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The Continuous Classical Boundary Optimal Control of Triple Nonlinear Elliptic Partial Differential Equations with State Constraints

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Abstract
Our aim in this work is to study the classical continuous boundary control vector problem for triple nonlinear partial differential equations of elliptic type involving a Neumann boundary control. At first, we prove that the triple nonlinear partial differential equations of elliptic type with a given classical continuous boundary control vector have a unique "state" solution vector, by using the Minty-Browder Theorem. In addition, we prove the existence of a classical continuous boundary optimal control vector ruled by the triple nonlinear partial differential equations of elliptic type with equality and inequality constraints. We study the existence of the unique solution for the triple adjoint equations related with the triple state equations. The Fréchet derivative is obtained. Finally we prove the theorems of both the necessary and sufficient conditions for optimality of the triple nonlinear partial differential equations of elliptic type through the Kuhn-Tucker-Lagrange's Multipliers theorem with equality and inequality constraints.

Keywords: optimal control vector, triple nonlinear elliptic equations, necessary and sufficient conditions for optimality

مسألة السيطرة الحدودية الامثلية التقليدية المستمرة لثلاثي من المعادلات التفاضلية الجزئية الغير خطية من النوع الاهليجي بوجود قيود الحالة

جميل أمير علي الهواسي* و نبيل عدنان ذياب العجيلي
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الخلاصه
هدفنا في هذا العمل هو دراسة مسالة متجه السيطرة الحدودية الامثلية التقليدية المستمرة ثلاثي من المعادلات التفاضلية الجزئية الغير خطية من النوع الاهليجي تحوي شروط حدودية "متجه سيطرة" من نوع نيومان . في البداية وباستخدام مبرهنة مانتي – بروادير برهنا وجود ووحدانية حل المتجه للحالة لثلاثي من المعادلات التفاضلية الجزئية الغير خطية من النوع الاهليجي عندما يكون متجه السيطرة الحدودية التقليدية ثابتا". ايضا" تم برهنا وجود متجه سيطرة حدودية امثلية مستمرة تقليدية لهذه المسالة وبوجود قيدي التساوي وعدم التساوي. كذلك درسنا وجود و وحدانية الحل لثلاثي من المعادلات المرافقة المصاحبة لمعادلات الثلاثية للحالة. تم اشتقاق مشتقة فريشيه الخاصة بهذه المسالة. وفي النهاية تم برهان مبرهنتا الشروط الضرورية والكافية لوجود متجه سيطرة امثلية مستمرة تقليدية بوجود قيدي التساوي وعدم التساوي من خلال استخدام مبرهنة كهان –تاكر –لاكرانج.

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1. Introduction

In many fields, the optimal control problems play a significant role in life. Different examples of the applications of such problems are presented in medicine [1], aircraft industry [2], electric power production [3], economic growth [4], and many other fields.

All these applications pushed many investigators to a higher level of interest in studying the optimal control problem for nonlinear ordinary differential equations [5], for different types of linear partial differential equations, including the hyperbolic, parabolic and elliptic [6- 8], or for couple nonlinear partial differential equations of these three types [9-11]. While other authors [12, 13] studied these three types but included a Neumann boundary control. More recently, optimal control problems were studied for triple partial differential equations of these three types [14-16]. Also, the optimal control problem involving Neumann boundary control for triple partial differential equations of parabolic type was also recently investigated [17]. All these investigations motivated us to seek the optimal control problem, involving Neumann boundary control ruled by the triple nonlinear partial differential equations of elliptic type.

At first, our aim in this work is to prove that system of the triple nonlinear partial differential equations of elliptic type with a given classical continuous boundary control vector, which has a unique "state" solution vector, by using the Minty-Browder Theorem. Then, we prove the existence of a classical continuous boundary optimal control vector, ruled by the triple nonlinear partial differential equations of elliptic type with equality and inequality constraints.

We study the existence of the unique solution for the system of the triple adjoint equations related with the triple state equations. At the end, we prove the theorems of both the necessary and sufficient conditions for optimality of the triple nonlinear partial differential equations of elliptic type through the Kuhn-Tucker-Lagrange's Multipliers with equality and inequality constraints.

2. Problem Description

Let Ψ be a bounded and open connected subset in R^2 with Lipshitz boundary $\partial\Psi$. The optimal control problem is considered by the "state vector equation" which consists of the TNLEPDEs triple nonlinear elliptic partial differential equations with the Neumann boundary control.

$$A_1 b_1 + b_1 - b_2 - b_3 + m_1(\text{ӽ}, b_1) = \text{ƙ}_1(\text{ӽ}), \text{in } \Psi \tag{1}$$

$$A_2 b_2 + b_1 + b_2 + b_3 + m_2(\text{ӽ}, b_2) = \text{ƙ}_2(\text{ӽ}), \text{in } \Psi \tag{2}$$

$$A_3 b_3 + b_1 - b_2 + b_3 + m_3(\text{ӽ}, b_3) = \text{ƙ}_3(\text{ӽ}), \text{in } \Psi \tag{3}$$

$$\sum_{\sigma,j=1}^{2} a_{1_{\sigma j}} \frac{\partial b_1}{\partial n_1} = d_1, \text{on } \partial\Psi \tag{4}$$

$$\sum_{\sigma,j=1}^{2} a_{2_{\sigma j}} \frac{\partial b_2}{\partial n_2} = d_2, \text{on } \partial\Psi \tag{5}$$

$$\sum_{\sigma,j=1}^{2} a_{3_{\sigma j}} \frac{\partial b_3}{\partial n_3} = d_3, \text{on } \partial\Psi \tag{6}$$

where

$A_r b_r = -\sum_{\sigma,j=1}^{2} \frac{\partial}{\partial \text{ӽ}_\sigma}\left(a_{r_{\sigma j}}(\text{ӽ}) \frac{\partial b_r}{\partial \text{ӽ}_j}\right)$, $r = 1,2,3$ $a_{r_{\sigma j}} = a_{r_{\sigma j}}(\text{ӽ}_{\sigma j}) \in C^\infty(\Psi)$, for $\sigma, j = 1,2$

$(d_1, d_2, d_3) = (d_1(\text{ӽ}), d_2(\text{ӽ}), d_3(\text{ӽ})) \in (L_2(\partial\Psi))^3$ is the Neumann boundary control vector. The correspond "state" solution vector to the Neumann boundary control vector is $(b_1, b_2, b_3) = (b_1(\text{ӽ}), b_2(\text{ӽ}), b_3(\text{ӽ})) \in (H^1(\Psi))^3$, $(m_1, m_2, m_3) = (m_1(\text{ӽ}, b_1), m_2(\text{ӽ}, b_2), m_3(\text{ӽ}, b_3))$, $(\text{ƙ}_1, \text{ƙ}_2, \text{ƙ}_3) = (\text{ƙ}_1(\text{ӽ}), \text{ƙ}_2(\text{ӽ}), \text{ƙ}_3(\text{ӽ})) \in (L_2(\Psi))^3$, which is a vector of functions.

The control constraints are

$\vec{d} \in \vec{E}, \vec{E} \subset (L_2(\partial\Psi))^3$, where $\vec{d} = (d_1, d_2, d_3)$ and $\vec{E} = E_1 \times E_2 \times E_3$, with

$\vec{E} = \vec{E}_{\vec{D}} = \left\{\vec{E} \in (L_2(\partial\Psi))^3 | \vec{E} = (E_1, E_2, E_3) \in \vec{D} \text{ a.e in } \partial\Psi \right\}$,

where $\vec{D} = D_1 \times D_2 \times D_3$, with $\vec{D} \subset R^3$ is a convex and compact set .

The cost function and the equality and inequality constraints are given by:

$$\text{Ʊ}_0(\vec{d}) = \iint_\Psi [\text{ʊ}_{01}(\text{ӽ}, b_1) + \text{ʊ}_{02}(\text{ӽ}, b_2) + \text{ʊ}_{03}(\text{ӽ}, b_3)]\, d\text{ӽ}_1 d\text{ӽ}_2 + \int_{\partial\Psi} [\text{ʊ}_{04}(\text{ӽ}, d_1) + \text{ʊ}_{05}(\text{ӽ}, d_2) + \text{ʊ}_{06}(\text{ӽ}, d_3)] d\gamma \tag{7}$$

$$\mho_1(\vec{d}) = \iint_\Psi [\upsilon_{11}(\chi, b_1) + \upsilon_{12}(\chi, b_2) + \upsilon_{13}(\chi, b_3)]\, d\chi_1 d\chi_2 + \int_{\partial\Psi} [\upsilon_{14}(\chi, d_1) + \upsilon_{15}(\chi, d_2) + \upsilon_{16}(\chi, d_3)] d\gamma = 0 \quad (8)$$

$$\mho_2(\vec{d}) = \iint_\Psi [\upsilon_{21}(\chi, b_1) + \upsilon_{22}(\chi, b_2) + \upsilon_{23}(\chi, b_3)]\, d\chi_1 d\chi_2 + \int_{\partial\Psi} [\upsilon_{24}(\chi, d_1) + \upsilon_{25}(\chi, d_2) + \upsilon_{26}(\chi, d_3)] d\gamma \le 0 \quad (9)$$

The set of admissible control is

$$\vec{E}_A = \{\vec{d} \in \vec{E} \mid \mho_1(\vec{d}) = 0, \mho_2(\vec{d}) \le 0\} \quad (10)$$

The classical continuous boundary control vector problem is to minimize (7) subject to the state constraints (8) and (9), i.e. to find $\vec{d}$, such that

$\vec{d} \in \vec{E}_A$ and $\mho_0(\vec{d}) = \min_{\vec{e}\in\vec{E}} \mho_0(\vec{e})$.

