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Some Results on Fuzzy ω-Covering Dimension Function in Fuzzy Topological Space

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Abstract
The purpose of this paper is to study a new class of fuzzy covering dimension functions, called fuzzy **ω**-covering dimension function, on fuzzy topological space and investigate its relationships with fuzzy separation axioms and fuzzy **ω**-connected space. We also provide some characterization on **ω**-covering dimension function by using fuzzy countable sets, for which we obtained several properties.

Keywords: ω-$dim_f(X)$, fuzzy ω-open set, fuzzy locally countable set, fuzzy ω-disconnected space

بعض النتائج على البعد اوميكا- الغطائي الضبابي للداله في الفضاء التبولوجي الضبابي

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الخلاصة
الهدف في هذه الورقه هو دراسه نوع جديد من البعد الغطائي الضبابي للداله والذي يسمى البعد اوميكا- الغطائي الضبابي للداله على الفضاء التبولوجي الضبابي ودراسه العلاقات مع بديهيات الفصل الضبابيه وفضاء اوميكا- المتصل واعطينا بعض الخصائص على البعد اوميكا- الغطائي الضبابي للداله باستخدام المجاميع الضبابيه القابله للعد و حصلنا على العديد من الخصائص

I. Introduction

The concept considered in this paper is the so called "fuzzy sets", which is totally different from the classical concept called "crisp sets". The recent concept was introduced by Zadeh in 1965 [1], in which he defines fuzzy sets as a class of objects with a continuum of grades of membership. Such a set is characterized by a membership function that assigns to each object a grade of membership ranging between zero and one. In 1968, Chang [2] introduced the definition of fuzzy topological spaces and extended, in a straight forward manner, some concepts of crisp topological spaces to fuzzy topological spaces. Later, Lowen (1976) redefined what is now known as stratified fuzzy topology [3]. While Wong (1974) discussed and generalized some properties of fuzzy topological spaces [4]. The notion of fuzzy covering dimension of a fuzzy topological space $(X, \tilde{T})$ was introduced by Zougdani [5] in 1984. He used the notion of a shading family, which was first introduced by Gantner. *et al*. [6] in 1978, during their investigation of compactness in fuzzy topological spaces. Zougdani introduced that notion of $(X, \tilde{T})$ in the Chang's sense. Jackson [7] investigated Zougdani's paper and provided some counter-examples of which invalidated his results. Ajmal and Kohli (1994) introduced a notion of fuzzy covering dimension of $(X, \tilde{T})$, which is an extension of the concept of the covering dimension in classical topological spaces [8]. A distinctive feature of their study is that they deviated from the classical topological approach by employing the notion of quasi- coincidence, which was first

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introduced by Ming and Ming [9] in 1980, in defining the order of a family of fuzzy sets, which is instrumental in the definition of the dimension function. Ajmal and Kohli [8] restricted some results of their study to the Lowen's definition of (X, T̃). In this paper, we introduce the concepts of fuzzy **ω**-open set and fuzzy **ω**-covering dimension functions on fuzzy topological space and provide some characterization. Moreover, the study examines the relationships between these functions and fuzzy separation axioms and fuzzy **ω**-connected space. The current investigation could obtain several properties

II. Preliminaries

Definition 2.1 [1]

Let X be a non empty set and let I be the unit interval, i.e. I= [0, 1]. A fuzzy set in X is a function from X into the unit interval I ,Ã : X → [0,1] that is a function A fuzzy set Ã in X and can be represented by the following set of pairs:

Ã = {(x, $\mu_{\tilde{A}}(x)$): x ∈ X}; the family of all fuzzy sets in X is denoted by I^X.

Definition 2.2 [10]

A fuzzy point x_r is a fuzzy set such that :

$\mu_{x_r}(y) = r > 0$ if $x = y$, $\forall\ y \in X$ and

$\mu_{x_r}(y) = 0$ if $x \neq y$, $\forall\ y \in X$

The family of all fuzzy points of Ã will be denoted by FP(Ã).

Definition 2.4 [11]

A fuzzy point x_r is said to belong to a fuzzy set Ã in X

(denoted by : $x_r \in \tilde{A}$) if and only if $\mu_{x_r(x)} = r \leq \mu_{\tilde{A}}(x)$.

Proposition 2.5 [11]

Let Ã and B̃ be two fuzzy sets in X with membership functions $\mu_{\tilde{A}}$ and $\mu_{\tilde{B}}$, respectively. Then, for all x ∈ X:

1. $\tilde{A} \subseteq \tilde{B}$ if and only if $\mu_{\tilde{A}}(x) \leq \mu_{\tilde{B}}(x)$.
2. $\tilde{A} = \tilde{B}$ if and only if $\mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x)$.
3. $\tilde{C} = \tilde{A} \cap \tilde{B}$ if and only if $\mu_{\tilde{C}}(x) = \min\{ \mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x) \}$.
4. $\tilde{D} = \tilde{A} \cup \tilde{B}$ if and only if $\mu_{\tilde{D}}(x) = \max\{ \mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}$.

Definition 2.6 [12]

The support of a fuzzy set Ã, Supp (Ã), is the crisp set of all x ∈ X, such that $\mu_{\tilde{A}}(x) > 0$.

Definition 2.7 [2]

A fuzzy topology is a family T̃ of fuzzy subsets in X, satisfying the following conditions:

(a) $\emptyset$, $1_X \in \tilde{T}$.

(b) If $\tilde{A}, \tilde{B} \in \tilde{T}$, then $\tilde{A} \cap \tilde{B} \in \tilde{T}$.

(c) If $\tilde{A}_i \in \tilde{T}$, $\forall$ i ∈ J, where J is any index set, then $\bigcup_{i \in J} \tilde{A}_i \in \tilde{T}$.

T̃ is called fuzzy topology for X, and the pair (X, T̃) is a fuzzy topological space. Every member of T̃ is called open fuzzy set (T̃-open fuzzy set). A fuzzy set C̃ in 1_X is called closed fuzzy set (T̃-closed fuzzy set) if and only if its complement $\tilde{C}^c$ is T̃-open fuzzy set.

Definition 2.8 [13] :

If $\tilde{B} \in (X,\tilde{T})$, the complement of B̃ referred to 1_X denoted by $\tilde{B}^c$, is defined by $\tilde{B}^c = 1_X - \tilde{B}$

Definitions 2.9 [14, 9]

Let $\tilde{B}$, $\tilde{C}$ be a fuzzy set in a fuzzy topological space (X ,T̃), then:

- A fuzzy point x_r is said to be quasi coincident with the fuzzy set B̃ if there exists x ∈ X such that $\mu_{x_r}(x) + \mu_{\tilde{B}}(x) > \mu_{\tilde{A}}(x)$, and denoted by $x_r\ q\ \tilde{B}$. If $\mu_{x_r}(x) + \mu_{\tilde{B}}(x) \leq \mu_{\tilde{A}}(x)$ $\forall$ x ∈ X , then x_r is not quasi coincident with a fuzzy set B̃, and is denoted by $x_r\ \tilde{\bar{q}}\ \tilde{B}$.
- A fuzzy set $\tilde{B}$ is said to be quasi coincident (overlap) with a fuzzy set $\tilde{C}$ if there exists x ∈ X such that $\mu_{\tilde{B}}(x) + \mu_{\tilde{C}}(x) > \mu_{\tilde{A}}(x)$, and denoted by $\tilde{B}\ q\ \tilde{C}$. If $\mu_{\tilde{B}}(x) + \mu_{\tilde{C}}(x) \leq \mu_{\tilde{A}}(x)$ $\forall$ x ∈ X, then $\tilde{B}$ is not quasi coincident with a fuzzy set C̃, and is denoted by $\tilde{B}\ \tilde{\bar{q}}\ \tilde{C}$.

Definition 2.10 [11]

A fuzzy set B̃ in a fuzzy topological space (Ã ,T̃) is said to be a fuzzy neighborhood of a fuzzy point x_r in Ã if there is a fuzzy open set G̃ in Ã such that $\mu_{x_r}(x) \leq \mu_{\tilde{G}}(x) \leq \mu_{\tilde{B}}(x)$,, $\forall$ x ∈ X.

Definition 2.11 [15]

Let (X, $\tilde{T}$) be a fuzzy topological space and $\tilde{B} \in P(1_X)$, then the relative fuzzy topology for $\tilde{B}$ is defined by $\tilde{T}_B = \{ \tilde{B} \cap \tilde{G} : \tilde{G} \in \tilde{T} \}$.
The corresponding ($\tilde{B}$, $\tilde{T}_B$) is called fuzzy subspace of (X, $\tilde{T}$) .

Definition 2.12 [8]

Let X be a nonempty set. A family **U** $= \{\tilde{U}_\lambda\}_{\lambda \in \Lambda}$ of fuzzy sets in 1_X is said to be of order n ($n > -1$), written as ord_f**U** = n, if n is the greatest integer, such that there exists an overlapping subfamily of **U** having n + 1 elements.

