Some Results on Fuzzy $\omega$-Covering Dimension Function in Fuzzy Topological Space

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Abstract
The purpose of this paper is to study a new class of fuzzy covering dimension functions, called fuzzy $\omega$-covering dimension function, on fuzzy topological space and investigate its relationships with fuzzy separation axioms and fuzzy $\omega$-connected space. We also provide some characterization on $\omega$-covering dimension function by using fuzzy countable sets, for which we obtained several properties.

Keywords: $\omega$-dim$(X)$, fuzzy $\omega$-open set, fuzzy locally countable set, fuzzy $\omega$-disconnected space

I. Introduction
The concept considered in this paper is the so called “fuzzy sets”, which is totally different from the classical concept called “crisp sets”. The recent concept was introduced by Zadeh in 1965 [1], in which he defines fuzzy sets as a class of objects with a continuum of grades of membership. Such a set is characterized by a membership function that assigns to each object a grade of membership ranging between zero and one. In 1968, Chang [2] introduced the definition of fuzzy topological spaces and extended, in a straightforward manner, some concepts of crisp topological spaces to fuzzy topological spaces. Later, Lowen (1976) redefined what is now known as stratified fuzzy topology [3]. While Wong (1974) discussed and generalized some properties of fuzzy topological spaces [4]. The notion of fuzzy covering dimension of a fuzzy topological space $(X, \tilde{T})$ was introduced by Zougdani [5] in 1984. He used the notion of a shading family, which was first introduced by Gantner. et al. [6] in 1978, during their investigation of compactness in fuzzy topological spaces. Zougdani introduced that notion of $(X, \tilde{T})$ in the Chang’s sense. Jackson [7] investigated Zougdani’s paper and provided some counter-examples of which invalidated his results. Ajmal and Kohli (1994) introduced a notion of fuzzy covering dimension of $(X, \tilde{T})$, which is an extension of the concept of the covering dimension in classical topological spaces [8]. A distinctive feature of their study is that they deviated from the classical topological approach by employing the notion of quasi-coincidence, which was first
introduced by Ming and Ming [9] in 1980, in defining the order of a family of fuzzy sets, which is instrumental in the definition of the dimension function. Ajmal and Kohli [8] restricted some results of their study to the Lowen’s definition of (X, T). In this paper, we introduce the concepts of fuzzy ω-open set and fuzzy ω-covering dimension functions on fuzzy topological space and provide some characterization. Moreover, the study examines the relationships between these functions and fuzzy separation axioms and fuzzy ω-connected space. The current investigation could obtain several properties

II. Preliminaries

Definition 2.1 [1]
Let X be a non empty set and let I be the unit interval, i.e. I= [0, 1]. A fuzzy set in X is a function from X into the unit interval I, A : X → [0,1] that is a function. A fuzzy set A in X and can be represented by the following set of pairs:
A = { (x, μA(x)) : x ∈ X }; the family of all fuzzy sets in X is denoted by I.X.

Definition 2.2 [10]
A fuzzy point x_r is a fuzzy set such that:
μ_x_r(y) = r > 0 if x = y , ∀ y ∈ X and
μ_x_r(y) = 0 if x ≠ y , ∀ y ∈ X
The family of all fuzzy points of A will be denoted by FP(A).

Definition 2.4 [11]
A fuzzy point x_r is said to belong to a fuzzy set A in X (denoted by : x_r ∈ A) if and only if μ_x_r(x) = r ≤ μ_A(x).

Proposition 2.5 [11]
Let A and B be two fuzzy sets in X with membership functions μ_A and μ_B, respectively. Then, for all x ∈ X:
1. A ⊆ B if and only if μ_A(x) ≤ μ_B(x).
2. A = B if and only if μ_A(x) = μ_B(x).
3. C = A ∩ B if and only if μ_C(x) = min{μ_A(x), μ_B(x)}.
4. D = A ∪ B if and only if μ_D(x) = max{μ_A(x), μ_B(x)}.

Definition 2.6 [12]
The support of a fuzzy set A, Supp (A), is the crisp set of all x ∈ X, such that μ_A(x) > 0.

Definition 2.7 [2]
A fuzzy topology is a family T of fuzzy subsets in X, satisfying the following conditions:
(a) ∅, 1x ∈ T.
(b) If A, B ∈ T, then A ∩ B ∈ T.
(c) If A_i ∈ T, ∀ i ∈ J, where J is any index set, then U_{i ∈ J} A_i ∈ T.
T is called fuzzy topology for X, and the pair (X, T) is a fuzzy topological space. Every member of T is called open fuzzy set (T-open fuzzy set). A fuzzy set C in 1x is called closed fuzzy set (T-closed fuzzy set) if and only if its complement C_c is T-open fuzzy set.

Definition 2.8 [13]
If B ∈ (X,T), the complement of B referred to 1x denoted by B_c, is defined by B_c = 1x - B.

Definitions 2.9 [14, 9]
Let B, C be a fuzzy set in a fuzzy topological space (X, T), then:
• A fuzzy point x_r is said to be quasi coincident with the fuzzy set B if there exists x ∈ X such that μ_x_r(x) + μ_B(x) > μ_A(x), and denoted by x_r q B . If μ_x_r(x) + μ_B(x) ≤ μ_A(x) ∀ x ∈ X , then x_r is not quasi coincident with a fuzzy set B, and is denoted by x_r q B.
• A fuzzy set B is said to be quasi coincident (overlap) with a fuzzy set C if there exists x ∈ X such that μ_B(x) + μ_C(x) > μ_A(x), and denoted by B q C . If μ_B(x) + μ_C(x) ≤ μ_A(x) ∀ x ∈ X, then B is not quasi coincident with a fuzzy set C, and is denoted by B q C.

Definition 2.10 [11]
A fuzzy set B in a fuzzy topological space (A, T) is said to be a fuzzy neighborhood of a fuzzy point x_r in A if there is a fuzzy open set G in A such that μ_x_r(x) ≤ μ_G(x) ≤ μ_B(x), ∀ x ∈ X.
Definition 2.11 [15]
Let \((X, \tilde{T})\) be a fuzzy topological space and \(\tilde{B} \in \mathcal{P}(1_X)\), then the relative fuzzy topology for \(\tilde{B}\) is defined by \(\tilde{T}_B = \{ \tilde{B} \cap \tilde{G} : \tilde{G} \in \tilde{T} \}\).

The corresponding \((\tilde{B}, \tilde{T}_B)\) is called fuzzy subspace of \((X, \tilde{T})\).

Definition 2.12 [8]
Let \(X\) be a nonempty set. A family \(U = \{\tilde{U}_j\}_{\lambda \in \Lambda}\) of fuzzy sets in \(1_X\) is said to be of order \(n\) \((n > -1)\), written as \(\text{ord}_f U = n\), if \(n\) is the greatest integer, such that there exists an overlapping subfamily of \(U\) having \(n + 1\) elements.

Remark 2.13
From the above definition, if \(\text{ord}_f U = n\), then for each \(n + 2\) distinct indexes \(\lambda_1, \lambda_2, \ldots, \lambda_{n+2} \in \Lambda\), we have \(U_{\lambda_1} \cap U_{\lambda_2} \cap \ldots \cap U_{\lambda_{n+2}} = \emptyset\). Then, it is non-overlapping, in particular if \(\text{ord}_f U = -1\), then \(U\) consists of the empty fuzzy sets and \(\text{ord}_f U = 0\), then \(U\) consist of pairwise disjoint fuzzy sets which are not all empty.

Definition 2.14 [4]
Let \(B = \{\tilde{B}_\alpha : \alpha \in \Lambda\}\) and \(C = \{\tilde{C}_\beta : \beta \in \Lambda\}\) \((\beta < \alpha)\) be any two collections of fuzzy sets in \((X, \tilde{T})\), then \(C\) is a refinement of \(B\) if for each \(\beta \in \Lambda\) there exists \(\alpha \in \Lambda\) such that \(\mu_{\tilde{C}_\beta}(x) \leq \mu_{\tilde{B}_\alpha}(x)\).

Definition 2.15 [16]
A fuzzy topological space \((X, \tilde{T})\) is said to be fuzzy connected, if it has no proper fuzzy clopen set. Otherwise, it is called fuzzy disconnected.