Let $\vec{T} = (T)^3 = \left(H^1(\Psi)\right)^3$, the notations (t ,t)_(L_2 (Ψ)) , and ‖ T ‖_(L_2 (Ψ)) (‖ T ‖_(L_2 (∂Ψ))) refer to the inner product and the norm in L_2 (Ψ) (L_2 (∂Ψ)). The notations ⟦(t ,t) ⟧_(H^1 (Ψ)) and ‖T‖_(H^1 (Ψ)) refer to the inner product and the norm in H^1 (Ψ) , the notations (t ⃗ ,t ⃗)_(L_2 (Ψ))= ∑_(i=1)^2▒(t_i ,t_i) and ‖ t ⃗ ‖_((L_2 (Ψ))^3)= ∑_(i=1)^3▒‖ t_i ‖_(L_2 (Ψ)) refer to the inner product and the norm in (L_2 (Ψ))^3, while the notations (t ⃗ ,t ⃗)_(L_2 (Ψ))= ∑_(i=1)^3▒(t_i ,t_i) , and ‖t ⃗ ‖_((H^1 (Ψ))^3)=∑_(i=1)^3▒‖t_i ‖_(H^1 (Ψ)) refer to the inner product and the norm in T ⃗ , finally T ⃗^* is referred to the dual of T ⃗.

3. Weak formulation of the triple state equations

To find the weak formulation of problem (1-6) , let

$$\vec{T} = T_1 \times T_2 \times T_3 = H^1(\Psi) \times H^1(\Psi) \times H^1(\Psi)$$
$$= \{\vec{t} : \vec{t} = (t_1, t_2, t_3) \in \left(H^1(\Psi)\right)^3, \text{with } t_1, t_2, t_3 \text{ satisfy (4)-(6), respectively on } \partial\Psi\}.$$

By multiplying both sides of equations (1),(2) and (3) by $t_1 \in T_1, t_2 \in T_2$ $t_3 \in T_3$, respectively, integrating both sides of each one of the obtained equations with respect to χ, and then using the generalize Green's theorem, we get

$$a_1(b_1, t_1) - (b_2 + b_3, t_1)_{L_2(\Psi)} + (m_1(\chi, b_1), t_1)_{L_2(\Psi)} = (\kappa_1(\chi), t_1)_{L_2(\Psi)} + (d_1, t_1)_{L_2(\partial\Psi)}, \forall\, t_1 \in T_1 \quad (11)$$

$$a_2(b_2, t_2) + (b_1 + b_3, t_2)_{L_2(\Psi)} + (m_2(\chi, b_2), t_2)_{L_2(\Psi)} = (\kappa_2(\chi), t_2)_{L_2(\Psi)} + (d_2, t_2)_{L_2(\partial\Psi)}, \forall\, t_2 \in T_2 \quad (12)$$

and

$$a_3(b_3, t_3) + (b_1 - b_2, t_3)_{L_2(\Psi)} + (m_3(\chi, b_3), t_3)_{L_2(\Psi)} = (\kappa_3(\chi), t_3)_{L_2(\Psi)} + (d_3, t_3)_{L_2(\partial\Psi)}, \forall\, t_3 \in T_3 \quad (13)$$

By adding equations (11), (12) and (13), we get

$$a(\vec{b}, \vec{t}) + (m_1(\chi, b_1), t_1)_{L_2(\Psi)} + (m_2(\chi, b_2), t_2)_{L_2(\Psi)} + (m_3(\chi, b_3), t_3)_{L_2(\Psi)} = (\kappa_1(\chi), t_1)_{L_2(\Psi)} + (d_1, t_3)_{L_2(\partial\Psi)} + (\kappa_2(\chi), t_2)_{L_2(\Psi)} + (d_2, t_2)_{L_2(\partial\Psi)} + (\kappa_3(\chi), t_3)_{L_2(\Psi)} + (d_3, t_3)_{L_2(\partial\Psi)}, \forall\, (t_1, t_2, t_3) \in T \quad (14)$$

where

$$a(\vec{b}, \vec{t}) = a_1(b_1, t_1) - (b_2 + b_3, t_1)_{L_2(\Psi)} + a_2(b_2, t_2) + (b_1 + b_3, t_2)_{L_2(\Psi)} + a_3(b_3, t_3) + (b_1 + b_2, t_3)_{L_2(\Psi)}$$

with

$$a_r(b_r, t_r) = \int_\Psi \left(\sum_{i,j=1}^{2} a_{r_{ij}} \frac{\partial b_r}{\partial \chi_i} \frac{\partial t_r}{\partial \chi_j} + b_r t_r\right) d\chi$$

which satisfies

$a_r(b_r, t_r) \ge c_{1r} \|b_r\|^2_{H^1(\Psi)}$, where $c_{1r} \ge 0$, $r = 1, 2, 3$

$|a_r(b_r, t_r)| \le c_{2r} \|b_r\|^2_{H^1(\Psi)} \|T_r\|^2_{H^1(\Psi)}$, where $c_{2r} \ge 0$, $r = 1, 2, 3$.

The following assumptions are useful to prove the existence theorem of a unique solution of the weak form (14).

Assumption (I):

a) $a(\vec{b}, \vec{t})$ is coercive , i.e , $\frac{a(\vec{b}, \vec{b})}{\|\vec{b}\|_{(H^1(\Psi))^3}} \ge c \|\vec{b}\|_{(H^1(\Psi))^3} > 0$, $\forall \vec{b} \in \vec{T}$

b) $a(\vec{b},\vec{t})$ is continuous, i.e.

$|a(\vec{b},\vec{t})| \le \ell 1 \|\vec{b}\|_{(H^1(\Psi))^3} \|\vec{t}\|_{(H^1(\Psi))^3}$, $\ell 1 > 0$, $\forall \vec{b},\vec{t} \in \vec{T}$

c) m_1 , m_2 and m_3 are of Carathéodory type on $\Psi \times \mathbb{R}$ and the following sub linearity conditions with respect to b_1, b_2, b_3 are satisfied, respectively, i.e.

$|m_\sigma(x, b_\sigma)| \le \phi_\sigma(x) + \bar{c}_\sigma |b_\sigma|$, $\forall (x, b_\sigma) \in \Psi \times \mathbb{R}$ with $\phi_\sigma \in L_2(\Psi)$, $\bar{c}_i > 0$ for $\sigma = 1, 2, 3$

d) $m_\sigma(x, b_\sigma)$ are monotone with respect to b_σ for each $x \in \Psi$, and $m_\sigma(x, 0) = 0$, $\forall x \in \Psi$, $\sigma = 1, 2, 3$.

e) $ƙ_\sigma(x)$ are of the Carathéodory type on Ψ and satisfy $|ƙ_\sigma(x)| \le \phi j(x)$, $\forall x \in \Psi$, with $\phi j(x) \in L_2(\Psi)$, $\sigma, j = 1, 2, 3$

Theorem (1): If assumption (I) is hold, and if one of the functions m_1, m_2 or m_3 in (14) is strictly monotone, then for each fixed classical continuous boundary optimal control vector $\vec{d} \in \vec{E}_A$, the weak form of (14) has a unique "state" solution vector $\vec{b} \in \vec{T}$.

Proof: It is clear that the existence of a unique solution of (14) is obtained after the usage of assumptions (I), then theorem (1) in reference [18] is applied.

4. Existence of the Classical Continuous Boundary Optimal Control Vector

In this section, the theorem of the existence of a classical continuous boundary optimal control vector under the suitable assumptions is proved. However, before proving it, it is necessary to deal with the following lemmas and assumptions.

Lemma (1): If the assumption (I) is hold, the functions m_1, m_2, m_3 are Lipschitz continuous with respect to b_1, b_2, b_3, res respectively, and if $ƙ_1(x), ƙ_2(x), ƙ_3(x)$ are bounded, then the mapping $\vec{d} \longrightarrow \vec{b}_{\vec{d}}$ is Lipschitz continuous from $\vec{E}_{\vec{D}}$ into $(L_2(\Psi))^3$, i.e.