Remark *2.13*

From the above definition, if ord_f **U** = n, then for each n + 2 distinct indexes $\lambda_1, \lambda_2, \ldots, \lambda_{n+2} \in \Lambda$, we have $U_{\lambda_1} \cap U_{\lambda_2} \cap \ldots \cap U_{\lambda_{n+2}} = \varnothing$. Then, it is non-overlapping, in particular if ord_f **U** = –1, then **U** consists of the empty fuzzy sets and ord_f **U** = 0, then **U** consist of pairwise disjoint fuzzy sets which are not all empty.

Definition 2.14 [4]

Let B = $\{\tilde{B}_\alpha : \alpha \in \Lambda\}$ and **C** = $\{\tilde{C}_\beta : \beta \in \Lambda\}$ ($\beta < \alpha$) be any two collections of fuzzy sets in (X,$\tilde{T}$) , then C is a refinement of B if for each $\beta \in \Lambda$ there exists $\alpha \in \Lambda$ such that $\mu_{\tilde{C}_\beta}(x) \leq \mu_{\tilde{B}_\alpha}(x)$.

Definition 2.15 [16]

A fuzzy topological space (X,$\tilde{T}$) is said to be fuzzy connected, if it has no proper fuzzy clopen set. Otherwise, it is called fuzzy disconnected.

III. Properties Of Fuzzy ω-Open Set In Fuzzy Topological Space

Definition 3.1 [17]

A fuzzy set $\tilde{A}$ in a fuzzy topological space (X,$\tilde{T}$) is called a fuzzy uncountable if and only if supp($\tilde{A}$) is an uncountable subset of X

Definition 3.2

A fuzzy point x_r of a fuzzy topological space (X,$\tilde{T}$) is called a fuzzy condensation point of $\tilde{A} \subseteq 1_X$ if $\tilde{B} \cap \tilde{A}$ is fuzzy uncountable for each fuzzy open set $\tilde{B}$ containing x_r. And the set of all fuzzy condensation points of $\tilde{A}$ is denoted by Cond($\tilde{A}$).

Definition 3.3

A fuzzy subset $\tilde{A}$ in a fuzzy topological space (X,$\tilde{T}$) is called a fuzzy ω-closed set if it contains all its fuzzy condensation points. The complement fuzzy ω-closed sets are called fuzzy ω-open sets.

Theorem 3.4

A fuzzy subset $\tilde{G}$ of a fuzzy topological space (X,$\tilde{T}$) is fuzzy ω-open set if and only if for any $x_r \in \tilde{G}$ there exists a fuzzy open set $\tilde{U}$ such that $x_r \in \tilde{U}$ and $\tilde{U}$- $\tilde{G}$ is countable.

Proof

$\tilde{G}$ is fuzzy ω-open set if and only if 1_X - $\tilde{G}$ is fuzzy ω-closed set,
1_X - $\tilde{G}$ is fuzzy ω-closed set if and only if Cond(1_X - $\tilde{G}$) $\subseteq 1_X$ - $\tilde{G}$,
and Cond(1_X - $\tilde{G}$) $\subseteq 1_X$ - $\tilde{G}$ if and only if for each $x_r \in \tilde{G}$, $x_r \notin$ Cond(1_X - $\tilde{G}$).
Thus, for $x_r \notin$ Cond(1_X - $\tilde{G}$) there exists a fuzzy open set $\tilde{U}$ such that
$x_r \in \tilde{U}$ and $\tilde{U} \cap (1_X - \tilde{G}) = \tilde{U} - \tilde{G}$ is countable

Theorem 3.5

A fuzzy subset $\tilde{G}$ of a fuzzy topological space (X,$\tilde{T}$) is ω-open set if and only if for each $x_r \in \tilde{G}$ there exist an fuzzy open set $\tilde{U}$ containing x_r and countable fuzzy subset $\tilde{C}$ of 1_X such that $\tilde{U}$- $\tilde{C} \subseteq \tilde{G}$.

Proof

($\Longrightarrow$) suppose that $\tilde{G}$ is fuzzy ω-open set and let $x_r \in \tilde{G}$.
Then, there exist a fuzzy open set $\tilde{U}$ and $x_r \in \tilde{U}$ and $\tilde{U}$- $\tilde{G}$ is countable.
Set $\tilde{C} = \tilde{U} - \tilde{G}$, then $\tilde{C}$ is countable and $x_r \in \tilde{U} - \tilde{C} = \tilde{U} - (\tilde{U} - \tilde{G}) \subseteq \tilde{G}$.
($\Longleftarrow$) let $x_r \in \tilde{G}$ then, by assumption, there exist fuzzy open set $\tilde{U}$ containing x_r and countable fuzzy subset $\tilde{C}$ of 1_X such that $\tilde{U}$- $\tilde{C} \subseteq \tilde{G}$.
Since $\tilde{U}$- $\tilde{G} \subseteq \tilde{C}$, then $\tilde{U}$- $\tilde{G}$ is countable, hence $\tilde{G}$ is fuzzy ω-open set

Proposition 3.6

In a fuzzy topological space, every fuzzy open set is fuzzy ω-open set.

Proof

Let $\tilde{G}$ be fuzzy open set and $x_r \in \tilde{G}$. Set $\tilde{U} = \tilde{G}$, $\tilde{C} = \emptyset$, then $\tilde{U}$ is fuzzy open set and $\tilde{C}$ is a countable set , such that $x_r \in \tilde{U} - \tilde{C} \subseteq \tilde{G}$. Thus, $\tilde{G}$ is fuzzy ω-open set.

Remark 3.7

The converse of proposition (***3***.6) is not true in general, as the following examples show:

Examples 3.8

Let X = { a , b , c } and $\tilde{A}$, $\tilde{B}$ are fuzzy subsets in 1_X where

1_X= { (a , 1) , (b , 1) , (c , 1) }

$\tilde{A}$ = { (a , 0.6) , (b , 0.6) , (c , 0.7) }

$\tilde{B}$ = { (a , 0.5) , (b , 0.5) , (c , 0.4) }

Let $\tilde{T} = \{\emptyset , 1_X , \tilde{A}\}$ be a fuzzy topology on X.

Then the fuzzy set $\tilde{B}$ is a fuzzy ω – open set but not a fuzzy open set.

Definition 3.9

Let $\tilde{B}$ be a fuzzy set in a fuzzy topological space $(X,\tilde{T})$, then the **ω-interior** of $\tilde{B}$, which is denoted by

ω-Int($\tilde{B}$) and defined by ω-Int($\tilde{B}$) = ∪ { $\tilde{G}$: $\tilde{G}$, is a fuzzy ω-open set in 1_X , $\tilde{G} \subseteq \tilde{B}$ }.

Definition 3.10

Let $\tilde{B}$ be a fuzzy set in a fuzzy topological space $(X,\tilde{T})$, then the **ω-closure** of $\tilde{B}$, which is denoted by

ω-cl($\tilde{B}$) and defined by ω-cl($\tilde{B}$) = ⋂ { $\tilde{G}$: $\tilde{G}$, is a fuzzy ω-closed set in 1_X , $\tilde{B} \subseteq \tilde{G}$ }.

Definition 3.11

Let $\tilde{B}$ be a fuzzy set in a fuzzy topological space $(X,\tilde{T})$, then the **ω-boundary** of $\tilde{B}$ is denoted by ω-b($\tilde{B}$) and defined by ω-b($\tilde{B}$) = ω-cl($\tilde{B}$) - ω-Int($\tilde{B}$)

Theorem 3.12

Let $\tilde{A}$ be fuzzy subset of a fuzzy topological space $(X,\tilde{T})$.

Then, $(\tilde{T}_{\tilde{A}})^{\omega} = \tilde{T}^{\omega}_{\tilde{A}}$

Proof

To prove that $(\tilde{T}_{\tilde{A}})^{\omega} \subseteq \tilde{T}^{\omega}_{\tilde{A}}$, let $\tilde{B} \in (\tilde{T}_{\tilde{A}})^{\omega}$ and $x_r \in \tilde{B}$ **,** by **Theorem *3*.5.**

There exist fuzzy open set $\tilde{V}$ of $\tilde{T}_{\tilde{A}}$ and $\tilde{C}$ countable subset of $\tilde{T}_{\tilde{A}}$, such that $x_r \in \tilde{V} - \tilde{C} \subseteq \tilde{B}$. We choose $\tilde{U} \in \tilde{T}$ such that $\tilde{V} = \tilde{U} \cap \tilde{A}$, then $\tilde{U} - \tilde{C} \in \tilde{T}^{\omega}$, $x_r \in \tilde{U} - \tilde{C}$ and $\tilde{U} - \tilde{C} \cap \tilde{A} = \tilde{V} - \tilde{C} \subseteq \tilde{B}$.

