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SOME RESULTS ABOUT GENERALIZED SEMIDERIVATIONS IN 3-PRIME NEAR-RINGS

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Abstract

The purpose of this paper is to extend some results concerning generalized derivations to generalized semiderivations of 3-prime near rings.

Keywords: 3-prime near-rings, derivations, generalized derivations, semederivations, generalized semiderivations

بعض النتائج حول شبه الاشتقاقات المعممة في الحلقات المقتربة الاولية

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الخلاصة

الغرض من هذا البحث هو تعميم بعض النتائج المتعلقة بالاشتقاقات المعممة الى شبه الاشتقاقات المعممة في الحلقات المقتربة الاوليه الثلاثيه.

1. INTRODUCTION

In this paper, N is a zero-symmetric right near ring, i.e. non empty set, together with two binary operations " + " and "." such that: a- (N, +) is a group (not necessarily abelian), b- (N, .) is a semigroup, c- for all $n_1, n_2, n_3 \in N$, $(n_1+n_2)n_3 = n_1n_3 + n_2n_3$, according to the right distributive law, and d- $n0 = 0n = 0$ for all $n \in N$ [1]. A set Z(N) is called the multiplication center of the near ring N, if it contains the elements of N which commute with every element of N, that is, $Z(N) = \{x \in N : xy = yx$ for all $y \in N\}$. Note that $0 \in Z(N)$, so $Z(N) \neq \emptyset$. Usually N will be 3-prime near ring, that is, we will have the property that $xNy = \{0\}$ for $x, y \in N$ implies $x = 0$ or $y = 0$ [1]. Nonempty subset I of N is called a semigroup right ideal (resp. semigroup left ideal) if $IN \subseteq I$ (resp. $NI \subseteq I$); and I is said to be a semigroup ideal if it is both a semigroup right and a semigroup left ideal. An additive mapping $d : N \to N$ is a derivation if $d(xy) = xd(y) + d(x)y$ for all $x, y \in N$, or equivalently, as presented previously [12], that $d(xy) = d(x)y + xd(y)$ for all $x, y \in N$. Motivated by a definition given by Bergen [5] for rings, Asma *et al*. [3] defined semiderivation in near rings as follows: An additive mapping $d : N \to N$ is called a semiderivation if there exists a function $g : N \to N$ such that $d(xy) = d(x)y + g(x)d(y) = d(x)g(y) + xd(y)$ and $d(g(x)) = g(d(x))$ for all $x, y \in N$. Further, Boua *et al*. [10] defined the generalized semideivation as follows: An additive mapping $F : N \to N$ is called a generalized semiderivation associated with semiderivation d if $F(xy) = F(x)y + g(x)d(y) = d(x)g(y) + xF(y)$ and

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F(g(x)) = g(F(x)) for all x, y ∈ N. Clearly, every semiderivation is a generalized semiderivation. We will write, for all x, y ∈ N, that [x, y] = xy − yx and x ∘ y = xy + yx for the Lie and Jordan products, respectively. Let g : N → N be a function. We usually denote for all x, y ∈ N, that $[x, y]_g = g(x)y - yx$, and $(x \circ y)_g = g(x)y + yx$. In particular, $[x,y]_{id_N}$ = [x, y] and $(x \circ y)_{id_N}$ = (x ∘ y) for all x, y ∈ N, where id_N is the identity map on N. In the current paper, we will prove the commutativity of the Near-ring N admitting the generalized semiderivation F associated with a nonzero semiderivation d and an automorphism g, satisfying the following identities: ($F([x, y]_g) = \pm[F(x), y]_g$, $F([x, y]_g) = \pm[x, F(y)]_g$, $[F(x), F(y)]_g = 0$, $F([x, y]_g) = \pm x^k [x, y]_g x^l$) .

2. Preliminaries

Throughout the paper, N is a zero-symmetric near-ring and g : N → N is an automorphism.