$\|\overrightarrow{\Delta b}\|_{(L_2(\Psi))^3} \le L \|\overrightarrow{\Delta d}\|_{(L_2(\partial\Psi))^3}$, with $L > 0$.

Proof: Assume that $\vec{d}, \vec{d}' \in \vec{E}$ are two given controls, then there corresponding "state" solution vectors (of the weak form (14)) are $\vec{b}, \vec{b}'$. By subtracting the above three obtained weak forms from their corresponding ones in (14), putting $\overrightarrow{\Delta b} = \vec{b}' - \vec{b}$ and $\overrightarrow{\Delta d} = \vec{d}' - \vec{d}$, with $\vec{t} = \overrightarrow{\Delta b}$, then adding the obtained three equations, we get

$$a_1(\Delta b_1, \Delta b_1) + a_2(\Delta b_2, \Delta b_2) + a_3(\Delta b_3, \Delta b_3) + (m_1(x, b_1 + \Delta b_1) - m_1(x, b_1), \Delta b_1)_{L_2(\Psi)} + (m_2(x, b_2 + \Delta b_2) - m_2(x, b_2), \Delta b_2)_{L_2(\Psi)} + (m_3(x, b_3 + \Delta b_3) - m_3(x, b_3), \Delta b_3)_{L_2(\Psi)} = (\Delta d_1, \Delta b_1)_{L_2(\partial\Psi)} + (\Delta d_2, \Delta b_2)_{L_2(\partial\Psi)} + (\Delta d_3, \Delta b_3)_{L_2(\partial\Psi)} \quad (16)$$

By using assumption A-(a, d) , taking the absolute value for both sides of (16), it becomes

$$c\|\overrightarrow{\Delta b}\|^2_{(H^1(\Psi))^3} \le \theta_1 \|\Delta b_1\|^2_{H^1(\Psi)} + \theta_2 \|\Delta b_2\|^2_{H^1(\Psi)} + \theta_3 \|\Delta b_3\|^2_{H^1(\Psi)} \le |(\Delta d_1, \Delta b_1)_{L_2(\partial\Psi)}| + |(\Delta d_2, \Delta b_2)_{L_2(\partial\Psi)}| + |(\Delta d_3, \Delta b_3)_{L_2(\partial\Psi)}| \quad (17)$$

By using the Cauchy-Schwarz inequality and then the trace operator in the right side, on (17) , we obtain

$$c\|\overrightarrow{\Delta b}\|^2_{(H^1(\Psi))^3} \le 3c_1 \|\overrightarrow{\Delta d}\|_{(L_2(\partial\Psi))^3} + \|\overrightarrow{\Delta b}\|_{(H^1(\Psi))^3} \Rightarrow$$

$$\|\overrightarrow{\Delta b}\|_{(H^1(\Psi))^3} \le L^2 \|\overrightarrow{\Delta d}\|_{(L_2(\partial\Psi))^3} \text{, where } L^2 = \frac{3c_1}{c} \quad (18)$$

which gives

$$\|\overrightarrow{\Delta b}\|_{(L_2(\Psi))^3} \le L \|\overrightarrow{\Delta d}\|_{(L_2(\partial\Psi))^3} \quad (19)$$

Assumption (II):

Assume that τ_P1 ,τ_P2 ,τ_P3 on Ψ×R and τ_P4 ,τ_P5 ,τ_P6 on Ψ×D are of the Carathéodory type, then the following are satisfied for each P=0,1,2:

|τ_P1 (x ,b_1)|≤Y_(P_1) (x)+c_P1 b_1^2 , |τ_P2 (x ,b_2)|≤Y_(P_2) (x)+c_P2 b_2^2 ,

$|\tau_{P3}(x, b_3)| \le Y_{P_3}(x) + c_{P3} b_3^2$, $|\tau_{P4}(x, d_1)| \le Y_{P_4}(x) + c_{P4} d_1^2$,

$|\tau_{P5}(x, d_2)| \le Y_{P_5}(x) + c_{P5} d_2^2$, and $|\tau_{P6}(x, d_3)| \le Y_{P_6}(x) + c_{P6} d_3^2$, where $\Upsilon_{P_1}, \Upsilon_{P_2}, \Upsilon_{P_3} \in L_1(\Psi), \Upsilon_{P_4}, \Upsilon_{P_5}, \Upsilon_{P_6} \in L_1(\partial\Psi)$ and $c_{P_\sigma} \geq 0$ for $\sigma = 1,2,3,4,5,6$.

Lemma (2): If assumption (II) is held, then the functional Ⴀ_P (d ⃗) is continuous on (L_2 (∂Ψ))^3 for each P=0,1,2.Proof: For any $P = 0,1,2$, we set

$\mathfrak{p}_{P_1}(x,\vec{b}) = \mathfrak{v}_{P1}(x, b_1) + \mathfrak{v}_{P2}(x, b_2) + \mathfrak{v}_{P3}(x, b_3)$ and

$\mathfrak{p}_{P_2}(x,\vec{d}) = \mathfrak{v}_{P4}(x, d_1) + \mathfrak{v}_{P5}(x, d_2) + \mathfrak{v}_{P6}(x, d_3)$.

To prove the continuity for any one of the above two integrals, the used technique will be similar. Thus, it is enough to prove one of them, which is in this case the second integral. Hence, let $\vec{d} = (d_1, d_2, d_3)$, with $\mathfrak{p}_{P_2}: \Psi \times \mathbb{R}^3 \to \mathbb{R}$, then from assumption (II), we have

$$\| \mathfrak{p}_{P_2}(x,\vec{d}) \| \leq \Upsilon_{P_7}(x) + c_{P4}d_1^2 + c_{P5}d_2^2 + c_{P6}d_3^2$$
$$\leq \Upsilon_{P_7}(x) + c_{P7}\|\vec{d}\|^2$$

where $\Upsilon_{P_7} = \Upsilon_{P_4} + \Upsilon_{P_5} + \Upsilon_{P_6}$, $c_{P7} = \max(c_{P4}, c_{P5}, c_{P6})$, $\Upsilon_{P_7} \in L_1(\partial\Psi)$, $c_{P7} \in L_\infty(\mathbb{R})$.

Then, the $\int_{\partial\Psi} \mathfrak{p}_{P_2}(x,\vec{d})d\gamma$ is continuous on $\left(L_2(\partial\Psi)\right)^3$ (by using Proposition (1) in reference [19]). Hence,

$\mho_P(\vec{d}) = \iint_\Psi \mathfrak{p}_{P_1}(x,\vec{b})\,dx_1dx_2 + \int_{\partial\Psi}\mathfrak{p}_{P_2}(x,\vec{d})d\gamma$ is continuous on $\left(L_2(\partial\Psi)\right)^3$.

Theorem (2): If the assumptions (I) and (II) are hold, $\vec{E}_A \neq \emptyset$, m_1, m_2, m_3 are not dependent on d_1, d_2, d_3, respectively , and $\mathfrak{k}_1, \mathfrak{k}_2, \mathfrak{k}_3$ are bounded functions , so that,

$|m_1(x, b_1)| \leq \emptyset_{1(x)} + \bar{c}_1|b_1|$, $|m_2(x, b_2)| \leq \emptyset_{2(x)} + \bar{c}_2|b_2|$, $|m_3(x, b_3)| \leq \emptyset_{3(x)} + \bar{c}_3|b_3|$,

$|\mathfrak{k}_1(x)| \leq m_1$, $|\mathfrak{k}_2(x)| \leq m_2$, and $|\mathfrak{k}_3(x)| \leq m_3$,

where $\emptyset_\sigma \in L_2(\Psi)$, $\bar{c}_\sigma \geq 0$, and $m_i \geq 0$, for $\sigma = 1,2,3$.

$\mathfrak{v}_{11}, \mathfrak{v}_{12}, \mathfrak{v}_{13}$ are not dependent on $d_1, d_2\, d_3$, respectively . $\mathfrak{v}_{P4}, \mathfrak{v}_{P5}, \mathfrak{v}_{P6}$ $(P = 0,2)$ are convex with respect to d_1, d_2, d_3, respectively, for fixed x. Then there exists a continuous classical boundary optimal control vector.

Proof: The set E_σ and D_σ (∀ σ=1,2,3) is convex and bounded, then E_1×E_2×E_3is convex and bounded. On the other hand, by using theorem (2) in reference [19], E_σ ∀ σ=1,2,3 is closed, since D_σ is closed, then E_1×E_2×E_3 is closed, too. Therefore, we obtain that E_1×E_2×E_3 is weakly compact.