Therefore, $\tilde{B} \in \tilde{T}^{\omega}_{\tilde{A}}$. To prove that $\tilde{T}^{\omega}_{\tilde{A}} \subseteq (\tilde{T}_{\tilde{A}})^{\omega}$, let $\tilde{G} \in \tilde{T}^{\omega}_{\tilde{A}}$, then there exists $\tilde{H} \in \tilde{T}^{\omega}$ such that

$\tilde{G} = \tilde{H} \cap \tilde{A}$. If $x_r \in \tilde{G}$, then $x_r \in \tilde{H}$ and there exist fuzzy open set $\tilde{U}$ of

$\tilde{T}$ and $\tilde{D}$ countable subset of $\tilde{T}$ such that $x_r \in \tilde{U} - \tilde{D} \subseteq \tilde{H}$.

We put $\tilde{V} = \tilde{U} \cap \tilde{A}$, then $\tilde{V} \in \tilde{T}_{\tilde{A}}$ and $x_r \in \tilde{V} - \tilde{D} \subseteq \tilde{G}$. It follows that $\tilde{G} \in (\tilde{T}_{\tilde{A}})^{\omega}$

Definition 3.13

The fuzzy family $\{\tilde{B}_\alpha : \alpha \in \Lambda\}$ of the subset of a fuzzy topological space $(X,\tilde{T})$ is called

1- Fuzzy ω-locally finite, if for each $x_r \in 1_X$ there exists a fuzzy ω-open set $\tilde{G}$ containing x_r, such that the set $\{\tilde{G} \cap \tilde{B}_\alpha \neq \emptyset : \alpha \in \Lambda\}$ is finite.

2- Fuzzy ω-discrete, if for each $x_r \in 1_X$, there exists a fuzzy ω-open set $\tilde{G}$ containing x_r, such that the set $\{\tilde{G} \cap \tilde{B}_\alpha \neq \emptyset : \alpha \in \Lambda\}$ has at most one member.

proposition 3.14

Every fuzzy locally finite (resp.fuzzy discrete) family of any fuzzy topological space $(X,\tilde{T})$ is fuzzy ω-locally finite (resp.fuzzy ω-discrete).

Proof *:* It follows from the fact that every fuzzy open set is fuzzy ω-open set.

Definition 3.15

A fuzzy topological space $(X,\tilde{T})$ is called a fuzzy anti-locally-countable if each nonempty fuzzy open subset of 1_X is uncountable.

Definition 3.16

A fuzzy topological space $(X,\tilde{T})$ is said to be ω-$\tilde{T}_0$ if for each pair of distinct fuzzy points x_r and y_t of 1_X there exists a fuzzy ω-open set $\tilde{G}$, such that either $x_r \in \tilde{G}$ and $y_t \notin \tilde{G}$ or $y_t \in \tilde{G}$ and $x_r \notin \tilde{G}$.

Definition *3.17*

A fuzzy topological space (X,$\tilde{T}$) is said to be ω-$\tilde{T}_1$ if for each pair of distinct fuzzy points x_r and y_t of 1_X there exist fuzzy ω-open sets $\tilde{G}$ and $\widetilde{H}$, such that both $x_r \in \tilde{G}$ and $y_t \notin \tilde{G}$ as well as $y_t \in \widetilde{H}$ and $x_r \notin \widetilde{H}$ hold.

Definition 3.18

A fuzzy topological space (X,$\tilde{T}$) is said to be ω-$\tilde{T}_2$ if for each pair of distinct fuzzy points x_r and y_t of 1_X there exist disjoint fuzzy ω-open sets $\tilde{G}$ and $\widetilde{H}$ containing x_r and y_t, respectively .

Definition 3.19

A fuzzy topological space (X,$\tilde{T}$) is called a fuzzy ω-regular space if for each fuzzy ω-closed subset $\tilde{B}$ of 1_X , and a fuzzy point x_r in 1_X such that $x_r \notin \tilde{B}$, there exist disjoint fuzzy ω-open sets $\widetilde{U}$ and $\tilde{V}$ containing x_r and $\tilde{B}$, respectively

Definition 3.20

A fuzzy topological space (X,$\tilde{T}$) is called a fuzzy ω-Normal space if, for each pair of disjoint fuzzy ω-closed sets $\tilde{A}$ and $\tilde{B}$ in 1_X, there exist disjoint fuzzy ω-open sets $\widetilde{U}$ and $\tilde{V}$ containing $\tilde{A}$ and $\tilde{B}$, respectively.

Theorem 3.21

A fuzzy topological space (X,$\tilde{T}$) is a fuzzy ω-Normal if for each pair of fuzzy ω-open sets $\tilde{G}$ and $\widetilde{H}$ in 1_X, such that $1_X = \tilde{G} \cup \widetilde{H}$, there are fuzzy ω-closed sets $\widetilde{U}$ and $\tilde{V}$contained in $\tilde{G}$ and $\widetilde{H}$, respectively, such that

$1_X = \widetilde{U} \cup \tilde{V}$.

Proof : Obvious.

Theorem 3.22

Every fuzzy ω-closed subspace of fuzzy ω-Normal space is fuzzy ω-Normal space.

Proof : Obvious.

Definition 3.23

Two fuzzy families $\{\tilde{A}_\lambda\}_{\lambda\in\Lambda}$ and $\{\tilde{B}_\lambda\}_{\lambda\in\Lambda}$ of subset of a fuzzy space 1_X are said to be similar if, for every finite subset Δ of Λ, the fuzzy sets $\bigcap_{\lambda\in\Delta}\tilde{A}_\lambda$ and $\bigcap_{\lambda\in\Delta}\tilde{B}_\lambda$ are either empty or nonempty.

Definition 3.24

A fuzzy ω-open covering $\{\tilde{U}_\lambda\}_{\lambda\in\Lambda}$ of a fuzzy topological space (X,$\tilde{T}$) is said to be fuzzy ω-shrinkable if there exists a fuzzy ω-open covering $\{\tilde{V}_\lambda\}_{\lambda\in\Lambda}$ of 1_X, such that ω-cl($\tilde{V}_\lambda$) ⊆ $\tilde{U}_\lambda$ for each λ.

IV. Properties Of Fuzzy *ω*-Covering Dimension Function

Definition 4.1

The fuzzy ω-covering dimension function of a fuzzy topological space (X,$\tilde{T}$), denoted ω-$\dim_f$(**X**), is the least integer n, such that every finite ω-open cover of 1_X has a finite ω-open refinement of order not exceeding n or + ∞, if there exists no such integer.

Thus, it follows that ω-$\dim_f$(**X**) = −1 if and only if X = ∅, and ω-$\dim_f$(**X**) ≤ n if every finite ω-open cover of 1_X has a finite ω-open refinement of order ≤ n. We have ω-$\dim_f$(**X**) = n if it is true that ω-$\dim_f$(**X**) ≤ n, but it is false that ω-$\dim_f$(**X**) ≤ n – 1.

Finally, ω-$\dim_f$(**X**) = + ∞ if, for every positive integer n, it is false that ω-$\dim_f$(**X**) ≤ n.

Propositions 4. 2

Let ($\tilde{B}$,$\tilde{T}_B$) be a closed fuzzy subspace of a fuzzy topological space (X,$\tilde{T}$), then ω-$\dim_f$($\tilde{B}$) ≤ ω-$\dim_f$(**X**).

Proof

If ω-$\dim_f$(**X**) = ∞ or ω-$\dim_f$(**X**) = -1, then there is nothing to prove. So it is sufficient when we show that if ω-$\dim_f$(**X**) = n then ω-$\dim_f$($\tilde{B}$) ≤ n. For this, let $\{\widetilde{U}_i\}_{i=1}^{t}$ be fuzzy finite covering of $\tilde{B}$ by fuzzy ω-open sets of $\tilde{B}$. Then by ***Theorem 3.12***, there exist fuzzy ω-open sets $\tilde{V}_i$ in 1_X such that $\widetilde{U}_i = \tilde{V}_i \cap \tilde{B}$ for each i= 1,2,….., t . Hence, $\{\tilde{V}_i\}_{i=1}^{t} \cup \{1_X - \tilde{B}\}$ is fuzzy finite ω-open covering of 1_X . Since ω-$\dim_f$(**X**) = n, then there exists fuzzy ω-open refinement $\{\tilde{G}_\lambda\}_{\lambda\in\Lambda}$ of $\{\tilde{V}_i\}_{i=1}^{t} \cup \{1_X - \tilde{B}\}$ of order not exceeding n . Thus, $\{\tilde{G}_\lambda \cap \tilde{B}\}_{\lambda\in\Lambda}$ is fuzzy ω-open refinement of $\{\tilde{U}_i\}_{i=1}^{t}$ of order not exceeding n . This implies that ω-dim$_f$(B̃) ≤ n

Propositions 4. 3

If (X,T̃) is a fuzzy locally countable topological space, then ω-dim$_f$(**X**) = 0.