Lemma 1. [4, Lemma 1.5] Let N be a 3-prime near-ring. If N ⊆ Z(N), then N is a commutative ring.

Lemma 2. [2, Lemma 2.8] Let N be a 3-prime near-ring. If N admits a semiderivation d associated with an onto map g, then d(Z(N)) ⊆ Z(N).

Lemma 3. [9, Theorem 2] Let N be a 2-torsion free 3-prime near-ring admitting a generalized semiderivation F associated with a nonzero semiderivation d. If F(N) ⊆ Z(N), then N is a commutative ring.

Lemma 4. Let N be a 3-prime near-ring and F be a generalized semiderivation associated with a semiderivation d of N. Then N satisfies the following partial distributive laws

1. x(d(y)g(z) + yF(z)) = xd(y)g(z) + xyF(z) for all x, y, z ∈ N.
2. x(F(y)z + g(y)d(z)) = xF(y)z + xg(y)d(z) for all x, y, z ∈ N.

Proof: 1.) By using the definitions of d, F, and g, we have:

$$\begin{aligned} F(xyz) &= F((xy)z) \\ &= d(xy)g(z)+xyF(z) \\ &= (d(x)g(y)+xd(y))g(z)+xyF(z) \\ &= d(x)g(y)g(z)+ xd(y)g(z)+xyF(z) \\ &= d(x)g(yz) + xd(y)g(z)+xyF(z) \end{aligned}$$

On the other hand:

$$\begin{aligned} F(xyz) &= F(x(yz)) \\ &= d(x)g(yz)+xF(yz) \\ &= d(x)g(yz)+x(d(y)g(z)+yF(z) \end{aligned}$$

From the computation of F(x(yz)) and F((xy)z), we obtain:

$$d(x)g(yz)+x(d(y)g(z)+yF(z))= d(x)g(yz) + xd(y)g(z)+xyF(z)$$

then:

$$x(d(y)g(z) + yF(z)) = xd(y)g(z) + xyF(z) \text{ for all } x, y, z \in N.$$

2.) Using the same previous demonstrations with necessary changes, we can easily find the required result.

Lemma 5. [8, lemma 2.3] Let N be a near-ring. If N admits an additive mapping d, then the following statements are equivalent:

1. d is a semiderivation associated with an additive mapping g.

2. d(xy) = xd(y) + d(x)g(y) = g(x)d(y) + d(x)y and d(g(x)) = g(d(x)) for all x, y ∈ N.

3. Commutativity conditions involving generalized semiderivations

The present section is motivated by a previous work [5, Theorem 2]. Our aim is to extend these results on 3-prime near-rings admitting a nonzero generalized semiderivation F of N associated with a nonzero semiderivation d and an automorphism g such that [F(x), x] = 0 for all x ∈ N.

Theorem 1. Let N be a 3-prime near-ring and F be a generalized semiderivation of N associated with a nonzero semiderivation d and an automorphism g such that [F(x), x] = 0 for all x ∈ N. If $F([x, y]_g) = \pm[F(x), y]_g$ for all x, y ∈ N, then N is a commutative ring.

Proof: Assume that:

$$F([x, y]_g) = [F(x), y]_g \text{ for all } x, y \in N. \quad (1)$$

By taking yx instead of y in Equation (1) and noting that $[x, yx]_g = [x, y]_g x$, we get

$$F([x, y]_g)x + g([x, y]_g)d(x) = g(F(x))yx - yxF(x) \text{ for all } x, y \in N. \quad (2)$$