From the assumption on $\vec{E}_A$, there is an element $\vec{e} \in \vec{E}_A$. Then there is a minimum sequence $\{\vec{d}_n\} = \{(d_{1n}, d_{2n}, d_{3n})\} \in \vec{E}_A$ for each n, with $\mho_1(\vec{d}_n) = 0$, $\mho_2(\vec{d}_n) \leq 0$, , so that

$\lim_{n\to\infty} \mho_0(\vec{d}_n) = \inf_{\vec{e}\in\vec{E}} \mho_0(\vec{e})$.

But $\vec{E}$ is weakly compact, then there is a subsequence of $\{\vec{d}_n\}$, which will be symbolized again by $\{\vec{d}_n\}$, that converges weakly to $\vec{d}$ in $\vec{E}$.

Then, corresponding to the $\{\vec{d}_n\}$, there is the sequence of the "state" solution vector $\{\vec{b}_n\}$ of the sequence of the weak form. Then, from the proof of Theorem (3), we have:

$$a_1(b_{1n}, t_1) - (b_{2n} + b_{3n}, t_1)_{L_2(\Psi)} + a_2(b_{2n}, t_2) + (b_{1n} + b_{3n}, t_2)_{L_2(\Psi)} + a_3(b_{3n}, t_3) + (b_{1n} - b_{2n}, t_3)_{L_2(\Psi)} + (m_1(x, b_{1n}), t_1)_{L_2(\Psi)} + (m_2(x, b_{2n}), t_2)_{L_2(\Psi)} + (m_3(x, b_{3n}), t_3)_{L_2(\Psi)}$$
$$= (\mathfrak{k}_1(x), t_1)_{L_2(\Psi)} + (d_{1n}, t_3)_{L_2(\partial\Psi)} + (\mathfrak{k}_2(x), t_2)_{L_2(\Psi)} + (d_{2n}, t_2)_{L_2(\partial\Psi)} + (\mathfrak{k}_3(x), t_3)_{L_2(\Psi)} + (d_{3n}, t_3)_{L_2(\partial\Psi)} \quad (20)$$

With $\|\vec{b}_n\|_{\left(H^1(\Psi)\right)^3}$ for each n is bounded, then $\{\vec{b}_n\}$ has a subsequence, which will be symbolized again by $\{\vec{b}_n\}$, such that $\vec{b}_n \to \vec{b}$ weakly in $\vec{V}$ (Alaoglu theorem [20]).

Now, we have to show that (20) converges to

$$a_1(b_1, t_1) - (b_2 + b_3, t_1)_{L_2(\Psi)} + a_2(b_2, t_2) + (b_1 + b_3, t_2)_{L_2(\Psi)} + a_3(b_3, t_3) + (b_1 - b_2, t_3)_{L_2(\Psi)} + (m_1(x, b_1), t_1)_{L_2(\Psi)} + (m_2(x, b_2), t_2)_{L_2(\Psi)} + (m_3(x, b_3), t_3)_{L_2(\Psi)}$$
$$= (\mathfrak{k}_1(x), t_1)_{L_2(\Psi)} + (d_1, t_3)_{L_2(\partial\Psi)} + (\mathfrak{k}_2(x), t_2)_{L_2(\Psi)} + (d_2, t_2)_{L_2(\partial\Psi)} + (\mathfrak{k}_3(x), t_3)_{L_2(\Psi)} + (d_3, t_3)_{L_2(\partial\Psi)} \quad (21)$$

First, let $(t_1, t_2, t_3) \in (C(\overline{\Psi}))^3$, and, first for the left hand side, since $b_{\sigma n} \to b_\sigma$ weakly in T_σ, i.e $b_{\sigma n} \to b_\sigma$ weakly in $L_2(\Psi)$, for each $\sigma = 1,2,3$, then from the left hand side of (20) and (21) and by using the Cauchy-Schwarz inequality, one has

$$\big| a_1(b_{1n}, t_1) - (b_{2n} + b_{3n}, t_1)_{L_2(\Psi)} + a_2(b_{2n}, t_2) + (b_{1n} + b_{3n}, t_2)_{L_2(\Psi)} + a_3(b_{3n}, t_3) + (b_{1n} - b_{2n}, t_3)_{L_2(\Psi)} - a_1(b_1, t_1) + (b_2 + b_3, t_1)_{L_2(\Psi)} - a_2(b_2, t_2) - (b_1 + b_3, t_2)_{L_2(\Psi)} - a_3(b_3, t_3) - (b_1 - b_2, t_3)_{L_2(\Psi)} \big|$$
$$\leq (c_1\| b_{1n} - b_1\|_{H^1(\Psi)} + \| b_{2n} - b_2\|_{L_2(\Psi)} + \| b_{3n} - b_3\|_{L_2(\Psi)})\| t_1\|_{L_2(\Psi)} + $$
$$+(c_2\| b_{2n} - b_2\|_{H^1(\Psi)} + \| b_{1n} - b_1\|_{L_2(\Psi)} + \| b_{3n} - b_3\|_{L_2(\Psi)})\| t_2\|_{L_2(\Psi)} +$$
$$(c_3\| b_{3n} - b_3\|_{H^1(\Psi)} + \| b_{1n} - b_1\|_{L_2(\Psi)} + \| b_{2n} - b_2\|_{L_2(\Psi)})\| t_3\|_{L_2(\Psi)} \to 0 \quad (22)$$

i. From assumption (II) and Proposition (1), the functions $\iint_\Psi m_1(\mathfrak{x}, b_{1n})t_1 d\mathfrak{x}_1 d\mathfrak{x}_2$ $\iint_\Psi m_2(\mathfrak{x}, b_{2n})t_2\, d\mathfrak{x}_1 d\mathfrak{x}_2$ and $\iint_\Psi m_3(\mathfrak{x}, b_{3n})t_3 d\mathfrak{x}_1 d\mathfrak{x}_2$ are continuous with respect to b_{1n}, b_{2n} and b_{3n}, respectively.

But $\vec{b}_n \to \vec{b}$ weakly in $(L_2(\Psi))^3$, because $\vec{b}_n \to \vec{b}$ weakly in $\vec{T}$, then by using the Rellich-Kondrachov theorem in [21], we get that $\vec{b}_n \to \vec{b}$ strongly in $(L_2(\Psi))^3$, hence

$$(m_1(\mathfrak{x}, b_{1n}), t_1)_{L_2(\Psi)} + (m_2(\mathfrak{x}, b_{2n}), t_2)_{L_2(\Psi)} + (m_3(\mathfrak{x}, b_{3n}), t_3)_{L_2(\Psi)}$$
$$\longrightarrow (m_1(\mathfrak{x}, b_1), t_1)_{L_2(\Psi)} + (m_2(\mathfrak{x}, b_2), t_2)_{L_2(\Psi)} + (m_3(\mathfrak{x}, b_3), t_3)_{L_2(\Psi)} \quad (23a),$$

i.e. the left hand side of (20) ⟶ the left hand side of (21).

Second, since $d_{1n} \longrightarrow d_1, d_{2n} \longrightarrow d_2$ and $d_{3n} \longrightarrow d_3$ weakly in $L_2(\partial\Psi)$, then

$$(d_{1n} - d_1, t_1)_{L_2(\partial\Psi)} + (d_{2n} - d_2, t_2)_{L_2(\partial\Psi)} + (d_{3n} - d_3, t_3)_{L_2(\partial\Psi)} \longrightarrow 0 \quad (23b)$$

From (23a) and (23b), we obtain that (20) converges to (21).

Since $(C(\overline{\Psi}))^3$ is dense in $\vec{V}$, then this convergence satisfies for any $(t_1, t_2, t_3) \in \vec{T}$. This leads to $\vec{b}_n \longrightarrow \vec{b} = \vec{b}_{\vec{d}}$ is a solution of the weak form of the triple state equations.

From Lemma (2), the functional $\mho_P(\vec{d})$ is continuous on $(L_2(\partial\Psi))^3$, $\forall P = 0,1,2$.

