rho & \tilde{b}_{12} K_\rho \\ \tilde{a}_{21} & \tilde{a}_{22} & \tilde{b}_{2,1} K_\rho & \tilde{b}_{2,2} K_\rho \\ \tilde{a}_{31}(\tilde{b}_{11\rho} + \tilde{b}_{12\rho}) & \tilde{a}_{32}(\tilde{b}_{11\rho} + \tilde{b}_{12\rho}) & \sum_{j=1}^{2} \tilde{b}_{1j\rho} \tilde{b}_{31} K_\rho + \tilde{a}_{11\rho} & \sum_{j=1}^{2} \tilde{b}_{1j\rho} \tilde{b}_{32} K_\rho + \tilde{a}_{12\rho} \\ \tilde{a}_{31}(\tilde{b}_{21\rho} + \tilde{b}_{22\rho}) & \tilde{a}_{32}(\tilde{b}_{21\rho} + \tilde{b}_{22\rho}) & \sum_{j=1}^{2} \tilde{b}_{2j\rho} \tilde{b}_{31} K_\rho + \tilde{a}_{21\rho} & \sum_{j=1}^{2} \tilde{b}_{2j\rho} \tilde{b}_{32} K_\rho + \tilde{a}_{22\rho} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_{1\rho}(t) \\ x_{2\rho}(t) \end{bmatrix}$$

$$+ \begin{bmatrix} f_1(x_1, x_2, x_3) g_1 ({}_0^cD_t^{\beta} x_1, {}_0^cD_t^{\beta} x_2, {}_0^cD_t^{\beta} x_3) \\ f_2(x_1, x_2, x_3) g_2 ({}_0^cD_t^{\beta} x_1, {}_0^cD_t^{\beta} x_2, {}_0^cD_t^{\beta} x_3) \\ \tilde{b}_{12} f_3(x_1, x_2, x_3) g_3 ({}_0^cD_t^{\beta} x_1, {}_0^cD_t^{\beta} x_2, {}_0^cD_t^{\beta} x_3) \\ \tilde{b}_{22\rho} f_3(x_1, x_2, x_3) g_3 ({}_0^cD_t^{\beta} x_1, {}_0^cD_t^{\beta} x_2, {}_0^cD_t^{\beta} x_3) \end{bmatrix} \quad (19)$$

$$x_3(t) = [1 \quad 0 \quad 0 \quad 0]\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_{1\rho}(t) \\ x_{2\rho}(t) \end{bmatrix} + f_3(x_1, x_2, x_3)\mathrm{g}_3\ ({}_0^cD_t^{\beta} x_1, {}_0^cD_t^{\beta} x_2, {}_0^cD_t^{\beta} x_3) \quad (20)$$

$$\begin{bmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \tilde{b}_{11}K_\rho & \tilde{b}_{12}K_\rho \\ \tilde{a}_{21} & \tilde{a}_{22} & \tilde{b}_{2,1}K_\rho & \tilde{b}_{2,2}K_\rho \\ \tilde{a}_{31}(\tilde{b}_{11\rho} + \tilde{b}_{12\rho}) & \tilde{a}_{32}(\tilde{b}_{11\rho} + \tilde{b}_{12\rho}) & \sum_{j=1}^{2} \tilde{b}_{1j\rho}\tilde{b}_{31}K_\rho + \tilde{a}_{11\rho} & \sum_{j=1}^{2} \tilde{b}_{1j\rho}\tilde{b}_{32}K_\rho + \tilde{a}_{12\rho} \\ \tilde{a}_{31}(\tilde{b}_{21\rho} + \tilde{b}_{22\rho}) & \tilde{a}_{32}(\tilde{b}_{21\rho} + \tilde{b}_{22\rho}) & \sum_{j=1}^{2} \tilde{b}_{2j\rho}\tilde{b}_{31}K_\rho + \tilde{a}_{21\rho} & \sum_{j=1}^{2} \tilde{b}_{2j\rho}\tilde{b}_{32}K_\rho + \tilde{a}_{22\rho} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} f_1(x_1, x_2, x_3)\mathrm{g}_1\ ({}_0^cD_t^{\beta} x_1, {}_0^cD_t^{\beta} x_2, {}_0^cD_t^{\beta} x_3) \\ f_2(x_1, x_2, x_3)\mathrm{g}_2\ ({}_0^cD_t^{\beta} x_1, {}_0^cD_t^{\beta} x_2, {}_0^cD_t^{\beta} x_3) \\ \tilde{b}_{12\rho} f_3(x_1, x_2, x_3)\mathrm{g}_3\ ({}_0^cD_t^{\beta} x_1, {}_0^cD_t^{\beta} x_2, {}_0^cD_t^{\beta} x_3) \\ \tilde{b}_{22\rho} f_3(x_1, x_2, x_3)\mathrm{g}_3\ ({}_0^cD_t^{\beta} x_1, {}_0^cD_t^{\beta} x_2, {}_0^cD_t^{\beta} x_3) \end{bmatrix} = \begin{bmatrix} x_1x_2K_{g_1}(t)\ \mu_{g_1}(\|{}_0^cD_t^{\alpha} x\|) \\ x_1x_3\ K_{g_2}(t)\ \mu_{g_2}(\|{}_0^cD_t^{\alpha} x\|) \\ \tilde{b}_{12\rho}x_2x_3K_{g_3}(t)\ \mu_{g_3}(\|{}_0^cD_t^{\alpha} x\|) \\ \tilde{b}_{22\rho}x_2x_3K_{g_3}(t)\ \mu_{g_3}(\|{}_0^cD_t^{\alpha} x\|) \end{bmatrix}$$

To compute condition (1):

Since $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ and the eig(A) has values such as $\lambda_{1,2,3} = -2$ then

$Re(\mathrm{eig}(A)) < 0$ and $\omega = -\max Re(eig(A)) = 2 > \Gamma(\alpha) = 1.772$.

To compute condition (2):

$$\lim_{\|\bar{x}\|\to 0} \frac{\left\| f_1(x_1, x_2, x_3)\mathrm{g}_1\ ({}_0^cD_t^{\beta} x_1, {}_0^cD_t^{\beta} x_2, {}_0^cD_t^{\beta} x_3) \right\|}{\|\bar{x}(t)\|} = \lim_{\|\bar{x}\|\to 0} \frac{\sqrt{(x_1x_2)^2} K_{g_1}(t)\ \mu_{g_1}(\|{}_0^cD_t^{\alpha} x\|)}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \leq$$

$$\lim_{\|\bar{x}\|\to 0} \frac{\sqrt{x_1^2x_2^2}}{\sqrt{x_1^2}} = \lim_{\|\bar{x}\|\to 0} \sqrt{x_2^2} = \lim_{\|\bar{x}\|\to 0} \sqrt{x_1^2} = 0$$

$$\lim_{\|\bar{x}\|\to 0} \frac{\left\| f_2(x_1, x_2, x_3)\mathrm{g}_2\ ({}_0^cD_t^{\beta} x_1, {}_0^cD_t^{\beta} x_2, {}_0^cD_t^{\beta} x_3) \right\|}{\|\bar{x}(t)\|} = \lim_{\|\bar{x}\|\to 0} \frac{\sqrt{(x_1x_3)^2} K_{g_2}(t)\ \mu_{g_2}(\|{}_0^cD_t^{\alpha} x\|)}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$$

$$\leq \lim_{\|\bar{x}\|\to 0} \frac{\sqrt{x_1^2x_3^2}}{\sqrt{x_1^2}} = \lim_{\|\bar{x}\|\to 0} \sqrt{x_3^2} = \lim_{\|\bar{x}\|\to 0} \sqrt{x_3^2} = 0$$

$$\lim_{\|\bar{x}\|\to 0} \frac{\left\| f_3(x_1, x_2, x_3)\mathrm{g}_3\ ({}_0^cD_t^{\beta} x_1, {}_0^cD_t^{\beta} x_2, {}_0^cD_t^{\beta} x_3) \right\|}{\|\bar{x}(t)\|} = \lim_{\|\bar{x}\|\to 0} \frac{\sqrt{(x_2x_3)^2} K_{g_3}(t)\ \mu_{g_3}(\|{}_0^cD_t^{\alpha} x\|)}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$$

$$\leq \lim_{\|\bar{x}\|\to 0} \frac{\sqrt{x_2^2x_3^2}}{\sqrt{x_2^2}} = \lim_{\|\bar{x}\|\to 0} \sqrt{x_3^2} = \lim_{\|\bar{x}\|\to 0} \sqrt{x_3^2} = 0$$

and $\|x_3\| \to 0$. Then, by theorem (2.1.1) we show that the zero solution of the system (19-20) is asymptotically stable