Proof : Obvious.

Propositions 4. 4

A fuzzy topological space (X,T̃) with ω-dim$_f$(**X**) = 0 is fuzzy ω-Normal space.

Proof

Let ω-dim$_f$(**X**) = 0 and $\tilde{G}$, $\tilde{H}$ be two fuzzy ω-open sets of 1_X such that $\tilde{G} \cup \tilde{H} = 1_X$. Therefore, there exists a fuzzy ω-open refinement $\{\tilde{Z}_\lambda\}_{\lambda\in\Lambda}$ of the fuzzy cover $\{\tilde{G}\ ,\ \tilde{H}\}$ of order not exceeding 0. This means that the members of $\{\tilde{Z}_\lambda\}_{\lambda\in\Lambda}$ are fuzzy pairwise disjoint. Then, $\tilde{N} = \bigcup_{\lambda\in\Lambda}\{\tilde{Z}_\lambda : \tilde{Z}_\lambda \subseteq \tilde{G}\}$ and $\tilde{M} = \bigcup_{\lambda\in\Lambda}\{\tilde{Z}_\lambda : \tilde{Z}_\lambda \subseteq \tilde{H}\}$ are fuzzy disjoints and $\tilde{N} \cup \tilde{M} = 1_X$. Therefore, (X,T̃) is fuzzy ω-Normal space

Propositions 4. 5

A fuzzy topological space (X,T̃) with dim$_f$(**X**) = 0 is fuzzy Normal space.

Proof *:* Obvious.

Propositions *4. 6*

A fuzzy ω-$\tilde{T}_1$ space (X,T̃) with more than one fuzzy point and ω-dim$_f$(**X**) = 0 is fuzzy ω-disconnected space.

Proof

Let ω-dim$_f$(**X**) = 0 and let x_r and y_t be two fuzzy distinct points in 1_X, then $1_X - \{x_r\}$, $1_X - \{y_t\}$ is fuzzy finite ω-open covering of 1_X. So, by putting $\tilde{G} = 1_X - \{x_r\}$ and $\tilde{H} = 1_X - \{y_t\}$ in the proof of ***Propositions 4. 4,*** we obtain two fuzzy disjoint ω-clopen sets, i.e. $\tilde{N} \subseteq \tilde{G}$ and $\tilde{M} \subseteq \tilde{H}$, such that $\tilde{N} \cup \tilde{M} = 1_X$. Thus, $\tilde{N}$ and $\tilde{M}$ are fuzzy maximal proper ω-clopen subsets of 1_X . Hence, (X,T̃) is fuzzy ω-disconnected space

Theorem 4. 7

If (X,T̃) is a fuzzy anti-locally countable Normal space, then dim$_f$(**X**) ≤ ω-dim$_f$(**X**).

Proof

If either ω-dim$_f$(**X**) = ∞ or ω-dim$_f$(**X**) = -1, then there is nothing to prove.

Let n be any non-negative integer, such that ω-dim$_f$(**X**) = n, and let $\{\tilde{G}_i\}_{i=1}^{t}$ be any fuzzy finite open covering of 1_X . Since (X,T̃) is fuzzy Normal, then there exists a fuzzy open covering $\{\tilde{O}_i\}_{i=1}^{t}$ such that $cl(\tilde{O}_i) \subseteq \tilde{G}_i$ for each i=1,2,…..,t. Again, by Normality of 1_X and there exists a fuzzy family $\{\tilde{V}_i\}_{i=1}^{t}$ of fuzzy open subsets of 1_X such that $cl(\tilde{O}_i) \subseteq \tilde{V}_i \subseteq cl(\tilde{V}_i) \subseteq \tilde{G}_i$ for each i=1,2,…..,t. Also, the fuzzy families $\{cl(\tilde{V}_i)\}_{i=1}^{t}$ and $\{cl(\tilde{O}_i)\}_{i=1}^{t}$ are similar. Since ω-dim$_f$(**X**) = n, then there exists a fuzzy ω-open refinement $\tilde{Z}$ of $\{\tilde{V}_i\}_{i=1}^{t}$ of order not exceeding n . Let $\tilde{U}_i = \{\tilde{W} \in \tilde{Z} : \tilde{W} \subseteq \tilde{V}_i\}$ for each i=1,2,…..,t. Clearly, $\{\tilde{U}_i\}_{i=1}^{t}$ is of an order not exceeding n. Since (X,T̃) is fuzzy anti-locally countable and for each i=1,2,…..,t , $\tilde{U}_i$ is fuzzy ω-open set in 1_X, then we have $\tilde{U}_i \subseteq Int(cl(\tilde{U}_i)) \subseteq cl(\tilde{V}_i) \subseteq \tilde{G}_i$. Hence, $\{Int(cl(\tilde{U}_i))\}_{i=1}^{t}$ is fuzzy open refinement of $\{\tilde{G}_i\}_{i=1}^{t}$ of order not exceeding n. Therefore, dim$_f$(**X**) ≤ n = ω-dim$_f$(**X**)

Theorem 4. 8

If a fuzzy topological space (X,T̃) has the property that every fuzzy ω-open covering of 1_X has a fuzzy locally finite open refinement, then ω-dim$_f$(**X**) ≤ dim$_f$(**X**).

Proof

Let (X,T̃) be a fuzzy topological space with a given property. So, if either ω-dim$_f$(**X**) = ∞ or ω-dim$_f$(**X**) = -1, then there is nothing to prove.

Suppose that dim$_f$(**X**) = n and let $\{\tilde{U}_i\}_{i=1}^{t}$ be any fuzzy finite ω-open covering of 1_X, then by hypothesis, this ω-open covering has a fuzzy locally finite open refinement $\{\tilde{G}_\lambda\}_{\lambda\in\Lambda}$. Let $\tilde{G}_i = \bigcup \{\tilde{G}_\lambda : \tilde{G}_\lambda \subseteq \tilde{U}_i\}$ for each i, thus $\{\tilde{G}_i\}_{i=1}^{t}$ is fuzzy finite open covering of 1_X. Since dim$_f$(**X**) = n, then there exists a fuzzy open refinement $\{\tilde{Z}_\gamma\}_{\lambda\in\Gamma}$ of $\{\tilde{G}_i\}_{i=1}^{t}$(and hence of $\{\tilde{U}_i\}_{i=1}^{t}$) of order not exceeding n.

Therefore, ω-dim$_f$(**X**) ≤ n = dim$_f$(**X**)

Corollary 4. 9

Let $(X,\tilde{T})$ be a fuzzy anti-locally countable Normal space with the property that every fuzzy ω-open covering of 1_X has a fuzzy locally finite open refinement. Then, ω-dim$_f$(**X**) = dim$_f$(**X**).

Proof : Obvious.

Theorem 4.10:

For a fuzzy topological space $(X,\tilde{T})$, the following statements are equivalent:

1- ω-dim$_f$(**X**) ≤ n.

2- For every fuzzy ω-open cover $\{\tilde{U}_1,\ldots,\tilde{U}_K\}$ of 1_X, there exists a fuzzy finite ω-open cover $\{\tilde{\eta}_1,\ldots,\tilde{\eta}_K\}$ of 1_X of order less than or equal to n, and $\tilde{\eta}_i \leq \tilde{U}_i$ for i = 1,2,…,k.

3- If $\{\tilde{U}_1,\ldots,\tilde{U}_{n+2}\}$ is a fuzzy ω-open cover of 1_X, then there exists a non-overlapping fuzzy ω-open cover $\{\tilde{\eta}_1,\ldots,\tilde{\eta}_{n+2}\}$ of 1_X, such that $\tilde{\eta}_i \leq \tilde{U}_i$ for i = 1, 2, …, n + 2.

Proof

(1) ⟹ (2) suppose that ω-dim$_f$(**X**) ≤ n and let $\tilde{\mathbf{U}} = \{\tilde{U}_1,\ldots,\tilde{U}_K\}$ be a fuzzy ω-open cover of 1_X. Let $\tilde{\mathbf{V}}$ be a fuzzy finite ω-open refinement of $\tilde{\mathbf{U}}$ such that ord$_f$ $\tilde{\mathbf{V}} \leq$ n. If $\tilde{V} \in \tilde{\mathbf{V}}$, then $\tilde{V} \subseteq \tilde{U}_i$, for some i. Let each $\tilde{V} \in \tilde{\boldsymbol{V}}$ be associated with one of the fuzzy sets $\tilde{U}_i$ containing it. Let $\tilde{\eta}_i$ be the union of all those members of $\tilde{\mathbf{V}}$ and, thus, associated with $\tilde{U}_i$. Then, each $\tilde{\eta}_i$ is a fuzzy ω-open set and $\tilde{\eta}_i \subseteq \tilde{U}_i$.