III. Properties Of Fuzzy \(\omega\)-Open Set In Fuzzy Topological Space

Definition 3.1 [17]
a fuzzy set \(\tilde{A}\) in a fuzzy topological space \((X, \tilde{T})\) is called a fuzzy uncountable if and only if \(\text{supp}(\tilde{A})\) is an uncountable subset of \(X\)

Definition 3.2
A fuzzy point \(x_\tau\) of a fuzzy topological space \((X, \tilde{T})\) is called a fuzzy condensation point of \(\tilde{A} \subseteq 1_X\) if \(\tilde{B} \cap \tilde{A}\) is fuzzy uncountable for each fuzzy open set \(\tilde{B}\) containing \(x_\tau\). And the set of all fuzzy condensation points of \(\tilde{A}\) is denoted by \(\text{Cond}(\tilde{A})\).

Definition 3.3
A fuzzy subset \(\tilde{A}\) in a fuzzy topological space \((X, \tilde{T})\) is called a fuzzy \(\omega\)-closed set if it contains all its fuzzy condensation points. The complement fuzzy \(\omega\)-closed sets are called fuzzy \(\omega\)-open sets.

Theorem 3.4
A fuzzy subset \(\tilde{G}\) of a fuzzy topological space \((X, \tilde{T})\) is fuzzy \(\omega\)-open set if and only if for any \(x_\tau \in \tilde{G}\) there exists a fuzzy open set \(\tilde{U}\) such that \(x_\tau \in \tilde{U}\) and \(\tilde{U} \cap \tilde{G}\) is countable.

Proof
\(\tilde{G}\) is fuzzy \(\omega\)-open set if and only if \(1_{X - \tilde{G}}\) is fuzzy \(\omega\)-closed set, 

\(1_{X - \tilde{G}}\) is fuzzy \(\omega\)-closed set if and only if \(\text{Cond}(1_{X - \tilde{G}}) \subseteq 1_{X - \tilde{G}}\,

and \(\text{Cond}(1_{X - \tilde{G}}) \subseteq 1_{X - \tilde{G}}\) if and only if for each \(x_\tau \in \tilde{G}\), \(x_\tau \notin \text{Cond}(1_{X - \tilde{G}})\).

Thus, for \(x_\tau \notin \text{Cond}(1_{X - \tilde{G}})\) there exists a fuzzy open set \(\tilde{U}\) such that \(x_\tau \in \tilde{U}\) and \(\tilde{U} \cap (1_{X - \tilde{G}}) = \tilde{U} \cap \tilde{G}\) is countable.

Theorem 3.5
A fuzzy subset \(\tilde{G}\) of a fuzzy topological space \((X, \tilde{T})\) is \(\omega\)-open set if and only if for each \(x_\tau \in \tilde{G}\) there exist an fuzzy open set \(\tilde{U}\) containing \(x_\tau\) and countable fuzzy subset \(\tilde{C}\) of \(1_X\) such that \(\tilde{U} - \tilde{C} \subseteq \tilde{G}\).

Proof
\((\Rightarrow)\) suppose that \(\tilde{G}\) is fuzzy \(\omega\)-open set and let \(x_\tau \in \tilde{G}\).

Then, there exist a fuzzy open set \(\tilde{U}\) and \(x_\tau \in \tilde{U}\) and \(\tilde{U} \cap \tilde{G}\) is countable.

Set \(\tilde{C} = \tilde{U} - \tilde{G}\), then \(\tilde{C}\) is countable and \(x_\tau \in \tilde{U} - \tilde{C}\).

\((\Leftarrow)\) let \(x_\tau \in \tilde{G}\) then, by assumption, there exist fuzzy open set \(\tilde{U}\) containing \(x_\tau\) and countable fuzzy subset \(\tilde{C}\) of \(1_X\) such that \(\tilde{U} - \tilde{C} \subseteq \tilde{G}\).

Since \(\tilde{U} - \tilde{C} \subseteq \tilde{G}\), then \(\tilde{U} - \tilde{G}\) is countable, hence \(\tilde{G}\) is fuzzy \(\omega\)-open set.

Proposition 3.6
In a fuzzy topological space, every fuzzy open set is fuzzy \(\omega\)-open set.
Proof
Let \( G \) be fuzzy open set and \( x_r \in G \). Set \( U = G \), \( C = \emptyset \), then \( U \) is fuzzy open set and \( C \) is a countable set, such that \( x_r \in U \). \( C \subseteq G \). Thus, \( G \) is fuzzy \( \omega \)-open set.

Remark 3.7
The converse of proposition (3.6) is not true in general, as the following examples show:

Examples 3.8
Let \( X = \{ a, b, c \} \) and \( A, B \) are fuzzy subsets in \( 1_X \) where
\[
1_X = \{ (a, 1), (b, 1), (c, 1) \}
\]
\( A = \{ (a, 0.6), (b, 0.6), (c, 0.7) \} \)
\( B = \{ (a, 0.5), (b, 0.5), (c, 0.4) \} \)

Let \( T = \{ \emptyset, 1_X, A \} \) be a fuzzy topology on \( X \).

Then the fuzzy set \( B \) is a fuzzy \( \omega \)– open set but not a fuzzy open set.

Definition 3.9
Let \( B \) be a fuzzy set in a fuzzy topological space \( (X, T) \), then the fuzzy \( \omega \)-interior of \( B \), which is denoted by
\( \omega \)-Int(\( B \)) and defined by \( \omega \)-Int(\( B \)) = \( \bigcup \{ G : G \) is a fuzzy \( \omega \)-open set in \( 1_X \), \( G \subseteq B \} \).

Definition 3.10
Let \( B \) be a fuzzy set in a fuzzy topological space \( (X, T) \), then the fuzzy \( \omega \)-closure of \( B \), which is denoted by
\( \omega \)-cl(\( B \)) and defined by \( \omega \)-cl(\( B \)) = \( \bigcap \{ G : G \) is a fuzzy \( \omega \)-closed set in \( 1_X \), \( B \subseteq G \} \).

Definition 3.11
Let \( B \) be a fuzzy set in a fuzzy topological space \( (X, T) \), then the fuzzy \( \omega \)-boundary of \( B \) is denoted by \( \omega \)-b(\( B \)) and defined by \( \omega \)-b(\( B \)) = \( \omega \)-cl(\( B \)) - \( \omega \)-Int(\( B \))

Theorem 3.12
Let \( A \) be fuzzy subset of a fuzzy topological space \( (X, T) \).
Then, \( (\overline{T_A})^\omega = \overline{T_A}^\omega \)

Proof
To prove that \( (\overline{T_A})^\omega \subseteq \overline{T_A}^\omega \), let \( B \in (\overline{T_A})^\omega \) and \( x_r \in B \), by Theorem 3.5. There exist fuzzy open set \( V \) of \( T_A \) and \( C \) countable subset of \( V \), such that \( x_r \in V \). We choose \( U \subseteq V \) such that \( \overline{U} \cap C = \emptyset \). Then \( \overline{U} \cap C \subseteq B \).

Therefore, \( B \in (\overline{T_A})^\omega \). To prove that \( (\overline{T_A})^\omega \subseteq (\overline{T_A})^\omega \), let \( G \in (\overline{T_A})^\omega \), then there exists \( H \subseteq T^\omega \) such that \( G \subseteq H \). If \( x_r \in H \), then \( x_r \in H \) and there exist fuzzy open set \( U \) of \( H \) and \( D \) countable subset of \( H \) such that \( x_r \in U \). We put \( U \cap A \cap D \subseteq H \). It follows that \( G \in (\overline{T_A})^\omega \)

Definition 3.13
The fuzzy family \( \{ B_\alpha : \alpha \in A \} \) of the subset of a fuzzy topological space \( (X, T) \) is called
1- Fuzzy \( \omega \)-locally finite, if for each \( x_r \in 1_X \) there exists a fuzzy \( \omega \)-open set \( G \) containing \( x_r \) such that the set \( \{ G \cap B_\alpha = \emptyset : \alpha \in A \} \) is finite.
2- Fuzzy \( \omega \)-discrete, if for each \( x_r \in 1_X \) there exists a fuzzy \( \omega \)-open set \( G \) containing \( x_r \) such that the set \( \{ G \cap B_\alpha = \emptyset : \alpha \in A \} \) has at most one member.

proposition 3.14
Every fuzzy locally finite (resp.fuzzy discrete) family of any fuzzy topological space \( (X, T) \) is fuzzy \( \omega \)-locally finite (resp.fuzzy \( \omega \)-discrete).