2.2. Multi - Fractional Order Differential –Algebraic Control Equation

Consider the following nonlinear multi-fractional order differential – algebraic control system:

$$({}_0^cD_t^{\alpha_1})x_1(t) + ({}_0^cD_t^{\alpha_2})x_1(t) = \sum_{i=1}^{3} a_{1i}x_i(t) + \sum_{i=1}^{2} b_{1,i}\, u_i(t) + f_1(x_1, x_2, x_3)\mathrm{g}_1$$
$$({}_0^cD_t^{\beta} x_1, {}_0^cD_t^{\beta} x_2, {}_0^cD_t^{\beta} x_3 \quad (21)$$

$\left({}_0^cD_t^{\alpha_1}\right)x_2(t)+\left({}_0^cD_t^{\alpha_2}\right)x_2(t)=\sum_{i=1}^{3}a_{2i}x_i(t)+\sum_{i=1}^{2}b_{2,i}\,u_i(t)+f_2(x_1,x_2,x_3)\,g_2$
$({}_0^cD_t^{\beta}x_1, D_t^{\beta}x_2, D_t^{\beta}x_3)$ (22)

$x_3(t)=\sum_{i=1}^{3}a_{3i}x_i(t)+\sum_{i=1}^{2}b_{3i}\,u_i(t)+f_3(x_1,x_2,x_3)\,g_3({}_0^cD_t^{\beta}x_1, {}_0^cD_t^{\beta}x_2, {}_0^cD_t^{\beta}x_3)$ (23)

where $a_{1i}, a_{2i}, a_{3i}, b_{1,i}, b_{2i}$, and b_{3i} are constants, $x_i \in R$ are state vectors, i=1..3
$u_i(t) \in R$, i=1, 2 are control input functions, and f_i, g_i , i=1…3 are varying nonlinear time values .
System (21-23) yields a linear dynamical system, as follows

$\left({}_0^cD_t^{\alpha_1}\right)x_1(t)+\left({}_0^cD_t^{\alpha_2}\right)x_1(t)=\sum_{i=1}^{3}a_{1i}x_i(t)+\sum_{i=1}^{2}b_{1i}\,u_i(t)$
$\left({}_0^cD_t^{\alpha_1}\right)x_2(t)+\left({}_0^cD_t^{\alpha_2}\right)x_2(t)=\sum_{i=1}^{3}a_{2i}x_i(t)+\sum_{i=1}^{2}b_{2i}\,u_i(t)$
$x_3(t)=\sum_{i=1}^{3}a_{3i}x_i(t)+\sum_{i=1}^{2}b_{3i}\,u_i(t)$

We get

$$\left({}_0^cD_t^{\alpha_1}\right)x_1(t)+\left({}_0^cD_t^{\alpha_2}\right)x_1(t)=a_{11}x_1(t)+a_{12}x_2(t)+a_{13}\left[\frac{a_{31}x_1(t)}{1-a_{33}}+\frac{a_{32}x_2(t)}{1-a_{33}}+\frac{\sum_{i=1}^{2}b_{3i}u_i(t)}{1-a_{33}}\right]+\sum_{i=1}^{2}b_{1i}\,u_i(t)$$

$$\left({}_0^cD_t^{\alpha_1}\right)x_2(t)+\left({}_0^cD_t^{\alpha_2}\right)x_2(t)=a_{21}x_1(t)+a_{22}x_2(t)+a_{23}\left[\frac{a_{31}x_1(t)}{1-a_{33}}+\frac{a_{32}x_2(t)}{1-a_{33}}+\frac{\sum_{i=1}^{2}b_{3i}u_i(t)}{1-a_{33}}\right]+\sum_{i=1}^{2}b_{2,i}\,u_i(t)$$

$$x_3(t)=\left[\frac{a_{31}x_1(t)}{1-a_{33}}+\frac{a_{32}x_2(t)}{1-a_{33}}+\frac{\sum_{i=1}^{2}b_{3i}u_i(t)}{1-a_{33}}\right]$$

Assume that

$\tilde{a}_{11}=a_{11}+\frac{a_{13}a_{31}}{1-a_{33}}$, $\tilde{a}_{12}=\left[a_{12}+\frac{a_{13}a_{32}}{1-a_{33}}\right]$, $\tilde{a}_{21}=a_{21}+\frac{a_{23}a_{31}}{1-a_{33}}$, $\tilde{a}_{22}=\left[a_{22}+\frac{a_{23}a_{32}}{1-a_{33}}\right]$, $\tilde{a}_{31}=\frac{a_{31}}{1-a_{33}}$, $\tilde{a}_{32}=\frac{a_{32}}{1-a_{33}}$, $\tilde{b}_{11}=\frac{b_{3,1}}{1-a_{33}}+b_{11}$, $\tilde{b}_{12}=\frac{b_{3,2}}{1-a_{33}}+b_{12}$, $\tilde{b}_{21}=\frac{a_{2,3}b_{3,1}}{1-a_{33}}+b_{21}$, $\tilde{b}_{21}=\frac{a_{2,3}b_{3,2}}{1-a_{33}}+b_{21}$, $\tilde{b}_{31}=\frac{b_{3,1}}{1-a_{33}}$, $\tilde{b}_{32}=\frac{b_{3,2}}{1-a_{33}}$

Therefore

$\left({}_0^cD_t^{\alpha_1}\right)x_1(t)+\left({}_0^cD_t^{\alpha_2}\right)x_1(t)=\tilde{a}_{11}x_1(t)+\tilde{a}_{12}x_2(t)++\sum_{i=1}^{2}\tilde{b}_{1i}\,u_i(t)$
$\left({}_0^cD_t^{\alpha_1}\right)x_1(t)+\left({}_0^cD_t^{\alpha_2}\right)x_1(t)=\tilde{a}_{21}x_1(t)+\tilde{a}_{22}x_2(t)++\sum_{i=1}^{2}\tilde{b}_{2i}\,u_i(t)$
$x_3(t)=\tilde{a}_{31}x_1(t)+\tilde{a}_{32}\,x_2(t)+\sum_{i=1}^{2}\tilde{b}_{3i}\,u_i(t).$

Now, consider the following related linear feedback control system

$\left({}_0^cD_t^{\alpha_1}\right)x_{1,\rho}(t)+\left({}_0^cD_t^{\alpha_2}\right)x_{1\rho}(t)=\tilde{a}_{11\rho}x_{1\rho}(t)+\tilde{a}_{12\rho}x_{2\rho}(t)++\tilde{b}_{11\rho}x_3(t)+\tilde{b}_{12\rho}x_3(t)$
$\left({}_0^cD_t^{\alpha_1}\right)x_{2,\rho}(t)+\left({}_0^cD_t^{\alpha_2}\right)x_{2\rho}(t)=\tilde{a}_{21\rho}x_{1\rho}(t)+\tilde{a}_{22\rho}x_{2\rho}(t)++\tilde{b}_{21\rho}x_3(t)+\tilde{b}_{22\rho}x_3(t)$
$u_1(t)=K_\rho x_{1\rho}(t)$, $u_2(t)=K_\rho x_{2\rho}(t)$

where $x_{1\rho}, x_{2\rho} \in R$, and $\tilde{a}_{11\rho}, \tilde{a}_{12\rho}, \tilde{a}_{21\rho}, \tilde{a}_{22\rho}$, and K_ρ are constants.