Let $\tilde{\mathbf{N}} = \{\tilde{\eta}_1,\ldots,\tilde{\eta}_K\}$. We want to show that ord$_f$ $\tilde{\mathbf{N}} \leq$ n, that is, every overlapping subfamily of $\tilde{\mathbf{N}}$ contains at most n + 1 members.

Suppose, if possible, that there exists an overlapping subfamily $\tilde{N}_i$ of $\tilde{\mathbf{N}}$ containing (n + 2) members. Then, there exists $x_r \in 1_X$ such that $\mu_{\tilde{\eta}_\alpha}(x_r)+\mu_{\tilde{\eta}_\beta}(x_r) > 1$ for every pair $\tilde{\eta}_\alpha, \tilde{\eta}_\beta \in \tilde{N}_i$. Now, since

$\tilde{\eta}_\gamma = \bigcup\{\tilde{V}_{i_\gamma} \in \tilde{V} : \tilde{V}_{i_\gamma} \leq \tilde{U}_i$ is associated in the construction of $\tilde{\eta}_\gamma\}$

(γ = α, β), and since $\tilde{\mathbf{V}}$ is a fuzzy finite cover of 1_X $\mu_{\tilde{\eta}_\alpha}(x_r)=\max\{\mu_{\tilde{V}_{1\alpha}}(x_r), \ldots\ldots, \mu_{\tilde{V}_{r\alpha}}(x_r)\}$ and $\mu_{\tilde{\eta}_\beta}(x_r)=\max\{\mu_{\tilde{V}_{1\beta}}(x_r), \ldots\ldots, \mu_{\tilde{V}_{s\beta}}(x_r)\}$, we choose $\tilde{V}_{k\alpha}$ and $\tilde{V}_{t\beta}$ such that $\mu_{\tilde{\eta}_\alpha}(x_r) = \mu_{\tilde{V}_{k\alpha}}(x_r)$ and $\mu_{\tilde{\eta}_\beta}(x_r)=\mu_{\tilde{V}_{t\beta}}(x_r)$. Clearly, $\tilde{V}_{k\alpha}$ and $\tilde{V}_{t\beta}$ overlap at x_r. In this way, we obtain, corresponding to every overlapping pair $\tilde{\eta}_\alpha, \tilde{\eta}_\beta$ at x_r , a pair $\tilde{V}_{i\alpha}$ and $\tilde{V}_{j\beta}$ of $\tilde{V}$'s which are distinct in themselves, distinct from others, and overlap at x_r. The collection of all these members of $\tilde{\mathbf{V}}$, which were chosen as above, constitutes an overlapping subfamily of $\tilde{\mathbf{V}}$, having n + 2 members. This is a contradiction to the fact that ord$_f$ $\tilde{\mathbf{V}}$ ≤ n. Thus, ord$_f$ $\tilde{\mathbf{V}}$ ≤ n. The statements (2) ⇒ (1) and (2) ⇒ (3) are trivial.

To complete the proof, we will show that (3) ⇒ (2).

Let $(X,\tilde{T})$ be a fuzzy topological space satisfying (3), and

let$\{\tilde{U}_1,\ldots,\tilde{U}_K\}$ be a fuzzy ω-open cover of 1_X. Assume that k > n+1.

Let $\tilde{G}_i = \tilde{U}_i$, if 1 ≤ i ≤ n+1.

And let $\tilde{G}_{n+2} = \bigcup_{i=n+2}^{k} \tilde{U}_i$. Clearly, $\{\tilde{G}_1,\ldots,\tilde{G}_{n+2}\}$ is a fuzzy ω-open cover of 1_X . By hypothesis, there exists a fuzzy ω-open cover

$\tilde{\mathbf{N}} = \{\tilde{\eta}_1,\ldots,\tilde{\eta}_{n+2}\}$ of 1_X such that $\tilde{\eta}_i \leq \tilde{G}_i$ for each i, and $\tilde{\mathbf{N}}$ is a non-overlapping family.

We define fuzzy ω-open sets $\tilde{V}_i = \tilde{U}_i$ if 1 ≤ i ≤ n+1, and $\tilde{V}_i = \tilde{U}_i \cap \tilde{\eta}_{n+2}$ if i > n + 1.

Then $\tilde{\mathbf{V}} = \{\tilde{V}_1,\ldots,\tilde{V}_k\}$ is a fuzzy ω-open cover of 1_X such that $\tilde{V}_i \subseteq \tilde{U}_i$, for each i and the subfamily $\{\tilde{V}_1,\ldots,\tilde{V}_{n+2}\}$ of $\tilde{\mathbf{V}}$ is non-overlapping. If there is a fuzzy subset $\tilde{B}$ of {1,….,K} having n + 2 elements such that the family $\{\tilde{V}_j: j \in \tilde{B}\}$ is overlapping, then let the members of $\tilde{\mathbf{V}}$ be renumbered to give a family

$\tilde{\mathbf{P}} = \{\tilde{P}_1,\ldots,\tilde{P}_k\}$ such that the subfamily $\{\tilde{P}_1,\ldots,\tilde{P}_{n+2}\}$is overlapping. By applying the above construction to $\tilde{\mathbf{P}}$, we obtain a fuzzy ω-open cover $\tilde{\mathbf{M}} = \{\tilde{M}_1,\ldots,\tilde{M}_k\}$ of 1_X such that $\tilde{M}_i \subseteq \tilde{P}_i$ and the subfamily

$\{\tilde{M}_1,\ldots,\tilde{M}_{n+2}\}$ is non-overlapping. Clearly, if $\tilde{C}$ is a fuzzy subset of {1,…,K} with n+2 elements, such that the family $\{\tilde{P}_i: i \in \tilde{C}\}$ is non-overlapping, then so is the family $\{\tilde{M}_i: i \in \tilde{C}\}$.

Thus, by a finite number of repetitions of this process, we obtain a fuzzy ω-open cover

$\tilde{\mathbf{W}} = \{\tilde{W}_1,\ldots,\tilde{W}_k\}$ of 1_X such that $\tilde{W}_i \leq \tilde{U}_i$ for each i and ord$_f$ $\tilde{\mathbf{W}}$ ≤ n

Theorem 4.11

If $(X,\tilde{T})$ is a fuzzy ω-Normal space, then the following statements are equivalent:

1- ω-$\dim_f$(**X**) ≤ n.
2- For every fuzzy finite ω-open covering $\{\tilde{U}_i\}_{i=1}^{t}$ of 1_X, there is a fuzzy ω-open covering $\{\tilde{V}_i\}_{i=1}^{t}$, such that ω-cl($\tilde{V}_i$) ⊆ $\tilde{U}_i$, for each i = 1,2,…,t and the order {ω-cl $\tilde{V}_i\}_{i=1}^{t}$ that does not exceed n.
3- For every fuzzy finite ω-open covering $\{\tilde{U}_i\}_{i=1}^{t}$ of 1_X, there is a fuzzy ω-closed covering $\{\tilde{F}_i\}_{i=1}^{t}$ of order not exceeding n such that $\tilde{F}_i \subseteq \tilde{U}_i$ for each i = 1,2,…,t.
4- Every fuzzy finite ω-open covering of 1_X has a fuzzy finite ω-closed refinement of order that does not exceed n.
5- If $\{\tilde{U}_i\}_{i=1}^{n+2}$ is a fuzzy ω-open cover of 1_X, then there exists a fuzzy ω-closed covering $\{\tilde{F}_i\}_{i=1}^{n+2}$ such that $\bigcap_{i=1}^{n+2}\tilde{F}_i = \varnothing$