Proof : It follows from the fact that every fuzzy open set is fuzzy \( \omega \)-open set.

Definition 3.15
A fuzzy topological space \( (X, T) \) is called a fuzzy anti-locally-countable if each nonempty fuzzy open subset of \( 1_X \) is uncountable.

Definition 3.16
A fuzzy topological space \( (X, T) \) is said to be \( \omega \)-\( T_0 \) if for each pair of distinct fuzzy points \( x_r \) and \( y_t \) of \( 1_X \) there exists a fuzzy \( \omega \)-open set \( G \), such that either \( x_r \in G \) and \( y_t \in G \) or \( y_t \in G \) and \( x_r \in G \).
Definition 3.17
A fuzzy topological space \((X, \tilde{T})\) is said to be \(\omega\)-\(\tilde{T}_1\) if for each pair of distinct fuzzy points \(x_r\) and \(y_t\) of \(1_X\) there exist fuzzy open sets \(G\) and \(H\), such that both \(x_r \in G\) and \(y_t \notin G\) as well as \(y_t \in H\) and \(x_r \notin H\) hold.

Definition 3.18
A fuzzy topological space \((X, \tilde{T})\) is said to be \(\omega\)-\(T_2\) if for each pair of distinct fuzzy points \(x_r\) and \(y_t\) of \(1_X\) there exist disjoint fuzzy open sets \(G\) and \(H\) containing \(x_r\) and \(y_t\), respectively.

Definition 3.19
A fuzzy topological space \((X, \tilde{T})\) is called a fuzzy \(\omega\)-regular space if for each fuzzy \(\omega\)-closed subset \(B\) of \(1_X\), and a fuzzy point \(x_r\) in \(1_X\) such that \(x_r \notin B\), there exist disjoint fuzzy open sets \(\tilde{U}\) and \(\tilde{V}\) containing \(x_r\) and \(B\), respectively.

Definition 3.20
A fuzzy topological space \((X, \tilde{T})\) is called a fuzzy \(\omega\)-Normal space if, for each pair of disjoint fuzzy \(\omega\)-closed sets \(A\) and \(B\) in \(1_X\), there exist disjoint fuzzy open sets \(\tilde{U}\) and \(\tilde{V}\) containing \(A\) and \(B\), respectively.

Theorem 3.21
A fuzzy topological space \((X, \tilde{T})\) is a fuzzy \(\omega\)-Normal if for each pair of fuzzy \(\omega\)-open sets \(G\) and \(H\) in \(1_X\), such that \(1_X = \tilde{G} \cup \tilde{H}\), there are fuzzy \(\omega\)-closed sets \(\tilde{U}\) and \(\tilde{V}\) contained in \(\tilde{G}\) and \(\tilde{H}\), respectively, such that \(1_X = \tilde{U} \cup \tilde{V}\).

Proof: Obvious.

Theorem 3.22
Every fuzzy \(\omega\)-closed subspace of fuzzy \(\omega\)-Normal space is fuzzy \(\omega\)-Normal space.

Proof: Obvious.

Definition 3.23
Two fuzzy families \(\{\tilde{A}_\lambda\}_{\lambda \in \Lambda}\) and \(\{\tilde{B}_\lambda\}_{\lambda \in \Lambda}\) of subset of a fuzzy space \(1_X\) are said to be similar if, for every finite subset \(\Delta\) of \(\Lambda\), the fuzzy sets \(\bigcap\limits_{\lambda \in \Delta} \tilde{A}_\lambda\) and \(\bigcap\limits_{\lambda \in \Delta} \tilde{B}_\lambda\) are either empty or nonempty.

Definition 3.24
A fuzzy \(\omega\)-open covering \(\{\tilde{U}_\lambda\}_{\lambda \in \Lambda}\) of a fuzzy topological space \((X, \tilde{T})\) is said to be fuzzy \(\omega\)-shrinkable if there exists a fuzzy \(\omega\)-open covering \(\{\tilde{V}_\lambda\}_{\lambda \in \Lambda}\) of \(1_X\) such that \(\text{cl}(\tilde{V}_\lambda) \subseteq \tilde{U}_\lambda\) for each \(\lambda\).

IV. Properties Of Fuzzy \(\omega\)-Covering Dimension Function

Definition 4.1
The fuzzy \(\omega\)-covering dimension function of a fuzzy topological space \((X, \tilde{T})\), denoted \(\omega\text{-dim}(X)\), is the least integer \(n\), such that every finite \(\omega\)-open cover of \(1_X\) has a finite \(\omega\)-open refinement of order not exceeding \(n\) or + \(\infty\), if there exists no such integer.

Thus, it follows that \(\omega\text{-dim}(X) = -1\) if and only if \(X = \emptyset\), and \(\omega\text{-dim}(X) \leq n\) if every finite \(\omega\)-open cover of \(1_X\) has a finite \(\omega\)-open refinement of order \(\leq n\). We have \(\omega\text{-dim}(X) = n\) if it is true that \(\omega\text{-dim}(X) \leq n\), but it is false that \(\omega\text{-dim}(X) \leq n - 1\). Finally, \(\omega\text{-dim}(X) = + \infty\) if, for every positive integer \(n\), it is false that \(\omega\text{-dim}(X) \leq n\).

Propositions 4.2
Let \((\tilde{B}, \tilde{T}_B)\) be a closed fuzzy subspace of a fuzzy topological space \((X, \tilde{T})\), then \(\omega\text{-dim}(B) \leq \omega\text{-dim}(X)\).

Proof
If \(\omega\text{-dim}(X) = + \infty\) or \(\omega\text{-dim}(X) = -1\), there is nothing to prove. So it is sufficient when we show that if \(\omega\text{-dim}(X) = n\) then \(\omega\text{-dim}(B) \leq n\). For this, let \(\{\tilde{U}_\lambda\}_{\lambda = 1}^t\) be fuzzy finite covering of \(B\) by fuzzy \(\omega\)-open sets of \(B\). Then by Theorem 3.12, there exist fuzzy \(\omega\)-open sets \(\tilde{V}_i\) in \(1_X\) such that \(\bigcup\limits_{i = 1}^n \tilde{V}_i = \tilde{V}_i \cap \tilde{B}\) for each \(i = 1, 2, \ldots, t\). Hence, \(\{\tilde{V}_i\}_{i = 1}^t \cup \{1_X - \tilde{B}\}\) is fuzzy finite \(\omega\)-open covering of \(1_X\). Since \(\omega\text{-dim}(X) = n\), then there exists fuzzy \(\omega\)-open refinement \(\{\tilde{G}_\lambda\}_{\lambda \in \Lambda}\) of \(\{\tilde{V}_i\}_{i = 1}^t \cup \{1_X - \tilde{B}\}\) of order not
Therefore there exists a fuzzy family \( \{ \tilde{A}_\lambda \}_{\lambda \in \Lambda} \) of order not exceeding \( n \) such that \( \dim(X) = 0 \) and \( \tilde{A}_\lambda \subset \tilde{A}_\lambda \). Let \( \{ \tilde{A}_\lambda \}_{\lambda \in \Lambda} \) be any fuzzy finite open covering of \( X \). So, by putting \( \tilde{A} = \bigcup_{\lambda \in \Lambda} \tilde{A}_\lambda \) we have \( \tilde{A} \subseteq \tilde{A} \) and \( \tilde{A} \subseteq \tilde{A} \). Also, let \( \tilde{A} \subseteq \tilde{A} \) be any fuzzy finite open covering of \( X \). Then, \( \tilde{A} \subseteq \tilde{A} \subseteq \tilde{A} \subseteq \tilde{A} \). Again, by Normality of \( X \) and there exists a fuzzy family \( \{ \tilde{A}_i \}_{i=1}^t \) of fuzzy open subsets of \( X \) such that \( \dim(X) = 0 \) and \( \tilde{A}_i \subseteq \tilde{A}_i \). Finally, the fuzzy families \( \{ \tilde{A}_i \}_{i=1}^t \) are similar. Since \( \dim(X) = 0 \) and there exists a fuzzy \( \tilde{A}_i \subseteq \tilde{A}_i \) of order not exceeding \( n \). Let \( \tilde{A}_i = \bigcap_{j=1}^t \tilde{A}_j \) for each \( i=1,2,\ldots,t \). Clearly, \( \{ \tilde{A}_i \}_{i=1}^t \) is of order not exceeding \( n \). Since \( (X,\tilde{T}) \) is fuzzy anti-locally countable and for each \( i=1,2,\ldots,t \), \( \tilde{A}_i \subseteq \tilde{A}_i \). Hence, \( \tilde{A}_i \subseteq \tilde{A}_i \) is fuzzy open refinement of \( \tilde{A}_i \) of order not exceeding \( n \). Therefore, \( \dim(X) \leq n = \omega \dim(X) \)

**Theorem 4.8**

If a fuzzy topological space \( (X,\tilde{T}) \) has the property that every fuzzy \( \omega \)-open covering of \( X \) has a fuzzy locally finite open refinement, then \( \omega \dim(X) \leq \dim(X) \).