We have that

$$\begin{bmatrix}\left({}_0^cD_t^{\alpha_1}\right)x_1(t)+\left({}_0^cD_t^{\alpha_2}\right)x_1(t)\\ \left({}_0^cD_t^{\alpha_1}\right)x_2(t)+\left({}_0^cD_t^{\alpha_2}\right)x_2(t)\\ \left({}_0^cD_t^{\alpha_1}\right)x_{1,\rho}(t)+\left({}_0^cD_t^{\alpha_2}\right)x_{1,\rho}(t)\\ \left({}_0^cD_t^{\alpha_1}\right)x_{2,\rho}(t)+\left({}_0^cD_t^{\alpha_2}\right)x_{2,\rho}(t)\end{bmatrix}$$
$$=\left[\begin{bmatrix}\tilde{a}_{11} & \tilde{a}_{12} & \tilde{b}_{11}K_\rho & \tilde{b}_{12}K_\rho\\ \tilde{a}_{21} & \tilde{a}_{22} & \tilde{b}_{2,1}K_\rho & \tilde{b}_{22}K_\rho\\ \tilde{a}_{31}(\tilde{b}_{11\rho}+\tilde{b}_{12\rho}) & \tilde{a}_{32}(\tilde{b}_{11\rho}+\tilde{b}_{12\rho}) & \sum_{j=1}^{2}\tilde{b}_{1j\rho}\tilde{b}_{31}K_\rho+\tilde{a}_{11\rho} & \sum_{j=1}^{2}\tilde{b}_{1j\rho}\tilde{b}_{32}K_\rho+\tilde{a}_{12\rho}\\ \tilde{a}_{31}\tilde{b}_{2\rho} & \tilde{a}_{32}\tilde{b}_{2\rho} & \sum_{j=1}^{2}\tilde{b}_{2j\rho}\tilde{b}_{31}K_\rho+\tilde{a}_{21\rho} & \sum_{j=1}^{2}\tilde{b}_{2j\rho}\tilde{b}_{32}K_\rho+\tilde{a}_{22\rho}\end{bmatrix}\right]\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_{1\rho}(t)\\ x_{2\rho}(t)\end{bmatrix}$$
$$+\begin{bmatrix}f_1(x_1,x_2,x_3)g_1\,({}_0^cD_t^{\beta}x_1, {}_0^cD_t^{\beta}x_2, {}_0^cD_t^{\beta}x_3)\\ f_2(x_1,x_2,x_3)g_2\,({}_0^cD_t^{\beta}x_1, {}_0^cD_t^{\beta}x_2, {}_0^cD_t^{\beta}x_3)\\ \tilde{b}_{12\rho}f_3(x_1,x_2,x_3)g_3\,({}_0^cD_t^{\beta}x_1, {}_0^cD_t^{\beta}x_2, {}_0^cD_t^{\beta}x_3)\\ \tilde{b}_{22\rho}f_3(x_1,x_2,x_3)g_3\,({}_0^cD_t^{\beta}x_1, {}_0^cD_t^{\beta}x_2, {}_0^cD_t^{\beta}x_3)\end{bmatrix} \quad (24)$$

$$x_3(t) = \begin{bmatrix}\tilde{a}_{31} & \tilde{a}_{32} & \tilde{b}_{31}K_\rho & \tilde{b}_{32}K_\rho\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_{1\rho}(t)\\ x_{2\rho}(t)\end{bmatrix} + f_3(x_1,x_2,x_3)g_3\ (t, {}_0^cD_t^\beta x_1, {}_0^cD_t^\beta x_2, {}_0^cD_t^\beta x_3) \quad (25)$$

Then

$$\left({}_0^cD_t^{\alpha_1}\right)x(t) + \left({}_0^cD_t^{\alpha_2}\right)x(t) = Ax + F(x)G\ (t, {}_0^cD_t^\beta x) \quad (26)$$

$$x_3(t) = Bx + f_3(x)g_3\ (t,, {}_0^cD_t^\beta x),\ x = (x_1, x_2, x_3) \quad (27)$$

where

$$A = \begin{bmatrix}\tilde{a}_{11} & \tilde{a}_{12} & \tilde{b}_{11}K_\rho & \tilde{b}_{12}K_\rho\\ \tilde{a}_{21} & \tilde{a}_{22} & \tilde{b}_{2,1}K_\rho & \tilde{b}_{2,2}K_\rho\\ \tilde{a}_{31}(\tilde{b}_{11\rho}+\tilde{b}_{12\rho}) & \tilde{a}_{32}(\tilde{b}_{11\rho}+\tilde{b}_{12\rho}) & \sum_{j=1}^{2}\tilde{b}_{1j\rho}\tilde{b}_{31}K_\rho+\tilde{a}_{11\rho} & \sum_{j=1}^{2}\tilde{b}_{1j\rho}\tilde{b}_{32}K_\rho+\tilde{a}_{12\rho}\\ \tilde{a}_{31}(\tilde{b}_{21\rho}+\tilde{b}_{22\rho}) & \tilde{a}_{32}(\tilde{b}_{21\rho}+\tilde{b}_{22\rho}) & \sum_{j=1}^{2}\tilde{b}_{2j\rho}\tilde{b}_{31}K_\rho+\tilde{a}_{21\rho} & \sum_{j=1}^{2}\tilde{b}_{2j\rho}\tilde{b}_{32}K_\rho+\tilde{a}_{22\rho}\end{bmatrix}$$

$$B = \begin{bmatrix}\tilde{a}_{31} & \tilde{a}_{32} & \tilde{b}_{31}K_\rho & \tilde{b}_{32}K_\rho\end{bmatrix},\ F(x)G\left({}_0^cD_t^\beta x\right) = \begin{bmatrix}f_1(x_1,\ x_2,x_3)g_1\ ({}_0^cD_t^\beta x_1, {}_0^cD_t^\beta x_2, {}_0^cD_t^\beta x_3)\\ f_2(x_1,x_2,x_3)g_2\ ({}_0^cD_t^\beta x_1, {}_0^cD_t^\beta x_2, {}_0^cD_t^\beta x_3)\\ \tilde{b}_{12\rho}f_3(x_1,x_2,x_3)g_3\ ({}_0^cD_t^\beta x_1, {}_0^cD_t^\beta x_2, {}_0^cD_t^\beta x_3)\\ \tilde{b}_{22\rho}f_3(x_1,x_2,x_3)g_3\ ({}_0^cD_t^\beta x_1, {}_0^cD_t^\beta x_2, {}_0^cD_t^\beta x_3)\end{bmatrix}$$

$$f_3(x)g_3({}_0^cD_t^\alpha x) = f_3(x_1,x_2,x_3)g_3\ ({}_0^cD_t^\beta x_1, {}_0^cD_t^\beta x_2, {}_0^cD_t^\beta x_3).$$

2.2.3 The stable equivalent system

Theorem (2.2.1), [11]

Consider the following system of fractional differential equations

$$\left.\begin{aligned}{}_0^cD_t^{\alpha_1}x_{1_1}(t) &= f_1(\hat{x},t)\\ {}_0^cD_t^{\alpha_2}x_{2_1}(t) &= f_2(\hat{x},t)\\ &\vdots\\ {}_0^cD_t^{\alpha_2}x_{n_1}(t) &= f_n(\hat{x},t)\end{aligned}\right\}$$