From the assumptions on $\mho_{11}, \mho_{12}, \mho_{13}$, $\mho_1(\vec{d}_n)$ is continuous and from the strongly converged $b_{1n} \longrightarrow b_1, b_{2n} \longrightarrow b_2$ and $b_{3n} \longrightarrow b_3$ in $L_2(\Psi)$, we get

$\mho_1(\vec{d}) = \lim_{n\to\infty} \mho_1(\vec{d}_n) = 0.$

Also, from the assumptions on $\mho_{P1}(\mathfrak{x}, b_1)$ and $\mho_{P4}(\mathfrak{x}, d_1)$ $(\forall P = 0,2)$ and Lemma (2), the integrals $\iint_\Psi \mho_{P1}(\mathfrak{x}, b_1)d\mathfrak{x}_1 d\mathfrak{x}_2$ and $\int_{\partial\Psi}\mho_{P4}(\mathfrak{x}, d_1)d\gamma$ are continuous with respect to b_1 and d_1, respectively, but $\mho_{P4}(\mathfrak{x}, d_1)$, $(\forall P = 0,2)$ is convex with respect to d_1, then $\int_{\partial\Psi}\mho_{P4}(\mathfrak{x}, d_1)d\gamma$ is weakly lower semicontinuous with respect to d_1, i.e.

$$\iint_\Psi \mho_{P1}(\mathfrak{x}, b_1)d\mathfrak{x}_1 d\mathfrak{x}_2 + \int_{\partial\Psi}\mho_{P4}(\mathfrak{x}, d_1)d\gamma$$
$$\leq \iint_\Psi \mho_{P1}(\mathfrak{x}, b_1)d\mathfrak{x}_1 d\mathfrak{x}_2 + \underline{\lim}_{n\to\infty} \int_{\partial\Psi}\mho_{P4}(\mathfrak{x}, d_{1n})d\gamma$$
$$= \underline{\lim}_{n\to\infty} \iint_\Psi [\mho_{P1}(\mathfrak{x}, b_1) - \mho_{P1}(\mathfrak{x}, b_{1n})]d\mathfrak{x}_1 d\mathfrak{x}_2$$
$$+ \underline{\lim}_{n\to\infty} \int_{\partial\Psi}\mho_{P4}(\mathfrak{x}, d_{1n})d\gamma + \iint_\Psi \mho_{P1}(\mathfrak{x}, b_{1n})d\mathfrak{x}_1 d\mathfrak{x}_2$$
$$= \underline{\lim}_{n\to\infty} \left(\iint_\Psi \mho_{P1}(\mathfrak{x}, b_{1n})d\mathfrak{x}_1 d\mathfrak{x}_2 + \int_{\partial\Psi}\mho_{P4}(\mathfrak{x}, d_{1n})d\gamma\right)$$

By the same manner, and for each $P = 0,2$, we get the following two convergences:

$$\iint_\Psi \mho_{P2}(\mathfrak{x}, b_2)d\mathfrak{x}_1 d\mathfrak{x}_2 + \int_{\partial\Psi}\mho_{P5}(\mathfrak{x}, d_2)d\gamma$$
$$\leq \underline{\lim}_{n\to\infty} \left(\iint_\Psi \mho_{P2}(\mathfrak{x}, b_{2n})d\mathfrak{x}_1 d\mathfrak{x}_2 + \int_{\partial\Psi}\mho_{P5}(\mathfrak{x}, d_{2n})d\gamma\right)$$

and

$$\iint_\Psi \mho_{P3}(\mathfrak{x}, b_3)d\mathfrak{x}_1 d\mathfrak{x}_2 + \int_{\partial\Psi}\mho_{P6}(\mathfrak{x}, d_3)d\gamma$$
$$\leq \underline{\lim}_{n\to\infty} \left(\iint_\Psi \mho_{P3}(\mathfrak{x}, b_{3n})d\mathfrak{x}_1 d\mathfrak{x}_2 + \int_{\partial\Psi}\mho_{P6}(\mathfrak{x}, d_{3n})d\gamma\right)$$

From the above inequalities, one gets that $\mho_{P\ell}(\vec{d})$, $(\forall P = 0,2)$ is weakly lower semicontinuous with respect to $(\vec{b}, \vec{d})$. Thus $\mho_2(\vec{d}) \leq \underline{\lim}_{n\to\infty} \mho_2(\vec{d}_n) \leq 0$, and

$Ʊ_0(\vec{d}) \leq \underset{n\to\infty}{\underline{\lim}}\, Ʊ_0(\vec{d}_n) = \lim_{n\to\infty} Ʊ_0(\vec{d}_n) = \inf_{\vec{w}\in\vec{W}} Ʊ_0(\vec{w}) \ \Rightarrow$

$\vec{d}$ is a continuous classical boundary optimal control vector .

5. The Necessary and Sufficient Conditions for Optimality of the Continuous Classical Boundary Optimal Control Vector

The following assumptions are useful in this section to derive the Fréchet derivative of the Hamiltonian.

Assumption (III)

a) $m_{1_{b_1}}, m_{2_{b_2}}, m_{3_{b_3}}$ are of the Carathéodory type on $\Psi \times \mathbb{R}$ and satisfy

$\left|m_{1_{b_1}}(x, b_1)\right| \leq \breve{c}_1, \left|m_{2_{b_2}}(x, b_2)\right| \leq \breve{c}_2, \left|m_{3_{b_3}}(x, b_3)\right| \leq \breve{c}_3$, with $\breve{c}_1, \breve{c}_2, \breve{c}_3 \geq 0$

$m_{1_{b_1}}(x, b_1) \geq 0, m_{2_{b_2}}(x, b_2) \geq 0$ and $m_{3_{b_3}}(x, b_3) \geq 0$.

b) $ꞕ_1, ꞕ_2, ꞕ_3$ are of the Carathéodory type on Ψ and satisfy

$|ꞕ_1(x)| \leq \breve{c}_4, |ꞕ_2(x)| \leq \breve{c}_5$ and $|ꞕ_3(x)| \leq \breve{c}_6$, with $\breve{c}_4, \breve{c}_5, \breve{c}_6 \geq 0$.

c) $ʊ_{P1_{b_1}}, ʊ_{P2_{b_2}}, ʊ_{P3_{b_3}}, ʊ_{P4_{d_1}}, ʊ_{P5_{d_2}}, ʊ_{P6_{d_3}}$ $(\forall\, P = 0,1,2)$ are of the Carathéodory type on $\Psi \times \mathbb{R}$ and satisfy

$\left|ʊ_{P_{1_{b_1}}}\right| \leq \Upsilon_{P_1} + c_{P1}|b_1|, \left|ʊ_{P_{2_{b_2}}}\right| \leq \Upsilon_{P_2} + c_{P2}|b_1|, \left|ʊ_{P_{3_{b_3}}}\right| \leq \Upsilon_{\ell_3} + c_{P3}|b_1|,$

$\left|ʊ_{P_{4_{d_1}}}\right| \leq \Upsilon_{P_4} + c_{P4}|d_1|, \left|ʊ_{P_{5_{d_2}}}\right| \leq \Upsilon_{P_5} + c_{P5}|d_2|$, and $\left|ʊ_{P_{6_{d_3}}}\right| \leq \Upsilon_{P_6} + c_{P6}|d_3|$

where $c_{P\sigma} \geq 0$, $\Upsilon_{P_1}, \Upsilon_{P_2}, \Upsilon_{P_3} \in L_2(\Psi)$ and $\Upsilon_{P_4}, \Upsilon_{P_5}, \Upsilon_{P_6} \in L_2(\partial\Psi)$, for $\sigma = 1,2,3,4,5,6$ and $P = 0,1,2$.

Theorem (3): If the assumptions (I), (II), and (III) are hold, the Hamiltonian is given as:

$$\begin{aligned} &И(x, b_1, b_2, b_3, \zeta_1, \zeta_2, \zeta_3, d_1, d_2, d_3) \\ &= \zeta_1\big(ꞕ_1(x) - m_1(x, b_1)\big) + ʊ_{01}(x, b_1) + ʊ_{04}(x, d_1) + \zeta_2\big(ꞕ_2(x) - m_2(x, b_2)\big) \\ &\quad + ʊ_{02}(x, b_2) + ʊ_{05}(x, d_2) + \zeta_3\big(ꞕ_3(x) - m_3(x, b_3)\big) + ʊ_{03}(x, b_3) + ʊ_{06}(x, d_3) \end{aligned}$$

The triple adjoint equations of the triple state equations (1-6) are :

$$A_1 \zeta_1 + \zeta_1 + \zeta_2 + \zeta_3 + \zeta_1 m_{1b_1}(x, b_1) = ʊ_{01b_1}(x, b_1), \text{ in } \Psi \quad (24)$$

$$A_2 \zeta_2 - \zeta_1 + \zeta_2 - \zeta_3 + \zeta_2 m_{2b_2}(x, b_2) = ʊ_{02b_2}(x, b_2), \text{ in } \Psi \quad (25)$$

$$A_3 \zeta_3 - \zeta_1 + \zeta_2 + \zeta_3 + \zeta_3 m_{3b_3}(x, b_3) = ʊ_{03b_3}(x, b_3), \text{ in } \Psi \quad (26)$$

$$\frac{\partial \zeta_1}{\partial n_1} = 0, \text{ in } \partial\Psi \quad (27)$$

$$\frac{\partial \zeta_2}{\partial n_2} = 0, \text{ in } \partial\Psi \quad (28)$$

$$\frac{\partial \zeta_3}{\partial n_3} = 0, \text{ in } \partial\Psi \quad (29)$$

Then the Fréchet derivative of $Ʊ_0$ is

$\overrightarrow{Ʊ_0'}(\vec{d}).\overrightarrow{\Delta d} = \int_{\partial\Psi} И_{\vec{d}}^{\mathrm{T}}.\overrightarrow{\Delta d}\, d\gamma$, where

$И_{\vec{d}}^{\mathrm{T}} = \begin{pmatrix} И_{d_1}(x, \zeta_1, \zeta_2, \zeta_3, d_1, d_2, d_3) \\ И_{d_2}(x, \zeta_1, \zeta_2, \zeta_3, d_1, d_2, d_3) \\ И_{d_3}(x, \zeta_1, \zeta_2, \zeta_3, d_1, d_2, d_3) \end{pmatrix} = \begin{pmatrix} \zeta_1 + ʊ_{04d_1} \\ \zeta_2 + ʊ_{05d_2} \\ \zeta_3 + ʊ_{06d_3} \end{pmatrix}$ and $\vec{\zeta} = \vec{\zeta}_{\vec{d}}$ is the triple adjoint equation of the triple state equation $\vec{y}_{\vec{d}}$.