Proof

(1) ⟹ (2) let ω-$\dim_f$(**X**) ≤ n and $\{\tilde{U}_i\}_{i=1}^{t}$ be fuzzy finite ω-open covering of 1_X, then by ***Theorem 4.10*** there exists a fuzzy ω-open covering $\{\tilde{W}_i\}_{i=1}^{t}$ of 1_X of order not exceeding n, such that $\tilde{W}_i \subseteq \tilde{U}_i$ for each i . Since (X,T̃) is a fuzzy ω-Normal space, then by ***Theorem 3.22***, there exists a fuzzy ω-open covering $\{\tilde{V}_i\}_{i=1}^{t}$ of 1_X such that ω-cl($\tilde{V}_i$) ⊆ $\tilde{W}_i$ for each i. Thus, $\{\tilde{V}_i\}_{i=1}^{t}$ is the required fuzzy ω-open covering of 1_X .
(2) ⟹ (3) and (3) ⟹ (4) Obvious.
(4) ⟹ (5) let (X,T̃) be fuzzy ω-Normal space which satisfies (4).
Also, let $\{\tilde{U}_i\}_{i=1}^{n+2}$ be a fuzzy ω-open covering of 1_X, then, by hypothesis, there exists a fuzzy finite ω-closed refinement $\tilde{Z}$ of $\{\tilde{U}_i\}_{i=1}^{n+2}$ of order not exceeding n. Since for each $\tilde{F} \in \tilde{Z}$ there exist a unique i∈{1,2,…,n+2} such that $\tilde{F} \subseteq \tilde{U}_i$, then we set $\tilde{F}_i = \cup\{\tilde{F} \in \tilde{Z} : \tilde{F} \subseteq \tilde{U}_i\}$. since $\tilde{Z}$ is fuzzy finite, so for each i , $\tilde{F}_i$ is fuzzy ω-closed and $\tilde{F}_i \subseteq \tilde{U}_i$. Furthermore, $\{\tilde{F}_i\}_{i=1}^{n+2}$ is fuzzy cover of 1_X of order not exceeding n. Thus, $\bigcap_{i=1}^{n+2}\tilde{F}_i = \varnothing$
(5) ⟹ (1) let (X,T̃) be fuzzy ω-Normal space which satisfies (5).
Also, let $\{\tilde{U}_i\}_{i=1}^{n+2}$ be a fuzzy ω-open covering of 1_X, so by our hypothesis, there exist a fuzzy ω-closed covering $\{\tilde{F}_i\}_{i=1}^{n+2}$ of 1_X, such that for each i $\tilde{F}_i \subseteq \tilde{U}_i$ and $\bigcap_{i=1}^{n+2}\tilde{F}_i = \varnothing$, there exists a fuzzy ω-open covering $\{\tilde{V}_i\}_{i=1}^{n+2}$ of 1_X, such that for each i $\tilde{F}_i \subseteq \tilde{V}_i \subseteq$ ω-cl($\tilde{V}_i$) ⊆ $\tilde{W}_i$ and the fuzzy family $\{\tilde{F}_i\}_{i=1}^{n+2}$ is similar to {ω-cl($\tilde{V}_i$)$\}_{i=1}^{n+2}$, thus $\{\tilde{V}_i\}_{i=1}^{n+2}$ is fuzzy ω-open covering of 1_X, such that $\tilde{V}_i \subseteq \tilde{U}_i$ for each i and $\bigcap_{i=1}^{n+2}\tilde{V}_i = \varnothing$. Hence, by ***Theorem 4.10***, ω-$\dim_f$(**X**) ≤ n

Theorem 4.12

If (X,T̃) is a fuzzy ω-Normal space, then the following statements are equivalent:
1- ω-$\dim_f$(**X**) ≤ n.
2- For each fuzzy family $\{\tilde{F}_i\}_{i=1}^{n+1}$ of ω-closed sets and each fuzzy family $\{\tilde{U}_i\}_{i=1}^{n+1}$ of ω-open sets of 1_X, such that $\tilde{F}_i \subseteq \tilde{U}_i$ for each i, there is a fuzzy family $\{\tilde{V}_i\}_{i=1}^{n+1}$ of ω-open sets, such that $\tilde{F}_i \subseteq \tilde{V}_i \subseteq$ ω-cl($\tilde{V}_i$) ⊆ $\tilde{U}_i$ for each i and $\bigcap_{i=1}^{n+1}\omega - b(\tilde{V}_i) = \varnothing$
3- For each fuzzy family $\{\tilde{F}_i\}_{i=1}^{t}$ of ω-closed sets and each fuzzy family $\{\tilde{U}_i\}_{i=1}^{t}$ of ω-open sets of 1_X, such that $\tilde{F}_i \subseteq \tilde{U}_i$ for each i, there exist fuzzy families $\{\tilde{V}_i\}_{i=1}^{t}$ and $\{\tilde{W}_i\}_{i=1}^{t}$of ω-open sets, such that $\tilde{F}_i \subseteq \tilde{V}_i \subseteq$ ω-cl($\tilde{V}_i$) ⊆ $\tilde{W}_i \subseteq \tilde{U}_i$ for each i, and the order of {ω-cl($\tilde{W}_i$) - $\tilde{V}_i\}_{i=1}^{t}$ does not exceed n-1.
4- For each fuzzy family $\{\tilde{F}_i\}_{i=1}^{t}$ of ω-closed sets and each fuzzy family $\{\tilde{U}_i\}_{i=1}^{t}$ of ω-open sets of 1_X, such that $\tilde{F}_i \subseteq \tilde{U}_i$ for each i, there exists a fuzzy family $\{\tilde{V}_i\}_{i=1}^{t}$ of ω-open sets, such that $\tilde{F}_i \subseteq \tilde{V}_i \subseteq$ ω-cl($\tilde{V}_i$) ⊆ $\tilde{U}_i$ for each i, and the order of {ω-b($\tilde{V}_i$)$\}_{i=1}^{t}$ does not exceed n-1.

Proof

(1) ⟹ (2) suppose that ω-$\dim_f$(**X**) ≤ n. Let $\{\tilde{F}_i\}_{i=1}^{n+1}$ be fuzzy ω-closed sets and let $\{\tilde{U}_i\}_{i=1}^{n+1}$ be fuzzy ω-open sets, such that $\tilde{F}_i \subseteq \tilde{U}_i$ since ω-$\dim_f$(**X**) ≤ n the fuzzy ω-open covering of 1_X consisting of

fuzzy sets of the form $\bigcap_{i=1}^{n+1}\tilde{H}_i$ where $\tilde{H}_i = \tilde{U}_i$ or $\tilde{H}_i$ = 1_X - $\tilde{F}_i$ for each i has a finite fuzzy ω-open refinement $\{\tilde{W}_i\}_{i=1}^{t}$ of order not exceeding n since (X,T̃) a fuzzy ω-Normal space there exist an fuzzy ω-closed covering $\{\tilde{K}_i\}_{i=1}^{t}$ of 1_X such that $\tilde{K}_r \subseteq \tilde{W}_r$ for r = 1,……, t. We denote the set
N_r ={i : $\tilde{W}_r \cap \tilde{F}_i \neq \emptyset$} for r = 1,……, t. We can find fuzzy ω-open set
$\tilde{V}_{ir}$ for i in N_r such that $\tilde{K}_r \subseteq \tilde{V}_{ir} \subseteq$ ω-cl($\tilde{V}_{ir}$) ⊆ $\tilde{W}_r$ and ω-cl($\tilde{V}_{ir}$) ⊆ $\tilde{V}_{jr}$
if i ≤ j . Now, for each i = 1,……, n+1, let $\tilde{V}_i = \cup_r\{\tilde{V}_{ir}$: i ∈ N_r }. Then, $\tilde{V}_i$ is a fuzzy ω-open set and $\tilde{F}_i \subseteq \tilde{V}_i$. If $x_r \in \tilde{F}_i$ and $x_r \in \tilde{K}_r$, then i ∈ N_r, so that
$x_r \in \tilde{V}_{ir} \subseteq \tilde{V}_i$. Furthermore, if i ∈ N_r, so that $\tilde{W}_r \cap \tilde{F}_i \neq \emptyset$, then $\tilde{W}_r$ is not contained in 1_X - $\tilde{F}_i$, so that $\tilde{W}_r \subseteq \tilde{U}_i$ .Thus, if i ∈ N_r, then $\tilde{V}_{ir} \subseteq \tilde{U}_i$, so that since ω-cl($\tilde{V}_i$) = $\cup_r$\{ω-cl$(\tilde{V}_{ir})$: i ∈ N_r }, it follows that ω-cl($\tilde{V}_i$) ⊆ $\tilde{U}_i$. Finally, on the contrary, suppose that $\bigcap_{i=1}^{n+1}\omega-b(\tilde{V}_i)\neq\emptyset$, which implies that there is a fuzzy point x_r in $\bigcap_{i=1}^{n+1}\omega-b(\tilde{V}_i)$. But we have ω-b($\tilde{V}_i$) ⊆ $\cup_r${ ω-b($\tilde{V}_{ir}$) : i ∈ N_r }. Hence for each i = 1,……, n+1 there exists r_i ∈ {1,…,t} such that $x_r \in \omega-b(\tilde{V}_{ir_i})$. If i≠j, then $r_i \neq r_j$
for if $r_i = r_j$ = r then x_r ∈ ω-cl$(\tilde{V}_{ir})$ and x_r ∈ ω-cl$(\tilde{V}_{jr})$. But $x_r \notin \tilde{V}_{ir}$ and $x_r \notin \tilde{V}_{jr}$ which is irrational, since either ω-cl$(\tilde{V}_{ir}) \subseteq \tilde{V}_{jr}$ or ω-cl$(\tilde{V}_{jr}) \subseteq \tilde{V}_{ir}$ for each i $x_r \notin \tilde{V}_{ir_i}$, so that $x_r \notin \tilde{K}_{r_i}$. But $\tilde{K}_r$ is fuzzy ω-closed covering, so there exists r_0 that is different from each of the r_i , such that $x_r \in \tilde{K}_{r_0} \subseteq \tilde{W}_{r_0}$.
Since x_r ∈ ω-cl$(\tilde{V}_{ir_i})$, it follows that $x_r \in \tilde{W}_{r_i}$ for i = 1,……, n+1, so that $x_r \in \bigcap_{i=0}^{n+1}(\tilde{W}_{r_i})$.
Since the order of $\tilde{W}_r$ dose not exceed n, then this is irrational . Hence $\bigcap_{i=1}^{n+1}\omega-b(\tilde{V}_i)=\emptyset$