**Proof**

Let \( (X,\tilde{T}) \) be a fuzzy topological space with the given property. So, if either \( \omega \dim(X) = \infty \) or \( \omega \dim(X) = -1 \), then there is nothing to prove.

Suppose that \( \dim(X) = n \) and let \( \{ \tilde{A}_i \}_{i=1}^t \) be any fuzzy finite \( \omega \)-open covering of \( X \), then by hypothesis, this \( \omega \)-open covering has a fuzzy locally finite open refinement \( \{ \tilde{A}_i \}_{i=1}^t \subseteq \tilde{A}_i \). Let \( \tilde{A}_i = \bigcup_{j=1}^t \tilde{A}_j \) for each \( i=1,2,\ldots,t \). \( \{ \tilde{A}_i \}_{i=1}^t \) is fuzzy finite open covering of \( X \). Since \( \dim(X) = n \), then there exists a fuzzy open refinement \( \{ \tilde{A}_i \}_{i=1}^t \subseteq \tilde{A}_i \) of \( \{ \tilde{A}_i \}_{i=1}^t \) and hence of \( \{ \tilde{A}_i \}_{i=1}^t \) of order not exceeding \( n \). Therefore, \( \omega \dim(X) \leq n = \dim(X) \).
Corollary 4.9
Let \((X, \tilde{T})\) be a fuzzy anti-locally countable Normal space with the property that every fuzzy \(\omega\)-open covering of \(\mathcal{N}_X\) has a fuzzy locally finite open refinement. Then, \(\omega-\dim(X) = \dim(X)\).

Proof: Obvious.

Theorem 4.10:
For a fuzzy topological space \((X, \tilde{T})\), the following statements are equivalent:
1. \(\omega-\dim(X) \leq n\).
2. For every fuzzy \(\omega\)-open cover \(\{\tilde{U}_1, \ldots, \tilde{U}_K\}\) of \(\mathcal{N}_X\), there exists a fuzzy finite \(\omega\)-open cover \(\{\tilde{n}_1, \ldots, \tilde{n}_k\}\) of \(\mathcal{N}_X\) of order less than or equal to \(n\), and \(\tilde{n}_i \leq \tilde{U}_i\) for \(i = 1, 2, \ldots, k\).
3. If \(\{\tilde{U}_1, \ldots, \tilde{U}_{n+2}\}\) is a fuzzy \(\omega\)-open cover of \(\mathcal{N}_X\), then there exists a non-overlapping fuzzy \(\omega\)-open cover \(\{\tilde{n}_1, \ldots, \tilde{n}_{n+2}\}\) of \(\mathcal{N}_X\), such that \(\tilde{n}_i \leq \tilde{U}_i\) for \(i = 1, 2, \ldots, n + 2\).

Proof
(1) \(\Rightarrow\) (2) suppose that \(\omega-\dim(X) \leq n\) and let \(\tilde{U} = \{\tilde{U}_1, \ldots, \tilde{U}_K\}\) be a fuzzy \(\omega\)-open cover of \(\mathcal{N}_X\). Let \(\tilde{V}\) be a fuzzy finite \(\omega\)-open refinement of \(\tilde{U}\) such that \(\ord_i \tilde{V} \leq n\). If \(\tilde{V} \in \tilde{U}\), then \(\tilde{V} \subseteq \tilde{U}_i\), for some \(i\). Let each \(\tilde{V} \in \tilde{V}\) be associated with one of the fuzzy sets \(\tilde{U}_i\) containing it. Let \(\tilde{n}_i\) be the union of all those members of \(\tilde{V}\) and, thus, associated with \(\tilde{U}_i\). Then, each \(\tilde{n}_i\) is a fuzzy \(\omega\)-open set and \(\tilde{n}_i \leq \tilde{U}_i\).

Let \(\tilde{N} = \{\tilde{n}_1, \ldots, \tilde{n}_K\}\). We want to show that \(\ord_i \tilde{N} \leq n\), that is, every overlapping subfamily of \(\tilde{N}\) contains at most \(n + 1\) members.

Suppose, if possible, that there exists an overlapping subfamily \(\tilde{n}_i\) of \(\tilde{N}\) containing \((n + 2)\) members. Then, there exists \(x_r \in \mathcal{N}_X\) such that \(\mu_{\tilde{n}_a}(x_r) + \mu_{\tilde{n}_b}(x_r) > 1\) for every pair \(\tilde{n}_a, \tilde{n}_b \in \tilde{n}_i\). Now, since \(\tilde{n}_r = \bigcup \{\tilde{V}_y \in \tilde{V}: \tilde{V}_y \leq \tilde{U}_i\}\) is associated in the construction of \(\tilde{n}_r\),

\((\gamma = \alpha, \beta), \text{ and since } \tilde{V}\) is a fuzzy finite cover of \(\mathcal{N}_X\), \(\mu_{\tilde{n}_a}(x_r) = \max\{\mu_{\tilde{V}_1}(x_r), \ldots, \mu_{\tilde{V}_n}(x_r)\}\) and \(\mu_{\tilde{n}_b}(x_r) = \max\{\mu_{\tilde{V}_1}(x_r), \ldots, \mu_{\tilde{V}_n}(x_r)\}\), we choose \(\tilde{V}_{k\alpha}\) and \(\tilde{V}_{l\beta}\) such that \(\mu_{\tilde{n}_a}(x_r) = \mu_{\tilde{V}_{k\alpha}}(x_r)\) and \(\mu_{\tilde{n}_b}(x_r) = \mu_{\tilde{V}_{l\beta}}(x_r)\).

Clearly, \(\tilde{V}_{k\alpha}\) and \(\tilde{V}_{l\beta}\) overlap at \(x_r\). In this way, we obtain, corresponding to every overlapping pair \(\tilde{n}_a, \tilde{n}_b\) at \(x_r\), a pair \(\tilde{V}_{k\alpha}\) and \(\tilde{V}_{l\beta}\) of \(\tilde{V}\)’s which are distinct in themselves, distinct from others, and overlap at \(x_r\). The collection of all these members of \(\tilde{V}\), which were chosen as above, constitutes an overlapping subfamily of \(\tilde{V}\), having \(n + 2\) members. This is a contradiction to the fact that \(\ord_i \tilde{V} \leq n\). Thus, \(\ord_i \tilde{V} \leq n\). The statements (2) \(\Rightarrow\) (1) and (2) \(\Rightarrow\) (3) are trivial.

To complete the proof, we will show that (3) \(\Rightarrow\) (2).

Let \((X, \tilde{T})\) be a fuzzy topological space satisfying (3), and let \(\{\tilde{U}_1, \ldots, \tilde{U}_K\}\) be a fuzzy \(\omega\)-open cover of \(\mathcal{N}_X\). Assume that \(k > n+1\).

Let \(\tilde{G}_i = \tilde{U}_i\), if \(1 \leq i \leq n+1\).

And let \(\tilde{G}_{n+2} = \bigcup_{i=n+2}^k \tilde{U}_i\). Clearly, \(\{\tilde{G}_1, \ldots, \tilde{G}_{n+2}\}\) is a fuzzy \(\omega\)-open cover of \(\mathcal{N}_X\). By hypothesis, there exists a fuzzy \(\omega\)-open cover \(\tilde{N} = \{\tilde{n}_1, \ldots, \tilde{n}_{n+2}\}\) of \(\mathcal{N}_X\) such that \(\tilde{n}_i \leq \tilde{G}_i\) for each \(i\), and \(\tilde{N}\) is a non-overlapping family.