(28)

$x_{i_1}^{(k)}(0) = x_{i_0}^{(k)}$, i=1, 2,…, n; k=0,1,..,$m_i - 1$, $t \geq 0$,

(29)

$\hat{x} = [x_{1_1}(t), x_{2_1}(t), \ldots, x_{n_1}(t)]^T$,$f_i \in R$,

Where f_i , i =1,2,…, n are continuous functions , the fractional orders defend by $m_i - 1 < \alpha_i < m_i \in Z_+$, i=1,2,…, also n means the N- dimensional system of the following fractional differential equations

$$\left.\begin{aligned}
{}_0^cD_t^\sigma x_{1_1}(t) &= x_{1_2}(t),\\
&\vdots\\
{}_0^cD_t^\sigma x_{1_M}(t) &= x_{1_{M+1}}(t),\\
&\vdots\\
{}_0^cD_t^\sigma x_{1_{(m_1-1)M}}(t) &= x_{1_{(m_1-1)M+1}}(t),\\
&\vdots\\
{}_0^cD_t^\sigma x_{1_{\alpha_1 M}}(t) &= f_1(\hat{x},t),\\
{}_0^cD_{0,t}^\sigma x_{2_1}(t) &= x_{2_2}(t),\\
&\vdots\\
{}_0^cD_t^\sigma x_{2_M}(t) &= x_{2_{M+1}}(t),\\
&\vdots\\
{}_0^cD_t^\sigma x_{2_{(m_2-1)M}}(t) &= x_{2_{(m_2-1)M+1}}(t),\\
&\vdots\\
{}_0^cD_{0,t}^\sigma x_{2_{\alpha_2 M}}(t) &= f_2(\hat{x},t),\\
&\vdots\\
{}_0^cD_{0,t}^\sigma x_{n_1}(t) &= x_{n_2}(t),\\
&\vdots\\
{}_0^cD_{0,t}^\sigma x_{n_M}(t) &= x_{n_{M+1}}(t)\\
&\vdots\\
{}_0^cD_t^\sigma x_{n_{(m_n-1)M}}(t) &= x_{n_{(m_n-1)M+1}}(t),\\
&\vdots\\
{}_0^cD_{0,t}^\sigma x_{1_{\alpha_n M}}(t) &= f_n(\hat{x},t),
\end{aligned}\right\} \quad (30)$$

where $\alpha_i=\frac{c_i}{d_i}$, g.c.d(c_i, d_i)=1, and i=1,2. Also, M is a lower common multiple of c_i and d_i, $\sigma=\frac{1}{M}$, N=$(\alpha_1+\alpha_2)M$.

$$x_{i_j}=\begin{cases} x_{i0}^{(k)} & j=kM+1; k=0,1,\dots,m_i-1\\ 0 & otherwise\end{cases}, \text{ i=1,2,\dots,n.} \quad (31)$$

whenever $\left[x_{1_1}(t),x_{1_2}(t),\dots,x_{1_{\alpha_1 M}}(t),x_{2_1}(t),x_{2_2}(t),\dots,x_{2_{\alpha_2 M}}(t),x_{n_1}(t),x_{n_2}(t),\dots,x_{n_{\alpha_n M}}(t)\right]^T$ is a solution of system (27-28) and $\left[x_{1_1}(t),x_{1_2}(t),\dots,x_{n_1}(t)\right]^T \epsilon C^{m_1}[0,b]\times C^{m_2}[0,b]\times\dots\times C^{m_n}[0,b]$, solved system (28-29).

Whenever $\left[x_{1_1}(t),x_{1_2}(t),\dots,x_{n_1}(t)\right]^T \epsilon C^{m_1}[0,b]\times C^{m_2}[0,b]\times\dots\times C^{m_n}[0,b]$ is a solution to system (28-29), then $\left[x_{1_1}(t),x_{1_2}(t),\dots,x_{1_{\alpha_1 M}}(t),x_{2_1}(t),x_{2_2}(t),\dots,x_{2_{\alpha_2 M}}(t),x_{n_1}(t),x_{n_2}(t),\dots,x_{n_{\alpha_n M}}(t)\right]^T=$ $\left[x_{1_1}(t),{}_0^cD_t^\sigma x_{1_1}(t),\dots,{}_0^cD_t^{(\alpha_1 M-1)\sigma}x_{1_1}(t),x_{2_1}(t),{}_0^cD_t^\sigma x_{2_1}(t),\dots,{}_0^cD_t^{(\alpha_1 M-1)\sigma}x_{2_1}(t),\dots,\right.$ $\left.x_{n_1}(t),{}_0^cD_t^\sigma x_{n_1}(t),\dots,{}_0^cD_t^{(\alpha_1 M-1)\sigma}x_{n_1}(t)\right]^T$ satisfies system (27-28).

Consider the following multi- fractional differential equation:

$${}_0^cD_t^{\alpha_n}x(t)+b_1\,{}_0^cD_t^{\alpha_{n-1}}x(t)+\dots+b_{n-1}\,{}_0^cD_t^{\alpha_n}x(t)+\dots+b_n=f\big(x(t)\big),\ t>0 \quad (32)$$

$$x^{(k)}(0)=x_0^{(k)}, \text{ k=0,1,\dots,}m_n\text{-1.} \quad (33)$$

where $x(t)\in R$, $f\colon D\to R$ is a continues function $D\subseteq R$ and b_i, i=1,…,n are constant numbers. The order $\alpha_i, i=1,2,\dots,n$ are rational numbers such that $m_i-1<\alpha_i<m_i\in Z_+$, i=1,2,…, n, $\alpha_n>\alpha_{n-1}>\dots>\alpha_1$ and $x(t)\in C[a,b]$ is a solution of system (27-28).

Corollary (2.2.2), [11]

Suppose that $f\big(x(t)\big)$ is a real valued continuous function such that $f\big(x(t)\big)=b_0x(t)$ and equation (32-33) has a unique solution $x(t)\in C[a,\infty]$. Then, the zero solution to equation (32-33) is asymptotically stable if $|\arg(\lambda)|>\frac{\gamma\pi}{2}$, where λ is a solution to the characteristic equation $\det(\lambda I-A)=0$. $\gamma=\frac{1}{M}$, and

$$A=\begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0\\ 0 & 0 & 1 & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & 0\\ a_{N1} & a_{N2} & a_{N3} & a_{N4} & \cdots & a_{NN}\end{pmatrix}, \text{ and } \quad a_{Nj}=\begin{cases} b_0 & j=1\\ -b_{n-i} & j=\alpha_i M+1, i=1,2,\dots,n-1\\ 0 & otherwise\end{cases}$$

Theorem (2.2.3)

Consider the nonlinear multi-fractional order differential-algebraic feedback control system (26-27), such that it satisfies the following conditions:

1. $|\arg(\lambda_k)|>\frac{\gamma\pi}{2}$, where λ is a solution to the characteristic equation $\det(\lambda_k I-A_k)=0$.