(2) ⟹ (3) let (2) holds, $\{\tilde{F}_i\}_{i=1}^{t}$ be a family of fuzzy ω-closed sets, and
$\{\tilde{U}_i\}_{i=1}^{t}$ be a family of fuzzy ω-open sets of 1_X , such that $\tilde{F}_i \subseteq \tilde{U}_i$ for each i
If t ≤n+1. Then there is nothing to prove. So, we suppose that t > n+1. Let the fuzzy subset of {1,2,…,t} containing n+1 elements be enumerated as $\tilde{C}_1$, $\tilde{C}_2$, ….. , $\tilde{C}_k$. Then by our hypothesis, for each i ∈ $\tilde{C}_1$ we can find fuzzy ω-open subsets $\tilde{V}_{i1}$ of 1_X , such that $\tilde{F}_i \subseteq \tilde{V}_{i1} \subseteq$ ω-cl$(\tilde{V}_{i1}) \subseteq \tilde{U}_i$ and $\bigcap_{i\in\tilde{C}_1}\omega-b(\tilde{V}_{i1})=\emptyset$. Also, we have that ω-b$(\tilde{V}_{i1})$ is fuzzy ω-closed subset of a fuzzy ω-Normal space (X,T̃), such that ω-b$(\tilde{V}_{i1}) \subseteq \tilde{U}_i$ for each i ∈ $\tilde{C}_1$. Therefore, for each i ∈ $\tilde{C}_1$ there exists a fuzzy ω-open subset $\tilde{G}_i$ of 1_X such that ω-b$(\tilde{V}_{i1}) \subseteq \tilde{G}_i \subseteq$ ω-cl$(\tilde{G}_i) \subseteq \tilde{U}_i$. Also, the fuzzy families $\{\omega-\text{cl}(\tilde{G}_i)\}_{i\in\tilde{C}_1}$ and $\{\omega-\text{b}(\tilde{V}_{i1})\}_{i\in\tilde{C}_1}$ are similar so that, in particular, $\bigcap_{i\in\tilde{C}_1}\omega-cl(\tilde{G}_i)=\emptyset$. Let $\tilde{W}_{i1} = \tilde{V}_{i1} \cup \tilde{G}_i$ if i ∈ $\tilde{C}_1$, then ω-cl$(\tilde{V}_{i1}) \subseteq \tilde{W}_{i1} \subseteq$ ω-cl$(\tilde{W}_{i1}) \subseteq \tilde{U}_i$. Since ω-cl$(\tilde{W}_{i1})$ - $\tilde{V}_{i1}$ ⊆ ω-cl$(\tilde{G}_i)$, then we have $\bigcap_{i\in\tilde{C}_1}\omega-cl(\tilde{W}_{i1})-\tilde{V}_{i1}=\emptyset$. If i ∉ $\tilde{C}_1$. Let $\tilde{V}_{i1}$ be fuzzy ω-open set such that $\tilde{F}_i \subseteq \tilde{V}_{i1} \subseteq$ ω-cl$(\tilde{V}_{i1}) \subseteq \tilde{U}_i$ and let $\tilde{W}_{i1} = \tilde{U}_i$. Then for i=1,…,t, we have fuzzy ω-open sets $\tilde{V}_{i1}$ and $\tilde{W}_{i1}$ such that $\tilde{F}_i \subseteq \tilde{V}_{i1} \subseteq$ ω-cl$(\tilde{V}_{i1}) \subseteq \tilde{W}_{i1} \subseteq \tilde{U}_i$ and $\bigcap_{i=1}^{t}\omega-cl(\tilde{W}_{i1})-\tilde{V}_{i1}=\emptyset$. Suppose that, for each m∈{2,3,….,k}, we can find, using the same technique above, the fuzzy ω-open sets $\tilde{V}_{i(m-1)}$ and $\tilde{W}_{i(m-1)}$, such that $\tilde{F}_i \subseteq \tilde{V}_{i(m-1)} \subseteq$ ω-cl$(\tilde{V}_{i(m-1)}) \subseteq \tilde{W}_{i(m-1)} \subseteq \tilde{U}_i$ and $\bigcap_{i\in\tilde{C}j}\omega-cl(\tilde{W}_{i(m-1)})-\tilde{V}_{i(m-1)}=\emptyset$,

for each j∈{1,2,…..,m-1}. Also, by the above argument, we can find fuzzy ω-open sets $\tilde{V}_{im}$ and $\tilde{W}_{im}$, such that
ω-cl$(\tilde{V}_{i(m-1)}) \subseteq \tilde{V}_{im} \subseteq$ ω-cl$(\tilde{V}_{im}) \subseteq \tilde{W}_{im} \subseteq \tilde{W}_{i(m-1)}$ and $\bigcap_{i\in\tilde{C}j}\omega-cl(\tilde{W}_{im})-\tilde{V}_{im}=\emptyset$, for each j∈{1,2,…..,m}. Thus, by induction, for each i∈{1,2,…..,t}, we can find fuzzy ω-open sets $\tilde{V}_i$ and $\tilde{W}_i$

($\tilde{V}_{ik}$ and $\widetilde{W}_{ik}$, respectively), such that $\tilde{F}_i \subseteq \tilde{V}_i \subseteq$ ω-cl$(\tilde{V}_i) \subseteq \widetilde{W}_i \subseteq \widetilde{U}_i$ and $\bigcap_{i \in \tilde{C}j} \omega - cl\,(\tilde{W}_i) - \tilde{V}_i = \emptyset$ for each j∈{1,2,…..,k}. Hence, the order of $\{\omega - \text{cl}(\widetilde{W}_i) - \tilde{V}_i\}_{i=1}^{t}$ does not exceed n-1.

(3) ⟹ (4) Obvious, since ω-b$(\tilde{V}_i)$ = ω-cl$(\tilde{V}_i)$ - $\tilde{V}_i \subseteq$ ω-cl$(\widetilde{W}_i)$ - $\tilde{V}_i$.

(4) ⟹ (1) Let $\{\widetilde{U}_i\}_{i=1}^{n+2}$ be a fuzzy ω-open covering of fuzzy ω-Normal space (X,T̃). Then, there exists fuzzy ω-closed covering $\{\tilde{F}_i\}_{i=1}^{n+2}$ of 1_X such that $\tilde{F}_i \subseteq \widetilde{U}_i$ for each i . So, by hypothesis, there exists a fuzzy family $\{\tilde{V}_i\}_{i=1}^{n+2}$ of fuzzy ω-open sets in 1_X , such that $\tilde{F}_i \subseteq \tilde{V}_i \subseteq$ ω-cl$(\tilde{V}_i) \subseteq \widetilde{U}_i$ for each i. Also, the order of $\{\omega\text{-b}(\tilde{V}_i)\}_{i=1}^{n+2}$ does not exceed n-1 and the fuzzy family $\{\tilde{V}_i\}_{i=1}^{n+2}$ is fuzzy ω-open covering of1_X. Let $\tilde{L}_j$ = ω-cl$(\tilde{V}_j)$ - $\bigcup_{i<j} \tilde{V}_i$ for j=1,2,…..,n+2. For each j , $\tilde{L}_j$ is fuzzy ω-closed and $\{\tilde{L}_j\}_{j=1}^{n+2}$ is a fuzzy ω-closed covering of 1_X, for if $x_r \in 1_X$, there exists j such that $x_r \in \tilde{V}_j$ and $x_r \notin \tilde{V}_i$ for i < j , so that $x_r \in \tilde{L}_j$.