We define fuzzy \(\omega\)-open sets \(\tilde{V}_i = \tilde{U}_i\) if \(1 \leq i \leq n+1\), and \(\tilde{V}_i = \tilde{U}_i \cap \tilde{n}_{n+2}\) if \(i > n+1\).

Then \(\tilde{V} = \{\tilde{V}_1, \ldots, \tilde{V}_k\}\) is a fuzzy \(\omega\)-open cover of \(\mathcal{N}_X\) such that \(\tilde{V}_i \subseteq \tilde{U}_i\), for each \(i\) and the subfamily \(\{\tilde{V}_1, \ldots, \tilde{V}_{n+2}\}\) of \(\tilde{V}\) is non-overlapping. If there is a fuzzy subset \(\tilde{B}\) of \(\{1, \ldots, K\}\) having \(n + 2\) elements such that the family \(\{\tilde{V}_j: j \in \tilde{B}\}\) is overlapping, then let the members of \(\tilde{V}\) be renumbered to give a family \(\tilde{P} = \{\tilde{P}_1, \ldots, \tilde{P}_k\}\) such that the subfamily \(\{\tilde{P}_1, \ldots, \tilde{P}_{n+2}\}\) is overlapping. By applying the above construction to \(\tilde{P}\), we obtain a fuzzy \(\omega\)-open cover \(\tilde{M} = \{\tilde{M}_1, \ldots, \tilde{M}_k\}\) of \(\mathcal{N}_X\) such that \(\tilde{M}_i \subseteq \tilde{P}_i\) and the subfamily \(\{\tilde{M}_1, \ldots, \tilde{M}_{n+2}\}\) is non-overlapping. Clearly, if \(\tilde{C}\) is a fuzzy subset of \(\{1, \ldots, K\}\) with \(n+2\) elements, such that the family \(\{\tilde{P}_i: i \in \tilde{C}\}\) is non-overlapping, then so is the family \(\{\tilde{M}_i: i \in \tilde{C}\}\).

Thus, by a finite number of repetitions of this process, we obtain a fuzzy \(\omega\)-open cover \(\tilde{W} = \{\tilde{W}_1, \ldots, \tilde{W}_k\}\) of \(\mathcal{N}_X\) such that \(\tilde{W}_i \subseteq \tilde{U}_i\) for each \(i\) and \(\ord_i \tilde{W} \leq n\).

Theorem 4.11
If \((X, \tilde{T})\) is a fuzzy \(\omega\)-Normal space, then the following statements are equivalent:
1. \( \omega\text{-dim}(X) \leq n \).  
2. For every fuzzy finite \( \omega \)-open covering \( \{\bar{U}_i\}_{i=1}^t \) of \( 1_X \), there is a fuzzy \( \omega \)-open covering \( \{\bar{V}_i\}_{i=1}^t \) of \( 1_X \) such that \( \omega \text{-cl}(\bar{V}_i) \subseteq \bar{U}_i \) for each \( i = 1, 2, \ldots, t \) and the order \( \{\omega \text{-cl} \bar{V}_i\}_{i=1}^t \) does not exceed \( n \).  
3. For every fuzzy finite \( \omega \)-open covering \( \{\bar{U}_i\}_{i=1}^t \) of \( 1_X \), there is a fuzzy \( \omega \)-closed covering \( \{\bar{F}_i\}_{i=1}^t \) of \( 1_X \) such that \( \bar{F}_i \subseteq \bar{U}_i \) for each \( i = 1, 2, \ldots, t \).  
4. Every fuzzy finite \( \omega \)-open covering of \( 1_X \) has a fuzzy finite \( \omega \)-closed refinement of order that does not exceed \( n \).  
5. If \( \{\bar{U}_i\}_{i=1}^{n+2} \) is a fuzzy \( \omega \)-open cover of \( 1_X \), then there exists a fuzzy \( \omega \)-closed covering \( \{\bar{F}_i\}_{i=1}^{n+2} \) such that \( \bigcap_{i=1}^{n+2} \bar{F}_i = \emptyset \).

**Proof**

(1) \( \Rightarrow \) (2) let \( \omega\text{-dim}(X) \leq n \) and \( \{\bar{U}_i\}_{i=1}^t \) be fuzzy finite \( \omega \)-open covering of \( 1_X \), then by

**Theorem 4.10** there exists a fuzzy \( \omega \)-open covering \( \{\bar{W}_i\}_{i=1}^t \) of \( 1_X \) of order not exceeding \( n \), such that \( \bar{W}_i \subseteq \bar{U}_i \) for each \( i \). Since \((X, \bar{T})\) is a fuzzy \( \omega \)-Normal space, then by **Theorem 3.22**, there exists a fuzzy \( \omega \)-open covering \( \{\bar{V}_i\}_{i=1}^t \) of \( 1_X \) such that \( \omega \text{-cl}(\bar{V}_i) \subseteq \bar{W}_i \) for each \( i \). Thus, \( \{\bar{V}_i\}_{i=1}^t \) is the required fuzzy \( \omega \)-open covering of \( 1_X \).

(2) \( \Rightarrow \) (3) and (3) \( \Rightarrow \) (4) Obvious.

(4) \( \Rightarrow \) (5) let \((X, \bar{T})\) be fuzzy \( \omega \)-Normal space which satisfies (4).

Also, let \( \{\bar{U}_i\}_{i=1}^{n+2} \) be a fuzzy \( \omega \)-open covering of \( 1_X \), then by hypothesis, there exists a fuzzy finite \( \omega \)-closed refinement \( \bar{Z} \) of \( \{\bar{U}_i\}_{i=1}^{n+2} \) of order not exceeding \( n \). Since for each \( \bar{F} \in \bar{Z} \) there exists a unique \( i \in \{1, 2, \ldots, n+2\} \) such that \( \bar{F} \subseteq \bar{U}_i \), then we set \( \bar{F}_i = \bigcup \{\bar{F} \in \bar{Z} : \bar{F} \subseteq \bar{U}_i\} \), since \( \bar{Z} \) is fuzzy finite, so for each \( i \), \( \bar{F}_i \) is fuzzy \( \omega \)-closed and \( \bar{F}_i \subseteq \bar{U}_i \). Furthermore, \( \{\bar{F}_i\}_{i=1}^{n+2} \) is fuzzy cover of \( 1_X \) of order not exceeding \( n \). Thus, \( \bigcap_{i=1}^{n+2} \bar{F}_i = \emptyset \).

(5) \( \Rightarrow \) (1) let \((X, \bar{T})\) be fuzzy \( \omega \)-Normal space which satisfies (5).

Also, let \( \{\bar{U}_i\}_{i=1}^{n+2} \) be a fuzzy \( \omega \)-open covering of \( 1_X \), so by our hypothesis, there exist a fuzzy \( \omega \)-closed covering \( \{\bar{F}_i\}_{i=1}^{n+2} \) of \( 1_X \), such that for each \( i \)

\[ \bar{F}_i \subseteq \bar{U}_i \text{ and } \bigcap_{i=1}^{n+2} \bar{F}_i = \emptyset \], there exists a fuzzy \( \omega \)-open covering \( \{\bar{V}_i\}_{i=1}^{n+2} \) of \( 1_X \), such that for each \( i \)

\[ \bar{V}_i \subseteq \omega \text{-cl}(\bar{V}_i) \subseteq \bar{W}_i \] and the fuzzy family \( \{\bar{F}_i\}_{i=1}^{n+2} \) is similar to \( \{\omega \text{-cl}(\bar{V}_i)\}_{i=1}^{n+2} \), thus \( \{\bar{V}_i\}_{i=1}^{n+2} \) is fuzzy \( \omega \)-open covering of \( 1_X \), such that \( \bar{V}_i \subseteq \bar{U}_i \) for each \( i \) and \( \bigcap_{i=1}^{n+2} \bar{V}_i = \emptyset \).