Proof: Formulating the triple adjoint equations (24-29) by their weak forms, then adding them, and then setting $\vec{t} = \overrightarrow{\Delta b}$ in the resulting equation, yield

$$\begin{aligned} &a_1(\zeta_1, \Delta b_1) + (\zeta_2 + \zeta_3, \Delta b_1)_{L_2(\Psi)} + a_2(\zeta_2, \Delta b_2) - (\zeta_1 + \zeta_3, \Delta b_2)_{L_2(\Psi)} + a_3(\zeta_3, \Delta b_3) \\ &- (\zeta_1 - \zeta_2, \Delta b_3)_{L_2(\Psi)} + (\zeta_1 m_{1b_1}(b_1), \Delta b_1)_{L_2(\Psi)} + (\zeta_2 m_{2b_2}(b_2), \Delta b_2)_{L_2(\Psi)} \\ &+ (\zeta_3 m_{3b_3}(b_3), \Delta b_3)_{L_2(\Psi)} = (ʊ_{01b_1}(b_1), \Delta b_1)_{L_2(\Psi)} \\ &\qquad + (ʊ_{02b_2}(b_2), \Delta b_2)_{L_2(\Psi)} + (ʊ_{03b_3}(b_3), \Delta b_3)_{L_2(\Psi)} \end{aligned} \quad (30)$$

One can easily prove that the weak form (30), with fixed continuous classical boundary optimal control vector $\vec{d} \in \vec{E}$, has a unique "state" solution vector $\vec{\zeta} = \vec{\zeta}_{\vec{d}}$, by applying the same manner employed in the proof of theorem (3).

Now, by setting once the solution b_1 in the weak forms of the state equations (11) and once again the solution $b_1 + \Delta b_1$, then subtracting the 1st obtained weak form from the other one, we obtain

$$a_1(\Delta b_1, t_1) - (\Delta b_2 + \Delta b_3, t_1)_{L_2(\Psi)} + (m_1(b_1 + \Delta b_1) - m_1(b_1), t_1)_{L_2(\Psi)} = (\Delta d_1, t_1)_{L_2(\partial\Psi)} \ \forall t_1 \in T_1 \quad (31)$$

The same above substituting and subtracting are repeated but from a side with the solutions b_2 and $b_2 + \Delta b_2$ in the weak form of equation (12) and from thither side with the solutions b_3 and $b_3 + \Delta b_3$ in the weak form of the state equation (13), respectively, to obtain

$$a_2(\Delta b_2, t_2) + (\Delta b_1 + \Delta b_3, t_2)_{L_2(\Psi)} + (m_2(b_2 + \Delta b_2) - m_2(b_2), t_2)_{L_2(\Psi)} = (\Delta d_2, t_2)_{L_2(\partial\Psi)} \ \forall t_2 \in T_2 \quad (32)$$

$$a_3(\Delta b_3, t_3) + (\Delta b_1 - \Delta b_3, t_3)_{L_2(\Psi)} + (m_3(b_3 + \Delta b_3) - m_3(b_3), t_3)_{L_2(\Psi)} = (\Delta d_3, t_3)_{L_2(\partial\Psi)} \ \forall t_3 \in T_3 \quad (33)$$

Adding (31),(32) and (33), then substituting $\vec{t} = (\zeta_1, \zeta_2, \zeta_3)$ in the resulting equation, yield

$$a_1(\Delta b_1, \zeta_1) - (\Delta b_2 + \Delta b_3, \zeta_1)_{L_2(\Psi)} + a_2(\Delta b_2, \zeta_2) + (\Delta b_1 + \Delta b_3, \zeta_2)_{L_2(\Psi)}$$
$$+ a_3(\Delta b_3, \zeta_3) + (\Delta b_1 - \Delta b_2, \zeta_3)_{L_2(\Psi)} + ((m_1(x, b_1 + \Delta b_1), \zeta_1) - m_1(x, b_1), \zeta_1)_{L_2(\Psi)}$$
$$+ ((m_2(x, b_2 + \Delta b_2), \zeta_2) - m_2(x, b_2), \zeta_2)_{L_2(\Psi)} + ((m_3(x, b_3 + \Delta b_3), \zeta_3) - m_3(x, b_3), \zeta_3)_{L_2(\Psi)}$$
$$= (\Delta d_1, \zeta_1)_{L_2(\partial\Psi)} + (\Delta d_2, \zeta_2)_{L_2(\partial\Psi)} + (\Delta d_3, \zeta_3)_{L_2(\partial\Psi)}, \forall = (\zeta_1, \zeta_2, \zeta_3) \in \vec{V} \quad (34)$$

From the assumptions on m_1, m_2, m_3 and by using Proposition (2) in reference [19], the Fréchet derivative of m_1, m_2, m_3 exists. Hence, from Lemma (1) and the Minkowski inequality, (34) becomes

$$a_1(\Delta b_1, \zeta_1) - (\Delta b_2 + \Delta b_3, \zeta_1)_{L_2(\Psi)} + a_2(\Delta b_2, \zeta_2) + (\Delta b_1 + \Delta b_3, \zeta_2)_{L_2(\Psi)}$$
$$a_3(\Delta b_3, \zeta_3) + (\Delta b_1 - \Delta b_2, \zeta_3)_{L_2(\Psi)} + (m_{1_{b_1}}\Delta b_1, \zeta_1)_{L_2(\Psi)} + \tilde{\varepsilon}_1(\overrightarrow{\Delta u})\|\overrightarrow{\Delta u}\|_{(L_2(\partial\Psi))^3}$$
$$+(m_{2_{b_2}}\Delta b_2, \zeta_2)_{L_2(\Psi)} + \tilde{\varepsilon}_2(\overrightarrow{\Delta u})\|\overrightarrow{\Delta u}\|_{(L_2(\partial\Psi))^3} + (m_{3_{b_3}}\Delta b_3, \zeta_3)_{L_2(\Psi)}$$
$$+ \tilde{\varepsilon}_3(\overrightarrow{\Delta u})\|\overrightarrow{\Delta u}\|_{(L_2(\partial\Psi))^3}$$
$$= (\Delta d_1, \zeta_1)_{L_2(\partial\Psi)} + (\Delta d_2, \zeta_2)_{L_2(\partial\Psi)} + (\Delta d_3, \zeta_3)_{L_2(\partial\Psi)} \quad (35)$$

where $\tilde{\varepsilon}_1(\overrightarrow{\Delta d}), \tilde{\varepsilon}_2(\overrightarrow{\Delta d})$ and $\tilde{\varepsilon}_3(\overrightarrow{\Delta d}) \longrightarrow 0$ and $\|\overrightarrow{\Delta d}\|_{(L_2(\partial\Psi))^3} \longrightarrow 0$ as $\overrightarrow{\Delta d} \longrightarrow 0$.