But [F(x), x] = 0 for all x ∈ N, so Equations (1) and (2) give:

$$g([x, y]_g)d(x) = 0 \text{ for all } x, y \in N. \quad (3)$$

Since g is automorphism, we get:

$$([x, y]_g (g^{-1}(d(x))) = 0 \text{ for all } x, y \in N. \quad (4)$$

This implies that:

$$g(x)yg^{-1}(d(x)) = yxg^{-1}(d(x)) \text{ for all } x, y \in N. \quad (5)$$

By substituting ty for y, where $t \in N$, in Equation (5) and using it, we get:

$$tyxg^{-1}(d(x)) = g(x)tyg^{-1}(d(x)) \text{ for all } x, y, t \in N$$

and we have also:

$$tyxg^{-1}(d(x)) = tg(x)yg^{-1}(d(x)) \text{ for all } x, y, t \in N.$$

Both expressions give:

$$[g(x), t]Ng^{-1}(d(x)) = \{0\} \text{ for all } x, y, t \in N. \quad (6)$$

Since N is 3-prime, $d \neq 0$, and g is an automorphism, then we obtain $g(x) \in Z(N)$ for all $x \in N$. Again, using the fact that g is an automorphism, we have $N \subseteq Z(N)$. Hence N is a commutative ring by Lemma 1.

Similarly, we can get the result for the case $F([x, y]_g) = -[F(x), y]_g$ for all $x, y \in N$.

By putting $g = id_N$, we get the following corollary.

Corollary 1. [5, Theorem 2] Let N be a 2-torsion free 3-prime near-ring. If F is a generalized derivation of N associated with a nonzero derivation d such that $F([x, y]) = [F(x), y]$ for all $x, y \in N$, then N is a commutative ring.

Theorem 2. Let N be a 3-prime near-ring and F be a generalized semiderivation of N associated with a nonzero semiderivation d and an automorphism g such that $[y, F(y)]_g = 0$ for all $y \in N$. If $F([x, y]_g) = \pm[x, F(y)]_g$ for all $x, y \in N$, then N is a commutative ring.

Proof: Assume that:

$$F([x, y]_g) = [x, F(y)]_g \text{ for all } x, y \in N. \quad (7)$$

By substituting xy instead of x in Equation (7) we arrive at:

$$F([x, y]_g)y + g([x, y]_g)d(y) = g(xy)F(y) - F(y)xy \text{ for all } x, y \in N. \quad (8)$$

But $g(y)F(y) = F(y)y$ for all $y \in N$, and g is an automorphism, so Equations (7) and (8) yield that

$$F([x, y]_g)y + g([x, y]_g)d(y) = [x, F(y)]_g y \text{ for all } x, y \in N. \quad (9)$$

Then by Equation (7), we get

$$g([x, y]_g)d(y) = 0 \text{ for all } x, y \in N. \quad (10)$$

But Equation (10) is like Equation (3) in the previous theorem, then we conclude that N is a commutative ring.

If $F([x, y]_g) = -[x, F(y)]_g$ for all $x, y \in N$, then using the similar techniques as above, we can get the required result.

Corollary 2. [5, Theorem 3] Let N be a 2-torsion free 3-prime near-ring. If F is a generalized derivation of N associated with a nonzero derivation d such that $F([x, y]) = [x, F(y)]$ for all $x, y \in N$, then N is a commutative ring.

Theorem 3. Let N be a 2-torsion free 3-prime near-ring and F be a generalized semiderivation of N associated with a semiderivation d and an automorphism g such that $d(Z(N)) \neq \{0\}$. If $[F(x), F(y)]_g = 0$ for all $x, y \in N$, then N is a commutative ring.

Proof: Let $z \in Z(N)$ such that $d(z) \neq 0$. Suppose that:

$$[F(x), F(y)]_g) = 0 \text{ for all } x, y \in N. \quad (11)$$

So,

$$g(F(x))F(y) = F(y)F(x) \text{ for all } x, y \in N. \quad (12)$$

By substituting yz instead of y in Eq.(12) and using Lemma 4, we get:

$$g(F(x))F(y)z + g(F(x))g(y)d(z) = F(y)zF(x) + g(y)d(z)F(x). \quad (13)$$

But by Equations (12) and (13) and Lemma 2, we get:

$$g(F(x))g(y)d(z) = g(y)F(x)d(z) \text{ for all } x, y \in N. \quad (14)$$

This implies that:

$$[F(x), g(y)]_g N\, d(z) = 0 \text{ for all } x, y \in N. \quad (15)$$

Since N is 3-prime and $d(z) \neq 0$, we have:

$$[F(x), g(y)]_g = 0 \text{ for all } x, y \in N. \quad (16)$$

We conclude that $F(N) \subseteq Z(N)$ and N is a commutative ring by Lemma 3.