Hence, by **Theorem 4.10**, \( \omega\text{-dim}(X) \leq n \)

**Theorem 4.12**

If \((X, \bar{T})\) is a fuzzy \( \omega \)-Normal space, then the following statements are equivalent:

1. \( \omega\text{-dim}(X) \leq n \).
2. For each fuzzy family \( \{\bar{F}_i\}_{i=1}^{n+1} \) of \( \omega \)-closed sets and each fuzzy family \( \{\bar{U}_i\}_{i=1}^{n+1} \) of \( \omega \)-open sets of \( 1_X \), such that \( \bar{F}_i \subseteq \bar{U}_i \) for each \( i \), there is a fuzzy family \( \{\bar{V}_i\}_{i=1}^{n+1} \) of \( \omega \)-open sets, such that

\[ \bar{F}_i \subseteq \bar{V}_i \subseteq \omega \text{-cl}(\bar{V}_i) \subseteq \bar{U}_i \text{ for each } i \text{ and } \bigcap_{i=1}^{n+1} \omega \text{-b}(\bar{V}_i) = \emptyset \]

3. For each fuzzy family \( \{\bar{F}_i\}_{i=1}^{n+1} \) of \( \omega \)-closed sets and each fuzzy family \( \{\bar{U}_i\}_{i=1}^{n+1} \) of \( \omega \)-open sets of \( 1_X \), such that \( \bar{F}_i \subseteq \bar{U}_i \) for each \( i \), there exist fuzzy families \( \{\bar{V}_i\}_{i=1}^{n+1} \) and \( \{\bar{W}_i\}_{i=1}^{n+1} \) of \( \omega \)-open sets, such that

\[ \bar{F}_i \subseteq \bar{V}_i \subseteq \omega \text{-cl}(\bar{V}_i) \subseteq \bar{W}_i \subseteq \bar{U}_i \text{ for each } i \text{ and the order of } \{\omega \text{-cl}(\bar{W}_i) - \bar{V}_i\}_{i=1}^{n+1} \text{ does not exceed } n-1 \].

4. For each fuzzy family \( \{\bar{F}_i\}_{i=1}^{n+1} \) of \( \omega \)-closed sets and each fuzzy family \( \{\bar{U}_i\}_{i=1}^{n+1} \) of \( \omega \)-open sets of \( 1_X \), such that \( \bar{F}_i \subseteq \bar{U}_i \) for each \( i \), there exists a fuzzy family \( \{\bar{V}_i\}_{i=1}^{n+1} \) of \( \omega \)-open sets, such that

\[ \bar{F}_i \subseteq \bar{V}_i \subseteq \omega \text{-cl}(\bar{V}_i) \subseteq \bar{U}_i \text{ for each } i \text{ and the order of } \{\omega \text{-b}(\bar{V}_i)\}_{i=1}^{n+1} \text{ does not exceed } n-1 \].

**Proof**

(1) \( \Rightarrow \) (2) suppose that \( \omega\text{-dim}(X) \leq n \). Let \( \{\bar{F}_i\}_{i=1}^{n+1} \) be fuzzy \( \omega \)-closed sets and let \( \{\bar{U}_i\}_{i=1}^{n+1} \) be fuzzy \( \omega \)-open sets, such that \( \bar{F}_i \subseteq \bar{U}_i \) since \( \omega\text{-dim}(X) \leq n \) the fuzzy \( \omega \)-open covering of \( 1_X \) consisting of
fuzzy sets of the form $\bigcap_{i=0}^{n+1} \bar{F}_i$ where $\bar{F}_i = \bar{U}_i$ or $\bar{F}_i = 1_X - \bar{F}_i$ for each $i$ has a finite fuzzy $\omega$-open refinement $\{\tilde{W}_r\}_{r=1}^t$ of order not exceeding $n$ since $(X, \tilde{T})$ a fuzzy $\omega$-Normal space there exist a fuzzy $\omega$-closed covering $\{\bar{R}_r\}_{r=1}^t$ of $1_X$ such that $\bar{R}_r \subseteq \tilde{W}_r$ for $r = 1, \ldots, t$. We denote the set $\mathcal{N}_i = \{i: \tilde{W}_r \cap \bar{F}_i \neq \emptyset \}$ for $r = 1, \ldots, t$. We can find fuzzy $\omega$-open set $\tilde{V}_i$ for $i \in \mathcal{N}_i$ such that $\bar{R}_r \subseteq \tilde{V}_i \subseteq \omega-\text{cl}(\tilde{V}_i) \subseteq \tilde{W}_r$ and $\omega-\text{cl}(\tilde{V}_i) \subseteq \tilde{V}_i$. 

if $i \leq j$. Now, for each $i = 1, \ldots, n+1$, let $\tilde{U}_r = \cup_{r} (\tilde{V}_i): i \in \mathcal{N}_r \}$. Then, $\tilde{U}_i$ is a fuzzy $\omega$-open set and $\tilde{F}_i \subseteq \tilde{U}_i$. If $x_i \in \tilde{F}_i$ and $x_i \in \bar{R}_r$, then $r \in \mathcal{N}_n$, so that $x_i \in \tilde{V}_i \subseteq \tilde{U}_i$. Furthermore, if $i \in \mathcal{N}_n$, so that $\tilde{W}_r \cap \bar{F}_i \neq \emptyset$, then $\tilde{W}_r$ is not contained in $1_X - \bar{F}_i$, so that $\tilde{W}_r \subseteq \tilde{U}_i$. Thus, if $i \in \mathcal{N}_n$, then $\tilde{V}_i \subseteq \tilde{U}_i$, so that since $\omega-\text{cl}(\tilde{V}_i) = \cup_{i} \{\omega-\text{cl}(\tilde{V}_i): i \in \mathcal{N}_i \}$, it follows that $\omega-\text{cl}(\tilde{V}_i) \subseteq \tilde{U}_i$. Finally, on the contrary, suppose that $\bigcap_{i=1}^{n+1} \omega-b(\tilde{V}_{i}) \neq \emptyset$, which implies that there is a fuzzy point $x_i$ in $\bigcap_{i=1}^{n+1} \omega-b(\tilde{V}_{i})$. But we have $\omega-b(\tilde{V}_i) \subseteq \bigcup_{i} \{\omega-b(\tilde{V}_i): i \in \mathcal{N}_r \}$. Hence for each $i = 1, \ldots, n+1$ there exists $r_i \in \{1, \ldots, t\}$ such that $x_i \in \omega-b(\tilde{V}_{r_i}$). If $i \neq j$, then $r_i \neq r_j$ for if $r_i = r_j$, then $x_i \in \omega-\text{cl}(\tilde{V}_{r_i})$ and $x_j \in \omega-\text{cl}(\tilde{V}_{r_j})$. But $x_i \in \tilde{V}_{r_i}$ and $x_j \in \tilde{V}_{r_j}$, which is irrational, since either $\omega-\text{cl}(\tilde{V}_{r_i}) \subseteq \tilde{V}_{r_j}$ or $\omega-\text{cl}(\tilde{V}_{r_j}) \subseteq \tilde{V}_{r_i}$ for each $i$, $x_i \in \tilde{V}_{r_i}$, so that $x_i \in \bar{R}_r$. But $\bar{R}_r$ is fuzzy $\omega$-closed covering, so there exists $r_0$ that is different from each of the $r_i$, such that $x_i \notin \bar{R}_r$, $r_0 \notin \tilde{W}_r$. 

Since $x_i \in \omega-\text{cl}(\tilde{V}_{r_i})$, it follows that $x_i \in \tilde{W}_r$ for $i = 1, \ldots, n+1$, so that $x_i \in \bigcap_{i=1}^{n+1} \omega-b(\tilde{V}_{i})$. 

Since the order of $\tilde{W}_r$ does not exceed $n$, then this is irrational. Hence $\bigcap_{i=1}^{n+1} \omega-b(\tilde{V}_{i}) = \emptyset$.