$$\gamma=\frac{1}{M} \text{ and } A_k=\begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0\\ 0 & 0 & 1 & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & 1\\ a_{N1} & a_{N2} & a_{N3} & a_{N4} & \cdots & a_{NN}\end{pmatrix}, \text{ k=1,2,3,4}$$

$$\text{and } \quad a_{Nj}=\begin{cases} -\tilde{a}_{11} & j=1\\ -\tilde{a}_{12} & j=2\\ -\tilde{b}_{11}K_\rho & j=3\\ -\tilde{b}_{12}K_\rho & j=4\\ -1 & j=\alpha_i M+1, i=1,2,\dots,n-1\end{cases} \qquad \text{For } A_1$$

$$a_{Nj}=\begin{cases} \tilde{a}_{21} & j=1\\ \tilde{a}_{22} & j=2\\ \tilde{b}_{21}K_\rho & j=3\\ \tilde{b}_{22}K_\rho & j=4\\ -1 & j=\alpha_i M+1, i=1,2,\dots,n-1\end{cases} \qquad \text{For } A_2$$

$$a_{Nj}=\begin{cases} \tilde{a}_{31}(\tilde{b}_{1,1,\rho}+\tilde{b}_{1,2,\rho}) & j=1\\ \tilde{a}_{32}(\tilde{b}_{1,1,\rho}+\tilde{b}_{1,2,\rho}) & j=2\\ \sum_{j=1}^{2}\tilde{b}_{1,j,\rho}\tilde{b}_{3,1}K_\rho+\tilde{a}_{1,1,\rho} & j=3\\ \sum_{j=1}^{2}\tilde{b}_{1,j,\rho}\tilde{b}_{3,2}K_\rho+\tilde{a}_{1,2,\rho} & j=4\\ -1 & j=\alpha_i M+1, i=1,2,\dots,n-1\end{cases} \qquad \text{For } A_3$$

$$a_{Nj}=\begin{cases} \tilde{a}_{31}\tilde{b}_{2,\rho} & j=1\\ \tilde{a}_{32}\tilde{b}_{2,\rho} & j=2\\ \sum_{j=1}^{2}\tilde{b}_{2,j,\rho}\tilde{b}_{3,1}K_\rho+\tilde{a}_{2,1,\rho}) & j=3\\ (\sum_{j=1}^{2}\tilde{b}_{2,j,\rho}\tilde{b}_{3,2}K_\rho+\tilde{a}_{2,2,\rho}) & j=4\\ -1 & j=\alpha_i M+1, i=1,2,\dots,n-1\end{cases} \qquad \text{For } A_4$$

2. $\|F(x)\|=o(\|x\|)$ as $\|x\|\to 0$.

where $F(x)=\begin{bmatrix} f_1(x_1,\ x_2,x_3)\mathrm{g}_1\ ({}_0^cD_t^\beta x_1, {}_0^cD_t^\beta x_2, {}_0^cD_t^\beta x_3)\\ f_2(x_1,x_2,x_3)\mathrm{g}_2\ ({}_0^cD_t^\beta x_1, {}_0^cD_t^\beta x_2, {}_0^cD_t^\beta x_3)\\ \tilde{b}_{2,2,\rho}f_3(x_1,x_2,x_3)\mathrm{g}_3\ ({}_0^cD_t^\beta x_1, {}_0^cD_t^\beta x_2, {}_0^cD_t^\beta x_3)\\ \tilde{b}_{1,2,\rho}f_3(x_1,x_2,x_3)\mathrm{g}_3\ ({}_0^cD_t^\beta x_1, {}_0^cD_t^\beta x_2, {}_0^cD_t^\beta x_3)\end{bmatrix}$ and K_{g_i} , i=1,..3 are the nonnegative continuous functions and μ_{g_i} are the continuous nondecreasing positive functions, such that $\|g_i(t, {}_0^cD_t^\alpha x)\|\le K_{g_i}(t)\ \mu_{g_i}(\|{}_0^cD_t^\alpha x\|)$, i=1,..3. Then, the system is a locally asymptotically stable.

Proof

From theorem (2.2.1), we can transform the system (24-25) which defined as folows:

$$\left({}_0^cD_t^{\alpha_1}\right)x_1(t)+\left({}_0^cD_t^{\alpha_2}\right)x_1(t)=\tilde{a}_{11}x_1(t)+\tilde{a}_{12}x_2(t)+\tilde{b}_{11}K_\rho x_{1,\rho}(t)+\tilde{b}_{12}K_\rho x_{2,\rho}(t)$$

into a system of a single fractional order equations

$${}_0^cD_t^{\sigma}x_{1_1}(t)=x_{1_2}(t),$$
$$\vdots$$
$${}_0^cD_t^{\sigma}x_{1_M}(t)=x_{1_{M+1}}(t),$$
$$\vdots$$
$${}_0^cD_t^{\sigma}x_{1_{(m_1-1)M}}(t)=x_{1_{(m_1-1)M+1}}(t),$$
$$\vdots$$
$${}_0^cD_t^{\sigma}x_{1_{\alpha_1 M}}(t)=-\tilde{a}_{11}x_1(t)-\tilde{a}_{12}x_2(t)-\tilde{b}_{11}K_\rho x_{1,\rho}(t)-\tilde{b}_{12}K_\rho x_{2,\rho}(t)$$

such that

$$A_1=\begin{bmatrix}0&1&0&0&\dots&0\\0&0&1&0&\dots&0\\\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&0&\cdots&1\\a_{N1}&a_{N2}&a_{N3}&a_{N4}&\dots&a_{NN}\end{bmatrix},\text{ and}$$

$$a_{Nj}=\begin{cases}-\tilde{a}_{11} & j=1\\-\tilde{a}_{12} & j=2\\-\tilde{b}_{11}K_\rho & j=3\\-\tilde{b}_{12}K_\rho & j=4\\-1 & j=\alpha_iM+1,i=1,2,\dots,n-1\end{cases}$$

Now, we transfer equation

$$\left({}_0^cD_t^{\alpha_1}\right)x_1(t)+\left({}_0^cD_t^{\alpha_2}\right)x_1(t)=\tilde{a}_{21}x_1(t)+\tilde{a}_{22}x_2(t)+\tilde{b}_{21}K_\rho x_{1,\rho}(t)+\tilde{b}_{22}K_\rho x_{2,\rho}(t)$$

into

$${}_0^cD_t^{\sigma}x_{2_1}(t)=x_{2_2}(t),$$
$$\vdots$$
$${}_0^cD_t^{\sigma}x_{2_M}(t)=x_{2_{M+1}}(t),$$
$$\vdots$$
$${}_0^cD_t^{\sigma}x_{2_{(m_1-1)M}}(t)=x_{2_{(m_1-1)M+1}}(t),$$
$$\vdots$$
$${}_0^cD_t^{\sigma}x_{2_{\alpha_1 M}}(t)=\tilde{a}_{21}x_1(t)+\tilde{a}_{22}x_2(t)+\tilde{b}_{21}K_\rho x_{1,\rho}(t)+\tilde{b}_{22}K_\rho x_{2,\rho}(t)$$

such that

$$A_2=\begin{bmatrix}0&1&0&0&\dots&0\\0&0&1&0&\dots&0\\\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&0&\cdots&1\\a_{N1}&a_{N2}&a_{N3}&a_{N4}&\dots&a_{NN}\end{bmatrix},\text{ and}$$

$$a_{Nj}=\begin{cases}\tilde{a}_{21} & j=1\\\tilde{a}_{22} & j=2\\\tilde{b}_{21}K_\rho & j=3\\\tilde{b}_{22}K_\rho & j=4\\-1 & j=\alpha_iM+1,i=1,2,\dots,n-1\end{cases}$$