The next theorem is a generalization of Theorem 1 in a previous work [10].

Theorem 4. Let N be a 3-prime near-ring. If k, l are non-negative integers and N admits a generalized semiderivation F of N associated with a nonzero semiderivation d and an automorphism g satisfying

$F([x, y]_g) = \pm x^k [x, y]_g x^l$ for all $x, y \in N$, then N is a commutative ring.

Proof: Assume that:

$$F([x, y]_g) = x^k [x, y]_g x^l \text{ for all } x, y \in N. \quad (17)$$

Since $[x, yx]_g = [x, y]_g x$ for all $x, y \in N$, by substituting yx for y in Eq. (17) we get:

$$\begin{aligned} F([x, yx]_g) &= F([x, y]_g x) \\ &= x^k [x, yx]_g x^l \\ &= x^k [x, y]_g x^{l+1} \text{ for all } x, y \in N. \end{aligned} \quad (18)$$

So, by the definition of F we have:

$F([x, yx]_g) = F([x, y]_g x) = F([x, y]_g)x + g([x, y]_g)d(x)$ for all $x, y \in N$.

By using Equations (17) and (18) we obtain:

$$x^k [x, y]_g x^{l+1} = x^k [x, y]_g x^{l+1} + g([x, y]_g)d(x) \text{ for all } x, y \in N. \quad (19)$$

This implies that:

$$g([x, y]_g)d(x) = 0 \text{ for all } x, y \in N. \quad (20)$$

But g is an automorphism, so we have:

$$([x, y]_g)g^{-1}(d(x)) = 0 \text{ for all } x, y \in N. \quad (21)$$

Thus:

$$g(x)y(g^{-1}(d(x))) = yx(g^{-1}(d(x))) \text{ for all } x, y \in N. \quad (22)$$

By substituting zy for y in Eq. (22), where $z \in N$, and using it, we have:

$$zyx(g^{-1}(d(x))) = zg(x)y(g^{-1}(d(x))) = g(x)zy(g^{-1}(d(x))) \text{ for all } x, y, z \in N. \quad (23)$$

Hence, $[g(x), z]Ng^{-1}(d(x)) = \{0\}$ for all $x, z \in N$, and by the 3-primeness of N we have either $[g(x), z] = 0$ or $g^{-1}(d(x)) = 0$ for all $x, z \in N$. But g is an automorphism and $d \neq 0$, thus $g(x) \in Z(N)$ for all $x \in N$, i.e. $N \subseteq Z(N)$, and N is a commutative ring by Lemma 1.

Similarly we can get the result in the case of $F([x, y]_g) = -x^k [x, y]_g x^l$ for all $x, y \in N$.

The next theorem is a generalization of Theorem 2 in a previous work [10].

Theorem 5. Let N be a 3-prime near-ring. If there exist non negative integers k and l, and if N admits a generalized semiderivation F of N associated with a nonzero semiderivation d and an automorphism g satisfying $F((x \circ y)_g) = \pm x^k (x \circ y)_g x^l$ for all $x, y \in N$, then N is a commutative ring.