(2) $\Rightarrow$ (3) let (2) holds. $\{\bar{F}_i\}_{i=1}^t$ be a family of fuzzy $\omega$-closed sets, and $\{\tilde{U}_i\}_{i=1}^t$ be a family of fuzzy $\omega$-open sets of $1_X$ such that $\bar{F}_i \subseteq \tilde{U}_i$ for each $i$ If $t \leq n+1$. Then there is nothing to prove. So, we suppose that $t > n+1$. Let the fuzzy subset of $\{1, 2, \ldots, t\}$ containing $n+1$ elements be enumerated as $\tilde{C}_1, \tilde{C}_2, \ldots, \tilde{C}_k$. Then by our hypothesis, for each $i \in \tilde{C}_1$ we can find fuzzy $\omega$-open subsets $\tilde{V}_{1i}$ of $1_X$ such that $\bar{F}_i \subseteq \tilde{V}_{1i} \subseteq \omega-\text{cl}(\tilde{V}_{1i}) \subseteq \tilde{U}_i$ and $\bigcap_{i=1}^{n+1} \omega-b(\tilde{V}_{i}) = \emptyset$. Also, we have that $\omega-b(\tilde{V}_{1i})$ is fuzzy $\omega$-closed subset of a fuzzy $\omega$-Normal space $(X, \tilde{T})$, such that $\omega-b(\tilde{V}_{1i}) \subseteq \tilde{U}_i$ for each $i \in \tilde{C}_1$. Therefore, for each $i \in \tilde{C}_1$ there exists a fuzzy $\omega$-open subset $\tilde{G}_i$ of $1_X$ such that $\omega-b(\tilde{V}_{1i}) \subseteq \tilde{G}_i \subseteq \omega-\text{cl}(\tilde{G}_i) \subseteq \tilde{U}_i$. Also, the fuzzy families $\{\omega-\text{cl}(\tilde{G}_i)\}_{i \in \tilde{C}_1}$ and $\{\omega-b(\tilde{V}_{1i})\}_{i \in \tilde{C}_1}$ are similar so that, in particular, $\bigcap_{i \in \tilde{C}_1} \omega-\text{cl}(\tilde{G}_i) = \emptyset$. Let $\tilde{W}_1 = \tilde{V}_{1i} \cup \tilde{C}_1$ if $i \in \tilde{C}_1$, then $\omega-\text{cl}(\tilde{V}_{1i}) \subseteq \tilde{W}_1 \subseteq \omega-\text{cl}(\tilde{W}_1) \subseteq \tilde{U}_i$. Since $\omega-\text{cl}(\tilde{W}_1) - \tilde{V}_{1i} \subseteq \omega-\text{cl}(\tilde{G}_i)$, then we have $\bigcap_{i \in \tilde{C}_1} \omega-\text{cl}(\tilde{W}_1) - \tilde{V}_{1i} = \emptyset$. If $i \notin \tilde{C}_1$. Let $\tilde{V}_{1i}$ be fuzzy $\omega$-open set such that $\bar{F}_i \subseteq \tilde{V}_{1i} \subseteq \omega-\text{cl}(\tilde{V}_{1i}) \subseteq \tilde{U}_i$ and let $\tilde{W}_1 = \tilde{V}_{1i}$. Then for $i = 1, \ldots, t$, we have fuzzy $\omega$-open sets $\tilde{V}_{1i}$ and $\tilde{W}_{1i}$ such that $\bar{F}_i \subseteq \tilde{V}_{1i} \subseteq \omega-\text{cl}(\tilde{V}_{1i}) \subseteq \tilde{W}_{1i} \subseteq \tilde{U}_i$ and $\bigcap_{i \in \tilde{C}_1} \omega-\text{cl}(\tilde{W}_1) - \tilde{V}_{1i} = \emptyset$. Suppose that, for each $m \in \{2, 3, \ldots, k\}$, we can find, using the same technique above, the fuzzy $\omega$-open sets $\tilde{V}_{1(m-1)}$ and $\tilde{W}_{1(m-1)}$, such that $\bar{F}_i \subseteq \tilde{V}_{1(m-1)} \subseteq \omega-\text{cl}(\tilde{V}_{1(m-1)}) \subseteq \tilde{W}_{1(m-1)} \subseteq \tilde{U}_i$ and $\bigcap_{i \in \tilde{C}_j} \omega-\text{cl}(\tilde{W}_{m}) - \tilde{V}_{1(m-1)} = \emptyset$, for each $j \in \{1, 2, \ldots, m-1\}$. Also, by the above argument, we can find fuzzy $\omega$-open sets $\tilde{V}_m$ and $\tilde{W}_m$, such that $\omega-\text{cl}(\tilde{V}_m) \subseteq \tilde{V}_m \subseteq \omega-\text{cl}(\tilde{V}_m) \subseteq \tilde{W}_m \subseteq \tilde{W}_m$ and $\bigcap_{i \in \tilde{C}_j} \omega-\text{cl}(\tilde{W}_m) - \tilde{V}_m = \emptyset$, for each $j \in \{1, 2, \ldots, m\}$. Thus, by induction, for each $i \in \{1, 2, \ldots, t\}$, we can find fuzzy $\omega$-open sets $\tilde{V}_i$ and $\tilde{W}_i$. 

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(ν_i and W_{ik} , respectively), such that \( F_i \subseteq V_i \subseteq \omega \text{-cl}(V_i) \subseteq W_i \subseteq U_i \) and \( \bigcap_{i \in I} \omega \text{-cl}(W_i) - V_i = \emptyset \) for each \( j \in \{1,2,\ldots,k\} \). Hence, the order of \( \{\omega \text{-cl}(W_i) - V_i\}_{i=1}^n \) does not exceed \( n \).-
(3) \( \Rightarrow \) (4) Obvious, since \( \omega \text{-b}(\tilde{V}_i) = \omega \text{-cl}(\tilde{V}_i) - \tilde{V}_i \subseteq \omega \text{-cl}(\tilde{V}_i) - V_i \).
(4) \( \Rightarrow \) (1) Let \( \{\tilde{U}_{ij}\}_{i=1}^{n+2} \) be a fuzzy \( \omega \)-open covering of fuzzy \( \omega \)-Normal space \( (X,\tilde{T}) \). Then, there exists fuzzy \( \omega \)-closed covering \( \{\tilde{F}_i\}_{i=1}^{n+2} \) of \( X \) such that \( \tilde{F}_i \subseteq \tilde{U}_i \) for each \( i \). So, by hypothesis, there exists a fuzzy family \( \{\tilde{V}_i\}_{i=1}^{n+2} \) of fuzzy \( \omega \)-open sets in \( 1_X \), such that \( \tilde{F}_i \subseteq \tilde{V}_i \subseteq \omega \text{-cl}(\tilde{V}_i) \subseteq \tilde{U}_i \) for each \( i \). Also, the order of \( \{\omega \text{-b}(\tilde{V}_i)\}_{i=1}^{n+2} \) does not exceed \( n \) and the fuzzy family \( \{\tilde{V}_i\}_{i=1}^{n+2} \) is fuzzy \( \omega \)-open covering of \( 1_X \). Let \( \tilde{L}_j = \omega \text{-cl}(\tilde{V}_j) - \bigcup_{i < j} \tilde{V}_i \) for \( j=1,2,\ldots,n+2 \). For each \( j \), \( \tilde{L}_j \) is fuzzy \( \omega \)-closed and \( \{\tilde{L}_j\}_{j=1}^{n+2} \) is a fuzzy \( \omega \)-closed covering of \( 1_X \), for if \( x \in 1_X \), there exists \( j \) such that \( x \in \tilde{V}_j \) and \( x \notin \tilde{V}_i \) for \( i < j \), so that \( x \in \tilde{L}_j \).

Now, \( \tilde{L}_j = \omega \text{-cl}(\tilde{V}_j) \cap \bigcap_{i < j} \tilde{V}_i \) so that \( \left( \bigcap_{j=3}^{n+2} \tilde{L}_j \right) = \left( \bigcap_{j=3}^{n+2} \omega \text{-cl}(\tilde{V}_j) \right) \cap \bigcap_{i=1}^{n+1} \tilde{V}_i \subseteq \bigcap_{j=3}^{n+2} \omega \text{-cl}(\tilde{V}_j) = \emptyset \). Hence, by Theorem 4.11, we have \( \omega \text{-dim}(X) \leq n \).

**Definition 4.13**

Let \( A \) and \( B \) be any two fuzzy disjoint sets in a fuzzy space \( 1_X \). A fuzzy subset \( \tilde{L} \) is called a fuzzy \( \omega \)-partition \( \tilde{A} \) and \( \tilde{B} \) if there exist two fuzzy disjoint \( \omega \)-open sets \( \tilde{U} \) and \( \tilde{W} \) such that \( \tilde{A} \subseteq \tilde{U} \), \( \tilde{B} \subseteq \tilde{W} \) and \( 1_X - L = \tilde{U} \cup \tilde{W} \).