Now, we transfer equation

$$\left({}_0^cD_t^{\alpha_1}\right)x_1(t)+\left({}_0^cD_t^{\alpha_2}\right)x_1(t)=\tilde{a}_{31}\tilde{a}_{31}\left(\tilde{b}_{1,1,\rho}+\tilde{b}_{1,2,\rho}\right)x_1(t)+\tilde{a}_{32}\left(\tilde{b}_{1,1,\rho}+\tilde{b}_{1,2,\rho}\right)x_2(t)+$$
$$\left(\sum_{j=1}^{2}\tilde{b}_{1,j,\rho}\tilde{b}_{3,1}K_\rho+\tilde{a}_{1,1,\rho}\right)x_{1,\rho}(t)+\left(\sum_{j=1}^{2}\tilde{b}_{1,j,\rho}\tilde{b}_{3,2}K_\rho+\tilde{a}_{1,2,\rho}\right)x_{2,\rho}(t)$$

into

$${}_0^cD_t^{\sigma}x_{3_1}(t)=x_{3_2}(t),$$
$$\vdots$$

$${}_0^cD_t^\sigma x_{3_M}(t) = x_{3_{M+1}}(t),$$
$$\vdots$$
$${}_0^cD_t^\sigma x_{31_{(m_1-1)M}}(t) = x_{3_{(m_1-1)M+1}}(t),$$
$$\vdots$$
$${}_0^cD_t^\sigma x_{2_{\alpha_1 M}}(t) = \tilde{a}_{31}\left(\tilde{b}_{1,1,\rho}+\tilde{b}_{1,2,\rho}\right)x_1(t)+\tilde{a}_{32}\left(\tilde{b}_{1,1,\rho}+\tilde{b}_{1,2,\rho}\right)x_2(t)+\left(\sum_{j=1}^{2}\tilde{b}_{1,j,\rho}\tilde{b}_{3,1}K_\rho+\tilde{a}_{1,1,\rho}\right)x_{1,\rho}(t)+\left(\sum_{j=1}^{2}\tilde{b}_{1,j,\rho}\tilde{b}_{3,2}K_\rho+\tilde{a}_{1,2,\rho}\right)x_{2,\rho}(t)$$
, such that

$$A_3=\begin{bmatrix}0&1&0&0&\dots&0\\0&0&1&0&\dots&0\\\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&0&\cdots&1\\a_{N1}&a_{N2}&a_{N3}&a_{N4}&\cdots&a_{NN}\end{bmatrix},$$ and

$$a_{Nj}=\begin{cases}\tilde{a}_{31}(\tilde{b}_{1,1,\rho}+\tilde{b}_{1,2,\rho}) & j=1\\ \tilde{a}_{32}(\tilde{b}_{1,1,\rho}+\tilde{b}_{1,2,\rho}) & j=2\\ \sum_{j=1}^{2}\tilde{b}_{1,j,\rho}\tilde{b}_{3,1}K_\rho+\tilde{a}_{1,1,\rho} & j=3\\ \sum_{j=1}^{2}\tilde{b}_{1,j,\rho}\tilde{b}_{3,2}K_\rho+\tilde{a}_{1,2,\rho} & j=4\\ -1 & j=\alpha_i M+1, i=1,2,\dots,n-1\end{cases}$$

Now, we transfer equation

$$\left({}_0^cD_t^{\alpha_1}\right)x_1(t)+\left({}_0^cD_t^{\alpha_2}\right)x_1(t)=\tilde{a}_{31}\tilde{b}_{2,\rho}x_1(t)+\tilde{a}_{32}\tilde{b}_{2,\rho}x_2(t)+\left(\sum_{j=1}^{2}\tilde{b}_{2,j,\rho}\tilde{b}_{3,1}K_\rho+\tilde{a}_{2,1,\rho}\right)x_{1,\rho}(t)+\left(\sum_{j=1}^{2}\tilde{b}_{2,j,\rho}\tilde{b}_{3,2}K_\rho+\tilde{a}_{2,2,\rho}\right)x_{2,\rho}(t)$$

into

$${}_0^cD_t^\sigma x_{4_1}(t) = x_{4_2}(t),$$
$$\vdots$$
$${}_0^cD_t^\sigma x_{4_M}(t) = x_{4_{M+1}}(t),$$
$$\vdots$$
$${}_0^cD_t^\sigma x_{4_{(m_1-1)M}}(t) = x_{4_{(m_1-1)M+1}}(t),$$
$$\vdots$$
$${}_0^cD_t^\sigma x_{4_{\alpha_1 M}}(t) = \tilde{a}_{31}\tilde{b}_{2,\rho}x_1(t)+\tilde{a}_{32}\tilde{b}_{2,\rho}x_2(t)+\left(\sum_{j=1}^{2}\tilde{b}_{2,j,\rho}\tilde{b}_{3,1}K_\rho+\tilde{a}_{2,1,\rho}\right)x_{1,\rho}(t)+\left(\sum_{j=1}^{2}\tilde{b}_{2,j,\rho}\tilde{b}_{3,2}K_\rho+\tilde{a}_{2,2,\rho}\right)x_{2,\rho}(t)$$

such that

$$A_4=\begin{bmatrix}0&1&0&0&\dots&0\\0&0&1&0&\dots&0\\\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&0&\cdots&1\\a_{N1}&a_{N2}&a_{N3}&a_{N4}&\cdots&a_{NN}\end{bmatrix}$$

and $$a_{Nj}=\begin{cases}\tilde{a}_{31}\tilde{b}_{2,\rho} & j=1\\ \tilde{a}_{32}\tilde{b}_{2,\rho} & j=2\\ \sum_{j=1}^{2}\tilde{b}_{2,j,\rho}\tilde{b}_{3,1}K_\rho+\tilde{a}_{2,1,\rho}) & j=3\\ (\sum_{j=1}^{2}\tilde{b}_{2,j,\rho}\tilde{b}_{3,2}K_\rho+\tilde{a}_{2,2,\rho}) & j=4\\ -1 & j=\alpha_i M+1, i=1,2,\dots,n-1\end{cases}$$

From condition (1), we have that $|\arg(\lambda_k)|>\frac{\gamma\pi}{2}$, then A_k , k=1, 2, 3, 4 are stable, and from condition (2), we have that $\left\|F(x(\tau))G({}_0^cD_t^\beta x(\tau)\right\| = \left\|f_1(x_1, x_2, x_3)g_1\left({}_0^cD_t^\beta x_1, {}_0^cD_t^\beta x_2, {}_0^cD_t^\beta x_3\right)\right\| + \left\|f_2(x_1, x_2, x_3)g_2\left({}_0^cD_t^\beta x_1, {}_0^cD_t^\beta x_2, {}_0^cD_t^\beta x_3\right)\right\|$

$$+\left\|\tilde{b}_{2,2,\rho}f_3(x_1,x_2,x_3)g_3\left({}_0^cD_t^\beta x_1, {}_0^cD_t^\beta x_2, {}_0^cD_t^\beta x_3\right)\right\|$$
$$+\left\|\tilde{b}_{12\rho}f_3(x_1,x_2,x_3)g_3\left({}_0^cD_t^\beta x_1, {}_0^cD_t^\beta x_2, {}_0^cD_t^\beta x_3\right)\right\|$$

$\leq \|f_1(x_1,\ x_2,x_3)\|K_{g_1}(t)\quad \mu_{g_1}(\|{}_0^cD_t^\alpha x\|)\quad +\|f_2(x_1,\ x_2,x_3)\|K_{g_2}(t)\ \mu_{g_2}(\|{}_0^cD_t^\alpha x\|)+|\tilde{b}_{22\rho}|\|f_3(x_1,\ x_2,x_3)\|K_{g_3}(t)\quad \mu_{g_3}(\|{}_0^cD_t^\alpha x\|)\quad +|\tilde{b}_{12\rho}|\|f_3(x_1,\ x_2,x_3)\|K_{g_3}(t)\ \mu_{g_3}(\|{}_0^cD_t^\alpha x\|)$.

since $\|F(x)G({}_0^cD_t^\alpha x)\| = o(\|x\|)$ as $\|x\| \to 0$, thus $\lim_{\|x\|\to 0}\frac{\|F(x)G({}_0^cD_t^\alpha x)\|}{\|x\|} = 0$, implies that $\frac{\|F(x)G({}_0^cD_t^\alpha x)\|}{\|x\|} < \frac{1}{L_3L_4}$, where $L_3, L_4 > 0$. Then, we can complete the proof as in theorem (2.1.1).
Moreover, $\|x_3(t)\| \leq (B + L_3L_4K_{g_3}(t)\ \mu_{g_3}(\|{}_0^cD_t^\alpha x\|))\|x\|$, hence $\|x_3(t)\| \to 0$.
Then, the solution of (24-25) is asymptotically stable.