Proof: By the hypothesis,

$$F((x \circ y)_g) = x^k (x \circ y)_g x^l \text{ for all } x, y \in N. \quad (24)$$

Since $(x \circ yx)_g = (x \circ y)_g x$ for all $x, y \in N$, then by replacing y by yx in Eq. (24), we obtain:

$$\begin{aligned} F((x \circ yx)_g) &= F((x \circ y)_g x) \\ &= x^k (x \circ yx)_g x^l \\ &= x^k (x \circ y)_g x^{l+1} \text{ for all } x, y \in N. \end{aligned} \quad (25)$$

So, by the definition of F, we have:

$$\begin{aligned} F((x \circ yx)_g) &= F((x \circ y)_g x) \\ &= F((x \circ y)_g)x + g((x \circ y)_g)d(x) \text{ for all } x, y \in N. \end{aligned}$$

By using Equations (24) and (25), we get:

$$x^k (x \circ y)_g x^{l+1} = x^k (x \circ y)_g x^{l+1} + g((x \circ y)_g)d(x) \text{ for all } x, y \in N. \quad (26)$$

This implies that:

$$g((x \circ y)_g)d(x) = 0 \text{ for all } x, y \in N. \quad (27)$$

But g is an automorphism, so we have:

$$((x \circ y)g(g^{-1}(d(x))) = 0 \text{ for all } x, y \in N \quad (28)$$

and:

$$g(x)y(g^{-1}(d(x))) = -yx(g^{-1}(d(x))) \text{ for all } x, y \in N. \quad (29)$$

By replacing y by zy in Equation (29) where $z \in N$, and using it, we get:

$$\begin{aligned} zyx(g^{-1}(d(x))) &= z(-g(x)y(g^{-1}(d(x)))) \\ &= z(-g(x))y(g^{-1}(d(x))) \\ &= (-g(x))zy(g^{-1}(d(x))) \quad \text{for all } x, y, z \in N. \end{aligned}$$

So:

$$z(-g(x)y(g^{-1}(d(x)))) = (-g(x))zy(g^{-1}(d(x))) \text{ for all } x, y, z \in N. \quad (30)$$

Hence,

$[-g(x), z]N(g^{-1}(d(x))) = \{0\}$ for all $x, z \in N$, and by the 3-primeness of N we have either $[-g(x), z] = 0$ or $g^{-1}(d(x)) = 0$ for all $x, z \in N$. But g is an automorphism and d 6= 0, thus $-g(x) = g(-x) \in Z(N)$ for all $x \in N$, i.e. $N \subseteq Z(N)$, and N is a commutative ring by Lemma 1.

Using the similar techniques as above in the case of $F((x \circ y)_g) = -x^k (x \circ y)_g x^l$ for all x, y ∈ N, we can get the results.

Remark 1. If we put $g = id_N$ in Theorems 4 and 5, we obtain Theorems 1 and 2 in a previous work [10], respectively, as a direct special case.

The following example shows that g to be an automorphism and N to be 3-prime cannot be omitted in the hypotheses of Theorems 1, 2, 3, 4 and 5.

Example 1. Let S be a 2-torsion free zero-symmetric right near ring. Let

$N = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} : 0, x, y, z \in S \right\}$. It can be easily seen that N is zero symmetric left near-ring with regard to matrix addition and matrix multiplication.

We define the mappings F, d, g : N ⟶ N by:

$$F\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} = \begin{pmatrix} 0 & -x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad d\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $g\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix}$

It is clear that N is a 2-torsion free near-ring which is not 3- prime, F is a generalized semiderivation on N associated with a semiderivation d and a non automorphism g satisfying the following conditions:

1. $[F(A), A] = 0$,
2. $F([A, B]_g) = \pm[F(A), B]_g$,
3. $[A, F(A)]_g = 0$,
4. $F([A, B]_g) = \pm[A, F(B)]_g$,
5. $d(Z(N)) \neq \{0\}$,
6. $[F(A), F(B)]_g = 0$,
7. $F([A, B]_g) = \pm A^k [A, B]_g A^l$,
8. $F((A \circ B)_g) = \pm A^k (A \circ B)_g A^l$ for all A, B ∈ N, and for some k, l ∈ N.

However, N is not a commutative ring.

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