Subtracting (30) from (35), to get

$$(\mho_{01b_1}(b_1), \Delta b_1)_{L_2(\Psi)} + (\mho_{02b_2}(b_2), \Delta b_2)_{L_2(\Psi)} + (\mho_{03b_3}(b_3), \Delta b_3)_{L_2(\Psi)}$$
$$+ \tilde{\varepsilon}_1(\overrightarrow{\Delta d})\|\overrightarrow{\Delta d}\|_{(L_2(\partial\Psi))^3} + \tilde{\varepsilon}_2(\overrightarrow{\Delta d})\|\overrightarrow{\Delta d}\|_{(L_2(\partial\Psi))^3} + \tilde{\varepsilon}_3(\overrightarrow{\Delta d})\|\overrightarrow{\Delta d}\|_{(L_2(\partial\Psi))^3}$$
$$= (\Delta d_1, \zeta_1)_{L_2(\partial\Psi)} + (\Delta d_2, \zeta_2)_{L_2(\partial\Psi)} + (\Delta d_3, \zeta_3)_{L_2(\partial\Psi)} \quad (36)$$

Now, from the assumptions on $\mho_{01}, \mho_{02}, \mho_{03}, \mho_{04}, \mho_{05}, \mho_{06}$, Proposition (2) in reference [19], and then using the result of Lemma (1), we have

$$\mho_0(\vec{d} + \overrightarrow{\Delta d}) - \mho_0(\vec{d}) = \iint_\Psi (\mho_{01b_1}(x, b_1)\Delta b_1 + \mho_{02b_2}(x, b_2)\Delta b_2 + \mho_{03b_3}(x, b_3)\Delta b_3)dx_1 dx_2$$
$$+ \int_{\partial\Psi} (\mho_{04d_1}(x, d_1)\Delta d_1 + \mho_{05d_2}(x, d_2)\Delta d_2 + \mho_{06d_3}(x, d_3)\Delta d_3)d\gamma + \varepsilon_4(\overrightarrow{\Delta d})\|\overrightarrow{\Delta d}\|_{(L_2(\Psi))^3}$$

where $\varepsilon_4(\overrightarrow{\Delta d}) \longrightarrow 0$, as $\overrightarrow{\Delta d} \longrightarrow 0$

By substituting (36) in the above equality, we get

$$\mho_0(\vec{d} + \overrightarrow{\Delta d}) - \mho_0(\vec{d}) = \int_{\partial\Psi}(\zeta_1, \mho_{04d_1})\Delta d_1 d\gamma + \int_{\partial\Psi}(\zeta_2, \mho_{05d_2})\Delta d_2 d\gamma + \int_{\partial\Psi}(\zeta_3, \mho_{06d_3})\Delta d_3 d\gamma + \tilde{\varepsilon}_5(\overrightarrow{\Delta d})\|\overrightarrow{\Delta d}\|_{(L_2(\partial\Psi))^3} \quad (37)$$

where

$\tilde{\varepsilon}_5(\overrightarrow{\Delta d}) = \tilde{\varepsilon}_4(\overrightarrow{\Delta d}) - \tilde{\varepsilon}_1(\overrightarrow{\Delta d}) - \tilde{\varepsilon}_2(\overrightarrow{\Delta d}) - \tilde{\varepsilon}_3(\overrightarrow{\Delta d}) \longrightarrow 0$, as $\overrightarrow{\Delta d} \longrightarrow 0$.

But from the definition of the Fréchet derivative of $\mho_0$, one gets

$$\overrightarrow{\mho_0'}(\vec{d})\overrightarrow{\Delta d} = \int_{\partial\Psi} И_{\vec{d}}^{T} \overrightarrow{\Delta d}\, d\gamma, \text{ where } И_{\vec{d}}^{T} \text{ is defined above.} \quad (38)$$

Note: In the proof of the above theorem, we have found the Fréchet derivative for the functional $\mho_0$, so the same technique is used to find the Fréchet derivative for $\mho_1$ and $\mho_2$.

Theorem (4):

(a) If assumptions (I) ,(II), and (III) are held, then $\vec{E}$ is convex, and if $\vec{d} \in \vec{E}_A$ is a continuous classical boundary optimal control vector , then $\forall$ P = 0,1,2 and there exist multipliers $\xi_P \in \mathbb{R}$, with $\xi_0 \geq 0$, $\xi_2 \geq 0$ $\sum_{P=1}^{2} |\xi_P| = 1$, so that the following Kuhn-Tucker- Lagrange's Multipliers conditions are held:

$$\int_{\partial\Psi} И_{\vec{d}}^{T} . \overrightarrow{\Delta d}\, d\gamma \geq 0 , \text{ with } \overrightarrow{\Delta d} = \vec{w} - \vec{d} , \forall \vec{e} \in \vec{E} \quad (39a)$$

where $Ʊ_{\sigma dj} = \sum_{P=0}^{2} \xi_P Ʊ_{\sigma dj}$ & $z_j = \sum_{P=0}^{2} \xi_P z_{Pj}$, (for j = 0,1,2 , σ = 4,5,6) in (Theorem (5)),

$$\xi_2 Ʊ_2(\vec{d}) = 0 , \text{(Transversality conditions)} \quad (39b)$$

(b) (Minimum Principle in point wise weak form): The inequality (39a) is equivalent to

$$И_{\vec{d}}^{T} \vec{d} = \min_{\vec{e} \in \vec{D}} И_{\vec{d}}^{T} \vec{E} \text{ a.e. in } \partial\Psi \quad (40)$$

Proof: (a) from Theorem (3) , $Ʊ_P(\vec{d})$ $\forall$P = 0,1,2 and at any $\vec{d} \in \vec{E}$) has a continuous Fréchet derivative. Since the continuous classical boundary optimal control vector $\vec{d} \in \vec{E}_A$ is optimal, then by using the Kuhn-Tucker- Lagrange's Multipliers theorem $\forall$ P = 0,1,2 , there exist multipliers $\xi_P \in \mathbb{R}$ with $\xi_0 \geq 0$, $\xi_2 \geq 0$, $\sum_{\ell=1}^{2} |\xi_P| = 1$ **,** such that

$$\left(\sum_{P=0}^{2} \xi_P \overrightarrow{Ʊ'_{P\vec{d}}}(\vec{d})\right)_{L_2(\partial\Psi)} . (\vec{e} - \vec{d}) \geq 0 , \forall \vec{e} \in \vec{E} \quad (41a)$$

$$\xi_2 Ʊ_2(\vec{d}) = 0 \quad (41b)$$

Then, from Theorem (3), (41a) with the setting $\Delta d_1 = e_1 - d_1$, $\Delta d_2 = e_2 - d_2$, $\Delta d_3 = e_3 - d_3$, we can rewrite $\forall \vec{e} \in \vec{E}$ as

$$\int_{\partial\Psi} [(z_1 + Ʊ_{4d_1}) \Delta d_1 + (z_2 + Ʊ_{5d_2}) \Delta d_2 + (z_3 + Ʊ_{6d_3}) \Delta d_3] d\gamma \geq 0$$

where $z_j = \sum_{P=0}^{2} \xi_P z_{jP}$, $Ʊ_{\sigma dj} = \sum_{P=0}^{2} \xi_P Ʊ_{\sigma dj}$, for j = 1,2,3 , σ = 4,5,6

$$\Rightarrow \int_{\partial\Psi} И_{\vec{d}}^{T} . \overrightarrow{\Delta d}\, d\gamma \geq 0 , \forall \vec{e} \in \vec{E} , \overrightarrow{\Delta d} = \vec{e} - \vec{d} .$$

(b) Let $\{\vec{d}_n\}$ be a sequence, dense in $\vec{E}_{\vec{D}}$, and $S \subset \partial\Psi$ be a measurable set, such that

$$\vec{E}(x) = \begin{cases} \vec{d}_n(x) , \text{for } x \text{ belong in } S \\ \vec{d}(x) , \text{for } x \text{ not belong in } S \end{cases}$$

Hence (41a) , becomes

$\int_S И_{\vec{d}}^{T} . (\vec{d}_n - \vec{d}) ds \geq 0$, for any $S \subset \partial\Psi$. Then, by using Theorem (2) in reference [19], we obtain $И_{\vec{d}}^{T} . (\vec{d}_n - \vec{d}) \geq 0$, a.e on $\partial\Psi$.

The above inequality satisfies on $\partial\Psi$, except in a subset $\partial\Psi_n$ with $\tau(\partial\Psi_n) = 0$, for each n , where τ is a Lebesgue measure, then this equality holds on $\partial\Psi$ except in $\cup_n \partial\Psi_n$ with $\tau(\cup_n \partial\Psi_n) = 0$. But $\{\vec{d}_n\}$ is a dense in $\vec{E}$, then there exists $\vec{d} \in \vec{E}$, such that

$$И_{\vec{d}}^{T} \vec{d} = \min_{\vec{e} \in \vec{D}} И_{\vec{d}}^{T} \vec{e} \text{ , a.e, on } \partial\Psi .$$

6. The Sufficient Conditions for Optimality of the Continuous Classical Boundary Optimal Control Vector

Theorem (7): In addition to assumptions (I), (II), and (III), if $m_1, m_2, m_3, Ʊ_{11}, Ʊ_{12}, Ʊ_{13}$ are affine with respect to $\vec{b}$, $Ʊ_{14}, Ʊ_{15}, Ʊ_{16}$ are affine with respect to $\vec{d}$, $ӄ_1, ӄ_2, ӄ_3$ are bounded functions for x. Also, if g_{P_σ}(P = 0,2 , σ = 1,2,3,4,5,6) are convex with respect to $b_1, b_2, b_3, d_1, d_2, d_3$, respectively, for each x, then the necessary and sufficient conditions for optimality in the previous theorem (6), with $\xi_0 > 0$, are sufficient .