**Theorem 4.14**

Let \( (X,\tilde{T}) \) be a fuzzy topological sum of the fuzzy family of spaces \( \{1_{X_{\lambda}}\}_{\lambda \in \Lambda} \). If \( \omega \text{-dim}(X_{\lambda}) \leq n \) for each \( \lambda \in \Lambda \), then \( \omega \text{-dim}(X) \leq n \).

**Proof**

Let \( \omega \text{-dim}(X_{\lambda}) \leq n \) for each \( \lambda \in \Lambda \) and let \( \{\tilde{U}_{ij}\}_{i=1}^{n+2} \) be fuzzy \( \omega \)-open covering of a fuzzy topological sum \( (X,\tilde{T}) \), then \( \{\tilde{U}_{ij} \cap X_{\lambda}\}_{i=1}^{n+2} \) is fuzzy \( \omega \)-open covering \( \{1_{X_{\lambda}}\}_{\lambda \in \Lambda} \). Therefore, by Theorem 4.10 , for each \( \lambda \in \Lambda \) there exists a fuzzy \( \omega \)-open covering \( \{\tilde{V}_{ij}\}_{i=1}^{n+2} \) of \( X_{\lambda} \) of order not exceeding \( n \), such that \( \tilde{V}_{ij} \subseteq \tilde{U}_{ij} \cap X_{\lambda} \) for each \( i \). Then, the fuzzy family \( \{\tilde{V}_{ij}\}_{i=1}^{n+2} \) is fuzzy \( \omega \)-open covering of \( X \), such that \( \tilde{W}_{ij} \subseteq \tilde{U}_{ij} \) for each \( i \) where \( \tilde{W}_{ij} = \bigcup_{\lambda \in \Lambda} \tilde{V}_{ij} \). If the order of \( \{\tilde{V}_{ij}\}_{i=1}^{n+2} \) does not exceed \( n \), then \( \omega \text{-dim}(X_{\lambda}) \leq n \) for each \( \lambda \in \Lambda \). Thus, the order of \( \{\tilde{W}_{ij}\}_{i=1}^{n+2} \) does not exceed \( n \), hence \( \omega \text{-dim}(X) \leq n \).

**Theorem 4.15**

If \( (X,\tilde{T}) \) is a fuzzy \( \omega \)-Normal space with the property that for each fuzzy \( \omega \)-closed set \( \tilde{F} \) and each fuzzy \( \omega \)-open set \( \tilde{U} \), such that \( \tilde{F} \subseteq \tilde{U} \) , there exists a fuzzy \( \omega \)-open set \( \tilde{V} \) in \( 1_X \) such that \( \tilde{F} \subseteq \tilde{V} \subseteq \tilde{U} \) and \( \omega \text{-dim}(\omega \text{-b}(\tilde{V})) \leq n \), then \( \omega \text{-dim}(X) \leq n+1 \).

**Proof**

Let \( \{\tilde{U}_{ij}\}_{i=1}^{n+2} \) be fuzzy \( \omega \)-open covering of \( X \). Since \( (X,\tilde{T}) \) is a fuzzy \( \omega \)-Normal space, so there exists a fuzzy \( \omega \)-closed covering \( \{\tilde{F}_i\}_{i=1}^{n+2} \) of \( 1_X \) such that \( \tilde{F}_i \subseteq \tilde{U}_i \) for each \( i \). Then, by hypothesis, there exists fuzzy \( \omega \)-open sets \( \tilde{V}_i \) such that \( \tilde{F}_i \subseteq \tilde{V}_i \subseteq \tilde{U}_i \) and \( \omega \text{-dim}(\omega \text{-b}(\tilde{V}_i)) \leq n \) for each \( i \). Again, since \( (X,\tilde{T}) \) is a fuzzy \( \omega \)-Normal space, so, by Theorem 3.12 and Theorem 3.22, the fuzzy set \( \tilde{E} = \bigcup_{i=1}^{n+2} \omega \text{-b}(\tilde{V}_i) \) is fuzzy \( \omega \)-Normal subspace of \( 1_X \). By Theorem 4.1, we have \( \omega \text{-dim}(\tilde{E}) \leq n \) and \( \{\tilde{E} \cap \tilde{U}_{ij}\}_{i=1}^{n+3} \) is fuzzy \( \omega \)-open covering of \( \tilde{E} \). Therefore, there exists a fuzzy \( \omega \)-closed covering \( \{\tilde{H}_i\}_{i=1}^{n+3} \) of \( \tilde{E} \) such that \( \tilde{H}_i \subseteq \tilde{E} \cap \tilde{U}_{ij} \) for each \( i \). Then, \( \tilde{H}_i \) is fuzzy \( \omega \)-closed subset of \( 1_X \) for each \( i \), then there exists a fuzzy family \( \{\tilde{W}_i\}_{i=1}^{n+3} \) of fuzzy \( \omega \)-open sets in \( 1_X \) such that \( \tilde{H}_i \subseteq \tilde{W}_i \subseteq \tilde{U}_i \) for each \( i \) and the fuzzy
families \( \{ \omega - \text{cl}(\tilde{W}_i) \}_{i=1}^{n+3} \) and \( \{ \tilde{H}_i \}_{i=1}^{n+3} \) are similar. Also we can obtain a fuzzy family \( \{ \tilde{G}_i \}_{i=1}^{n+3} \) of pairwise disjoint fuzzy \( \omega \)-open sets of \( 1_X \) where \( \tilde{G}_i = \tilde{V}_i - \bigcup_{j < i} \omega - \text{cl}(\tilde{V}_j) \) and it is clear that \( \tilde{G}_i \subseteq \tilde{U}_i \) for each \( i \). To show that \( 1_X - \tilde{E} \subseteq \bigcup_{i=1}^{n+3} \tilde{G}_i \), let \( x \in 1_X - \tilde{E} \), then \( x \in \omega - b(\tilde{V}_i) \) for each \( i \). Also, since \( \{ \tilde{V}_i \}_{i=1}^{n+3} \) is a fuzzy covering of \( 1_X \), then \( x \in \tilde{V}_i \) for some \( i \in \{1,2,\ldots,n+2,n+3\} \). Suppose that \( k \) is the first integer in which \( x \in \tilde{V}_k \). Thus for each \( m < k \) we have \( x \notin \tilde{V}_m \) and \( x \in \omega - b(\tilde{V}_m) \). Hence, \( x \notin \omega - \text{cl}(\tilde{V}_m) \). Thus, \( x \in \tilde{G}_k \subseteq \bigcup_{i=1}^{n+3} \tilde{G}_i \), hence \( 1_X - \tilde{E} \subseteq \bigcup_{i=1}^{n+3} \tilde{G}_i \). Let \( \tilde{O}_i = \tilde{G}_i \cup \tilde{W}_i \subseteq \tilde{U}_i \) for each \( i \). Clearly, \( \{ \tilde{O}_i \}_{i=1}^{n+3} \) is a fuzzy \( \omega \)-open covering of \( 1_X \). Suppose that \( x \notin \bigcap_{i=1}^{n+3} \tilde{O}_i \), then \( x \notin \tilde{G}_i \) or \( x \notin \tilde{W}_i \) for each \( i \).

Since \( \{ \tilde{G}_i \}_{i=1}^{n+3} \) is a pairwise disjoint fuzzy family of fuzzy \( \omega \)-open sets, then \( x \) cannot be contained in more than one of them. Therefore, there exists at most one \( k \in \{1,2,\ldots,n+2,n+3\} \) such that \( x \in \tilde{G}_k \), so that \( x \notin \tilde{G}_i \) for each \( i \neq k \). Hence, \( x \notin \tilde{W}_i \) for each \( i \in \{1,2,\ldots,n+2,n+3\} \), except at most for \( i = k \). Thus, \( x \) is contained in at least \((n+2)\) members of \( \{ \tilde{W}_i \}_{i=1}^{n+3} \). This implies that the order of \( [\tilde{W}_i]_{i=1}^{n+3} \) exceeds \( n \), which is a contradiction. Hence \( \bigcap_{i=1}^{n+3} \tilde{O}_i = \emptyset \). Therefore, by Theorem 4.1, we have \( \omega \text{-dim}(X) \leq n+1 \).

References