Now, $\tilde{L}_j$ = ω-cl$(\tilde{V}_j) \cap \bigcap_{i<j} 1_X - \tilde{V}_i$ so that $\bigcap_{j=1}^{n+2} \tilde{L}_j = \bigcap_{j=1}^{n+2} \omega - cl\,(\tilde{V}_j) \cap \bigcap_{i=1}^{n+1} 1_X - \tilde{V}_i \subseteq \bigcap_{j=1}^{n+1} \omega - b\,(\tilde{V}_j) = \emptyset$

. Thus, $\{\tilde{L}_j\}_{j=1}^{n+2}$ is fuzzy ω-closed covering of 1_X, $\tilde{L}_j \subseteq$ ω-cl$(\tilde{V}_j) \subseteq \widetilde{U}_j$ and $\bigcap_{j=1}^{n+2} \tilde{L}_j = \emptyset$. Hence, by ***Theorem 4.11***, we have ω-dim$_f$(**X**) ≤ n

Definition 4.13

Let $\tilde{A}$ and $\tilde{B}$ be any two fuzzy disjoint sets in a fuzzy spac1_X . A fuzzy subset $\tilde{L}$ is called an fuzzy ω-partition $\tilde{A}$ and $\tilde{B}$ if there exist two fuzzy disjoint ω-open sets $\widetilde{U}$ and $\widetilde{W}$ such that $\tilde{A} \subseteq \widetilde{U}$, $\tilde{B} \subseteq \widetilde{W}$ and 1_X - $\tilde{L} = \widetilde{U} \cup \widetilde{W}$.

Theorem 4.14

Let (X,T̃) be a fuzzy topological sum of the fuzzy family of spaces $\{1_{X_\lambda}\}_{\lambda\in\Lambda}$. If ω-dim$_f(X_\lambda)$ ≤ n for each $\lambda \in \Lambda$, then ω-dim$_f$(**X**) ≤ n.

Proof

Let ω-dim$_f(X_\lambda)$ ≤ n for each $\lambda \in \Lambda$ and let $\{\widetilde{U}_i\}_{i=1}^{t}$ be fuzzy ω-open covering of a fuzzy topological sum (X,T̃), then $\{\widetilde{U}_i \cap X_\lambda\}_{i=1}^{t}$ is fuzzy ω-open covering $\{1_{X_\lambda}\}_{\lambda\in\Lambda}$. Therefore, by ***Theorem 4.10*** , for each $\lambda \in \Lambda$ there exists a fuzzy ω-open covering $\{\tilde{V}_{i\lambda}\}_{i=1}^{t}$ of X_λ of order not exceeding n , such that $\tilde{V}_{i\lambda} \subseteq \widetilde{U}_i \cap X_\lambda$ for each i . Then, the fuzzy family $\{\widetilde{W}_i\}_{i=1}^{t}$ is fuzzy ω-open covering of 1_X , such that $\widetilde{W}_i \subseteq \widetilde{U}_i$ for each i where $\widetilde{W}_i = \bigcup_{\lambda\in\Lambda} \tilde{V}_{i\lambda}$. If the order of $\{\widetilde{W}_i\}_{i=1}^{t}$ is exceeding n, this means that there exists a subset ν of {1,2,…,t} of at least n+2 elements, such that $\emptyset = \bigcap_{i\in\nu} \widetilde{W}_i = \bigcap_{i\in\nu} \bigcup_{\lambda\in\Lambda} \tilde{V}_{i\lambda} = \bigcup_{\lambda\in\Lambda} \bigcap_{i\in\nu} \tilde{V}_{i\lambda}$. So, $\bigcap_{i\in\nu} \tilde{V}_{i\lambda} = \emptyset$ for some $\lambda \in \Lambda$. This contradicts the fact that the order of $\{\tilde{V}_{i\lambda}\}_{i=1}^{t}$ does not exceed n for each $\lambda \in \Lambda$. Thus, the order of $\{\widetilde{W}_i\}_{i=1}^{t}$ cannot exceed n, hence ω-dim$_f$(**X**) ≤ n

Theorem 4.15

If (X,T̃) is a fuzzy ω-Normal space with the property that for each fuzzy ω-closed set $\tilde{F}$ and each fuzzy ω-open set $\widetilde{U}$, such that $\tilde{F} \subseteq \widetilde{U}$, there exists a fuzzy ω-open set $\tilde{V}$ in 1_X such that $\tilde{F} \subseteq \tilde{V} \subseteq \widetilde{U}$ and ω-dim$_f$(ω-b($\tilde{V}$)) ≤ n, then ω-dim$_f$(**X**) ≤ n+1.

Proof

Let $\{\widetilde{U}_i\}_{i=1}^{t}$ be fuzzy ω-open covering of 1_X . Since (X,T̃) is a fuzzy ω-Normal space, so there exists a fuzzy ω-closed covering $\{\tilde{F}_i\}_{i=1}^{t}$ of 1_X such that $\tilde{F}_i \subseteq \widetilde{U}_i$ for each i. Then, by hypothesis, there exists fuzzy ω-open sets $\tilde{V}_i$ such that $\tilde{F}_i \subseteq \tilde{V}_i \subseteq \widetilde{U}_i$ and ω-dim$_f$(ω-b($\tilde{V}_i$)) ≤ n for each i . Again, since (X,T̃) is a fuzzy ω-Normal space, so, by ***Theorem 3.12*** and ***Theorem 3.22,*** the fuzzy set $\tilde{E} = \bigcup_{i=1}^{n+3} \omega - b\,(\tilde{V}_i)$ is fuzzy ω-Normal subspace of 1_X . By ***Theorem 4.1*** , we have ω-dim$_f(\tilde{E})$ ≤ n and $\{\tilde{E} \cap \widetilde{U}_i\}_{i=1}^{n+3}$ is fuzzy ω-open covering of $\tilde{E}$. Therefore, there exists a fuzzy ω-closed covering $\{\widetilde{H}_i\}_{i=1}^{n+3}$ of $\tilde{E}$ such that $\widetilde{H}_i \subseteq \tilde{E} \cap \widetilde{U}_i$ for each I. Then, $\widetilde{H}_i$ is fuzzy ω-closed subset of 1_X for each I, then there exists a fuzzy family $\{\widetilde{W}_i\}_{i=1}^{n+3}$ of fuzzy ω-open sets in 1_X such that $\widetilde{H}_i \subseteq \widetilde{W}_i \subseteq \widetilde{U}_i$ for each i and the fuzzy families $\{\omega - cl(\tilde{W}_i)\}_{i=1}^{n+3}$ and $\{\tilde{H}_i\}_{i=1}^{n+3}$ are similar. Also we can obtain a fuzzy family $\{\tilde{G}_i\}_{i=1}^{n+3}$ of pairwise disjoint fuzzy ω-open sets of 1_X where $\tilde{G}_i = \tilde{V}_i - \bigcup_{j<i} \omega - cl(\tilde{V}_j)$ and it is clear that $\tilde{G}_i \subseteq \tilde{U}_i$ for each i. To show that $1_X - \tilde{E} \subseteq \bigcup_{i=1}^{n+3} \tilde{G}_i$, let $x_r \in 1_X - \tilde{E}$, then $x_r \notin$ ω-b($\tilde{V}_i$) for each i. Also, since $\{\tilde{V}_i\}_{i=1}^{n+3}$ is a fuzzy covering of 1_X, then $x_r \in \tilde{V}_i$ for some i ∈ {1,2,…,n+2,n+3}. Suppose that k is the first integer in which $x_r \in \tilde{V}_k$, thus for each m<k we have $x_r \notin \tilde{V}_m$ and $x_r \notin$ ω-b($\tilde{V}_m$). Hence, $x_r \notin$ ω-cl($\tilde{V}_m$). Thus, $x_r \in \tilde{G}_k \subseteq \bigcup_{i=1}^{n+3} \tilde{G}_i$, hence $1_X - \tilde{E} \subseteq \bigcup_{i=1}^{n+3} \tilde{G}_i$. Let $\tilde{O}_i = \tilde{G}_i \cup \tilde{W}_i \subseteq \tilde{U}_i$ for each i . Clearly, $\{\tilde{O}_i\}_{i=1}^{n+3}$ is a fuzzy ω-open covering of 1_X. Suppose that $x_r \in \bigcap_{i=1}^{n+3} \tilde{O}_i$, then $x_r \in \tilde{G}_i$ or $x_r \in \tilde{W}_i$ for each i. Since $\{\tilde{G}_i\}_{i=1}^{n+3}$ is a pairwise disjoint fuzzy family of fuzzy ω-open sets, then x_r cannot be contained in more than one of them. Therefore, there exists at most one k ∈ {1,2,…,n+2,n+3} such that $x_r \in \tilde{G}_k$, so that $x_r \notin \tilde{G}_i$ for each i≠k. Hence, $x_r \in \tilde{W}_i$ for each i ∈ {1,2,…,n+2,n+3}, except at most for i=k. Thus, x_r is contained in at least (n+2) members of $\{\tilde{W}_i\}_{i=1}^{n+3}$. This implies that the order of $\{\tilde{W}_i\}_{i=1}^{n+3}$ exceeds n, which is a contradiction. Hence $\bigcap_{i=1}^{n+3} \tilde{O}_i = \varnothing$. Therefore, by ***Theorem 4. 1***, we have ω-$\dim_f$(**X**) ≤ n+1

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