Proof: Assume that $\vec{d} \in \vec{E}_A$, $\vec{d}$ satisfies the conditions (39a) and (39b).

Let

$$Ʊ(\vec{d}) = \sum_{P=0}^{2} \xi_P Ʊ(\vec{d}) \Rightarrow \overrightarrow{Ʊ'_0}(\vec{d}) \overrightarrow{\Delta d} = \sum_{P=0}^{2} \xi_P \int_{\partial\Psi} [((z_{P1} + Ʊ_{P4d_1}) \Delta d_1 + (z_{P2} + Ʊ_{P5d_2}) \Delta d_2 + (z_{P3} + Ʊ_{P6d_3}) \Delta d_3)] d\gamma$$

$$= \int_{\partial\Psi} И_{\vec{d}} (x, z_1, z_2, z_3, d_1, d_2, d_3) . \overrightarrow{\Delta d}\, d\gamma \geq 0 .$$

and

$m_1(x, b_1) = m_{11}(x) b_1 + m_{12}(x)$, $m_2(x, b_2) = m_{21}(x) b_2 + m_{22}(x)$,

$m_3(x, b_3) = m_{31}(x)b_1 + m_{32}(x)$, and $\mathfrak{h}_\sigma(x)$ are bounded for $\sigma = 1,2,3$, $\forall x \in \Psi$.
Let $\vec{d} = (d_1, d_2, d_3)$, $\vec{\bar{d}} = (\bar{d}_1, \bar{d}_2, \bar{d}_3)$ be given two continuous classical boundary optimal control vectors, then the corresponding "state" solution vector are $\vec{b} = (b_1, b_2, b_3)$, $\vec{\bar{b}} = (\bar{b}_1, \bar{b}_2, \bar{b}_3)$ By substituting $(\vec{b}, \vec{d})$ in (1)-(6) and multiplying the resulting equations by $\theta \in [0,1]$ once, we again the substitution of the pair $(\vec{\bar{b}}, \vec{\bar{d}})$ in (1)-(6). By multiplying the result by $\bar{\theta} = (1-\theta)$, and finally summing each pair from the corresponding equations together, we get:

$$A_1(\theta b_1 + \bar{\theta}\bar{b}_1) + (\theta b_1 + \bar{\theta}\bar{b}_1) - (\theta b_2 + \bar{\theta}\bar{b}_2) - (\theta b_3 + \bar{\theta}\bar{b}_3) + m_{11}(x)(\theta b_1 + \bar{\theta}\bar{b}_1)$$
$$+m_{12}(x) = \mathfrak{h}_1(x) \quad (42a)$$
$$\sum_{i,j=0}^{2} a_{1ij} \frac{\partial}{\partial n_1}(\theta b_1 + \bar{\theta}\bar{b}_1) = \theta d_1 + \bar{\theta}\bar{d}_1 \quad (42b)$$
$$A_2(\theta b_2 + \bar{\theta}\bar{b}_2) + (\theta b_1 + \bar{\theta}\bar{b}_1) + (\theta b_2 + \bar{\theta}\bar{b}_2) + (\theta b_3 + \bar{\theta}\bar{b}_3) + m_{21}(x)(\theta b_2 + \bar{\theta}\bar{b}_2)$$
$$+m_{22}(x) = \mathfrak{h}_2(x) \quad (43a)$$
$$\sum_{i,j=0}^{2} a_{2ij} \frac{\partial}{\partial n_2}(\theta b_2 + \bar{\theta}\bar{b}_2) = \theta u_2 + \bar{\theta}\bar{d}_2 \quad (43b)$$

and

$$A_3(\theta b_3 + \bar{\theta}\bar{b}_3) + (\theta b_1 + \bar{\theta}\bar{b}_1) - (\theta b_2 + \bar{\theta}\bar{b}_2) + (\theta b_3 + \bar{\theta}\bar{b}_3) + m_{31}(x)(\theta b_3 + \bar{\theta}\bar{b}_3)$$
$$+m_{32}(x) = \mathfrak{h}_3(x) \quad (44a)$$
$$\sum_{i,j=0}^{2} a_{3ij} \frac{\partial}{\partial n_3}(\theta b_3 + \bar{\theta}\bar{b}_3) = \theta d_3 + \bar{\theta}\bar{d}_3 \quad (44b)$$

Now, if we have the continuous classical boundary optimal control vector $\vec{\bar{\bar{d}}} = (\bar{\bar{d}}_1, \bar{\bar{d}}_2, \bar{\bar{d}}_3)$ with $\bar{\bar{d}}_1 = \theta u_1 + \tilde{\alpha}\bar{d}_1$, $\bar{\bar{d}}_2 = \theta u_2 + \bar{\theta}\bar{d}_2$ and $\bar{\bar{d}}_3 = \theta u_3 + \tilde{\alpha}\bar{d}_3$. Then, from (42a&b) , (43a&b), and (44a&b) , one gets that the "state" solution vector $(\bar{\bar{b}}_1 = b_{1\bar{\bar{d}}_1}, \bar{\bar{b}}_2 = b_{2\bar{\bar{d}}_2}, \bar{\bar{b}}_3 = b_{3\bar{\bar{d}}_3})$ with $\bar{\bar{b}}_1 = \theta b_{1_{u_1}} + \bar{\theta} b_{1_{\bar{d}_1}} = \theta b_1 + \bar{\theta}\bar{b}_1$ $\bar{\bar{b}}_2 = \theta b_{2_{u_2}} + \bar{\theta} b_{2_{\bar{d}_2}} = \theta b_2 + \bar{\theta}\bar{b}_2$ and $\bar{\bar{b}}_3 = \theta b_{3_{u_3}} + \bar{\theta} b_{2_{\bar{d}_3}} = \theta b_3 + \bar{\theta}\bar{b}_3$ are their corresponding solution, i.e. they satisfy (1-6), respectively . So, the operators $d_1 \mapsto b_{1_{d_1}}$, $d_2 \mapsto b_{2_{d_2}}$ and $d_3 \mapsto b_{3_{d_3}}$ are convex-linear with respect to (b_1, d_1) , (b_2, d_2) and (b_3, d_3), respectively.
Now, from this result and since $\mho_{11}, \mho_{12}, \mho_{13}, \mho_{14}, \mho_{15}, \mho_{16}$ are affine with respect to $b_1, b_2, b_3, d_1, d_2, d_3$, respectively, on Ψ, we get that $\forall x \in \Psi$, $\mho_P(\vec{d})$ is convex-linear in $(\vec{b}, \vec{d})$. Also, since ($\forall$ P=0,2 & $\forall x \in \Psi$) $\mho_{P1}, \mho_{P2}, \mho_{P3}, \mho_{P4}, \mho_{P5}, \mho_{P6}$ are convex with respect to b_1, b_2, b_3, d_1, d_2 and d_3 respectively, i.e. $\mho(\vec{d})$ is convex with respect to $\vec{b}$ and $\vec{d}$, then $\mho(\vec{d})$ is convex in $\vec{b}$ and $\vec{d}$ i n the convex set $\vec{E}$ and has a continuous Fréchet derivative that satisfies
$\overrightarrow{\mho_0'}(\vec{d}). \overrightarrow{\Delta d} \geq 0 \Longrightarrow \mho(\vec{d})$ has a minimum at $\vec{d}$, i.e. $\mho(\vec{d}) \leq \mho(\vec{w})$, $\forall \vec{e} \in \vec{E}$, then we have

$$\sum_{P=0}^{2} \xi_P \mho(\vec{d}) \leq \sum_{P=0}^{2} \xi_P \mho(\vec{e}) \quad (45)$$

Now, let $\vec{E}$ be also admissible and satisfies the Transversality condition, then (45) becomes $\mho_0(\vec{d}) \leq \mho_0(\vec{e})$, $\forall \vec{e} \in \vec{E}$, i.e.
$\vec{d}$ is a classical continuous boundary control vector problem .

Conclusions

The existence and uniqueness theorem for the "state" solution vector of the triple nonlinear partial differential equations of elliptic type is proved successfully, when the classical continuous boundary control vector is given. The proof of the existence of the classical continuous boundary control vector, ruling by the considered triple nonlinear partial differential equations of elliptic type, is demonstrated with the equality and inequality constraints. The studying of the existence solution of the triple adjoint equations related with the triple nonlinear partial differential equations of elliptic type is demonstrated with the equality and inequality constraints. Finally, the theorems of both the necessary and sufficient conditions for optimality of the triple nonlinear partial differential equations of elliptic type, through the Kuhn-Tucker-Lagrange's Multiplires with equality and inequality constraints, is demonstrated.

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