Example (2.2.4)

Consider the following nonlinear multi-fractional order differential- algebraic with feedback control system

$$\begin{bmatrix} ({}_0^cD_t^{\alpha_1})x_1(t) + ({}_0^cD_t^{\alpha_2})x_1(t) \\ ({}_0^cD_t^{\alpha_1})x_2(t) + ({}_0^cD_t^{\alpha_2})x_2(t) \\ ({}_0^cD_t^{\alpha_1})x_{1\rho}(t) + ({}_0^cD_t^{\alpha_2})x_{1\rho}(t) \\ ({}_0^cD_t^{\alpha_1})x_{2\rho}(t) + ({}_0^cD_t^{\alpha_2})x_{2\rho}(t) \end{bmatrix}$$
=
$$\left[\begin{bmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \tilde{b}_{11}K_\rho & \tilde{b}_{12}K_\rho \\ \tilde{a}_{21} & \tilde{a}_{22} & \tilde{b}_{2,1}K_\rho & \tilde{b}_{2,2}K_\rho \\ \tilde{a}_{31}(\tilde{b}_{11\rho}+\tilde{b}_{12\rho}) & \tilde{a}_{32}(\tilde{b}_{11\rho}+\tilde{b}_{12\rho}) & \sum_{j=1}^2 \tilde{b}_{1j\rho}\tilde{b}_{31}K_\rho + \tilde{a}_{11\rho} & \sum_{j=1}^2 \tilde{b}_{1j\rho}\tilde{b}_{32}K_\rho + \tilde{a}_{12\rho} \\ \tilde{a}_{31}(\tilde{b}_{21\rho}+\tilde{b}_{22\rho}) & \tilde{a}_{32}(\tilde{b}_{21\rho}+\tilde{b}_{22\rho}) & \sum_{j=1}^2 \tilde{b}_{2j\rho}\tilde{b}_{31}K_\rho + \tilde{a}_{21\rho} & \sum_{j=1}^2 \tilde{b}_{2j\rho}\tilde{b}_{32}K_\rho + \tilde{a}_{22\rho} \end{bmatrix}\right]\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_{1,\rho}(t) \\ x_{2,\rho}(t)\end{bmatrix}$$
$$+\begin{bmatrix} f_1(x_1,x_2,x_3)\mathrm{g}_1\ ({}_0^cD_t^\beta x_1, {}_0^cD_t^\beta x_2, {}_0^cD_t^\beta x_3) \\ f_2(x_1,x_2,x_3)\mathrm{g}_2\ ({}_0^cD_t^\beta x_1, {}_0^cD_t^\beta x_2, {}_0^cD_t^\beta x_3) \\ \tilde{b}_{12\rho}f_3(x_1,x_2,x_3)\mathrm{g}_3\ ({}_0^cD_t^\beta x_1, {}_0^cD_t^\beta x_2, {}_0^cD_t^\beta x_3) \\ \tilde{b}_{22\rho}f_3(x_1,x_2,x_3)\mathrm{g}_3\ ({}_0^cD_t^\beta x_1, {}_0^cD_t^\beta x_2, {}_0^cD_t^\beta x_3) \end{bmatrix} \tag{34}$$

$$x_3(t) = [\tilde{a}_{31}\quad \tilde{a}_{32}\quad \tilde{b}_{31}K_\rho\quad \tilde{b}_{32}K_\rho]\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_{1,\rho}(t) \\ x_{2,\rho}(t)\end{bmatrix} + f_3(x_1,x_2,x_3)\mathrm{g}_3\ (\mathrm{t}, {}_0^cD_t^\beta x_1, {}_0^cD_t^\beta x_2, {}_0^cD_t^\beta x_3) \tag{35}$$

where $\alpha_1 = \frac{1}{3}, \alpha_2 = \frac{1}{6}, \tilde{a}_{11} = \tilde{a}_{22} = \sum_{j=1}^2 \tilde{b}_{1j\rho}\tilde{b}_{31}K_\rho + \tilde{a}_{11\rho} = \sum_{j=1}^2 \tilde{b}_{2j\rho}\tilde{b}_{32}K_\rho + \tilde{a}_{22\rho} = -2$,
$\tilde{a}_{12} = \tilde{b}_{11}K_\rho = \tilde{b}_{12}K_\rho = \tilde{a}_{21} = \tilde{b}_{21}K_\rho = \tilde{b}_{22}K_\rho\tilde{a}_{31}(\tilde{b}_{11\rho}+\tilde{b}_{12\rho}) = \tilde{a}_{32}(\tilde{b}_{11\rho}+\tilde{b}_{12\rho}) = \sum_{j=1}^2 \tilde{b}_{1j\rho}\tilde{b}_{32}K_\rho + \tilde{a}_{12\rho} = \tilde{a}_{31}\tilde{b}_{2\rho} = \tilde{a}_{32}\tilde{b}_{2\rho} = \sum_{j=1}^2 \tilde{b}_{2j\rho}\tilde{b}_{31}K_\rho + \tilde{a}_{21\rho} = 0$,
$\tilde{a}_{31} = \tilde{a}_{32} = \tilde{b}_{31}K_\rho = 0$, $\tilde{b}_{32}K_\rho = \frac{1}{2}$.

$$\begin{bmatrix} f_1(x_1,x_2,x_3)\mathrm{g}_1\ ({}_0^cD_t^\beta x_1, {}_0^cD_t^\beta x_2, {}_0^cD_t^\beta x_3) \\ f_2(x_1,x_2,x_3)\mathrm{g}_2\ ({}_0^cD_t^\beta x_1, {}_0^cD_t^\beta x_2, {}_0^cD_t^\beta x_3) \\ \tilde{b}_{12\rho}f_3(x_1,x_2,x_3)\mathrm{g}_3\ ({}_0^cD_t^\beta x_1, {}_0^cD_t^\beta x_2, {}_0^cD_t^\beta x_3) \\ \tilde{b}_{22\rho}f_3(x_1,x_2,x_3)\mathrm{g}_3\ ({}_0^cD_t^\beta x_1, {}_0^cD_t^\beta x_2, {}_0^cD_t^\beta x_3) \end{bmatrix} = \begin{bmatrix} x_1x_2K_{g_1}(t)\ \mu_{g_1}(\|{}_0^cD_t^\alpha x\|))\|x\| \\ x_1x_3\ K_{g_2}(t)\ \mu_{g_2}(\|{}_0^cD_t^\alpha x\|))\|x\| \\ \tilde{b}_{12\rho}x_2x_3K_{g_3}(t)\ \mu_{g_3}(\|{}_0^cD_t^\alpha x\|))\|x\| \\ \tilde{b}_{22\rho}x_2x_3K_{g_3}(t)\ \mu_{g_3}(\|{}_0^cD_t^\alpha x\|))\|x\| \end{bmatrix}$$

We have that

$$\begin{bmatrix} \left({}_0^cD_t^{1/3}\right)x_1(t) + \left({}_0^cD_t^{1/6}\right)x_1(t) \\ \left({}_0^cD_t^{1/3}\right)x_2(t) + \left({}_0^cD_t^{1/6}\right)x_2(t) \\ \left({}_0^cD_t^{1/3}\right)x_{1,\rho}(t) + \left({}_0^cD_t^{1/6}\right)x_{1,\rho}(t) \\ \left({}_0^cD_t^{1/3}\right)x_{2,\rho}(t) + \left({}_0^cD_t^{1/6}\right)x_{2,\rho}(t) \end{bmatrix} = \begin{bmatrix} -9/64 & 0 & 0 & 0 \\ 0 & -9/64 & 0 & 0 \\ 0 & 0 & -9/64 & 0 \\ 0 & 0 & 0 & -9/64 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_{1,\rho}(t) \\ x_{2,\rho}(t)\end{bmatrix}$$

$$x_3(t) = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_{1\rho}(t) \\ x_{2\rho}(t) \end{bmatrix} + x_2 x_3 K_{g_3}(t)\, \mu_{g_3}(\|{}_0^c D_t^\alpha x\|))$$

Suppose that

$$\bar{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_{1\rho}(t) \\ x_{2\rho}(t) \end{bmatrix}, \quad {}_0^C D_t^{\frac{1}{3}} \bar{x} + {}_0^C D_t^{\frac{1}{6}} \bar{x} = \begin{bmatrix} \left({}_0^c D_t^{1/3}\right) x_1(t) + \left({}_0^c D_t^{1/6}\right) x_1(t) \\ \left({}_0^c D_t^{1/3}\right) x_2(t) + \left({}_0^c D_t^{1/6}\right) x_2(t) \\ \left({}_0^c D_t^{1/3}\right) x_{1\rho}(t) + \left({}_0^c D_t^{1/6}\right) x_{1\rho}(t) \\ \left({}_0^c D_t^{1/3}\right) x_{2\rho}(t) + \left({}_0^c D_t^{1/6}\right) x_{2\rho}(t) \end{bmatrix},$$

$$\bar{A} = \begin{bmatrix} -9/64 & 0 & 0 & 0 \\ 0 & -9/64 & 0 & 0 \\ 0 & 0 & -9/64 & 0 \\ 0 & 0 & 0 & -9/64 \end{bmatrix},$$

$$\bar{f}_1(\bar{x}) = \begin{bmatrix} x_1 x_2 K_{g_1}(t)\, \mu_{g_1}(\|{}_0^c D_t^\alpha x\|))\|x\| \\ x_1 x_3\, K_{g_2}(t)\, \mu_{g_2}(\|{}_0^c D_t^\alpha x\|))\|x\| \\ \tilde{b}_{12\rho} x_2 x_3 K_{g_3}(t)\, \mu_{g_3}(\|{}_0^c D_t^\alpha x\|))\|x\| \\ \tilde{b}_{22\rho} x_2 x_3 K_{g_3}(t)\, \mu_{g_3}(\|{}_0^c D_t^\alpha x\|))\|x\| \end{bmatrix},$$

$$\tilde{\tilde{A}}_{2,1} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}, \; \bar{f}_2(\bar{x}) = x_2 x_3 K_{g_3}(t)\, \mu_{g_3}(\|{}_0^c D_t^\alpha x\|))$$

$$\left({}_0^c D_t^{1/3}\right) x_1(t) + \left({}_0^c D_t^{1/6}\right) x_1(t) = -9/64\, x_1(t) + f_1(x_1, x_2, x_3) \mathrm{g}_1\, ({}_0^c D_t^\beta x_1, {}_0^c D_t^\beta x_2, {}_0^c D_t^\beta x_3) \quad (36)$$

From theorem (2.2.1), equation (36) becomes

$$\begin{aligned} {}_0^C D_t^{\frac{1}{6}} x_1(t) &= x_2(t), \\ {}_0^C D_t^{\frac{1}{6}} x_2(t) &= x_3(t), \end{aligned} \quad (37)$$

$${}_0^C D_t^{\frac{1}{6}} \bar{x}_3(t) = f_1(x_1, x_2, x_3) \mathrm{g}_1\, ({}_0^c D_t^\beta x_1, {}_0^c D_t^\beta x_2, {}_0^c D_t^\beta x_3) - x_3(\mathrm{t}) - 18 x_1(t)$$

$$\begin{bmatrix} {}_0^C D_t^{\frac{1}{6}} x_1(t) \\ {}_0^C D_t^{\frac{1}{6}} x_2(t)(t) \\ {}_0^C D_t^{\frac{1}{6}} x_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -9/64 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ f_1(x_1, x_2, x_3) \mathrm{g}_1\, ({}_0^c D_t^\beta x_1, {}_0^c D_t^\beta x_2, {}_0^c D_t^\beta x_3) \end{bmatrix} \quad (38)$$

The coefficient matrix of system (38) can be written as

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -9/64 & 0 & -1 \end{bmatrix}.$$

To obtain the eigenvalues of A_1,

$$\det(A_1 - \lambda I) = 0,$$

$$\det\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -9/64 & 0 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = 0,$$

$$\det\left(\begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -9/64 & 0 & -1-\lambda \end{bmatrix} \right) = 0,$$

$$\lambda^2(-1-\lambda) - {}^{9}/_{64} = 0$$
$$\lambda^3 + \lambda^2 + {}^{9}/_{64} = 0$$
$$\Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = \frac{-3}{4}$$
$$\Rightarrow \arg(\lambda_1) = \frac{\pi}{6}, \ \arg(\lambda_2) = \frac{-\pi}{6}$$
$$\Rightarrow |\arg(\lambda_i)| = \frac{\pi}{6}, \quad i = 1,2,3$$

and $\frac{\gamma\,\pi}{2} = \frac{\pi}{12} = 0.26167$

$$\Rightarrow |\arg(\lambda_i)| > \frac{\gamma\,\pi}{2}, \quad i = 1,2,3$$

Then, from corollary (2.2.2), we have that A_1 is stable.

It is easy to demonstrate that $f_1(x_1,x_2,x_3)\mathrm{g}_1\left({}_0^cD_t^\beta x_1, {}_0^cD_t^\beta x_2, {}_0^cD_t^\beta x_3\right) = x_1x_2K_{g_1}(t)\,\mu_{g_1}(\|{}_0^cD_t^\alpha x\|))$ satisfies the following

$$\lim_{\|\bar{x}\|\to 0}\frac{\left\|f_1(x_1,x_2,x_3)\mathrm{g}_1\left({}_0^cD_t^\beta x_1, {}_0^cD_t^\beta x_2, {}_0^cD_t^\beta x_3\right)\right\|}{\|\bar{x}(t)\|} = \lim_{\|\bar{x}\|\to 0}\frac{\sqrt{(x_1x_2)^2}K_{g_3}(t)\,\mu_{g_3}(\|{}_0^cD_t^\alpha x\|))}{\sqrt{(x_1)^2+(x_2)^2+(x_3)^2}}$$
$$\leq \lim_{\|\bar{x}\|\to 0}\frac{\sqrt{x_1^2x_2^2}}{\sqrt{x_1^2}} = \lim_{\|\bar{x}\|\to 0}\sqrt{x_2^2} = 0$$

that is $\left\|f_1(x_1,x_2,x_3)\mathrm{g}_1\left({}_0^cD_t^\beta x_1, {}_0^cD_t^\beta x_2, {}_0^cD_t^\beta x_3\right)\right\| = o\|x(t)\| \ as \ \|x\| \to 0.$

By continuing in this way with other equations of (3), and

$\|x_3(t)\| \leq \left[\left\|\begin{bmatrix}0 & 0 & 0 & \frac{1}{2}\end{bmatrix}\right\| + L_3L_4K_{g_3}(t)\,\mu_{g_3}(\|{}_0^cD_t^\alpha x\|)\right]\|x\|$, hence $\|x_3(t)\| \to 0$ as $\|x\| \to 0$.

By using theorem (2.2.3), then the zero solution of equation (34-35) is asymptotically stable.

Conclusions

We studied the asymptotic stability for the proposed multi- fractional differential-algebraic control systems, involving multi control inputs, which needed to be transformed to single-fractional differential systems, using sufficient and necessary conditions.

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