Numerical Solution to Recover Time-dependent Coefficient and Free Boundary from Nonlocal and Stefan Type Overdetermination Conditions in Heat Equation

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Abstract

This paper investigates the recovery for time-dependent coefficient and free boundary for heat equation. They are considered under mass/energy specification and Stefan conditions. The main issue with this problem is that the solution is unstable and sensitive to small contamination of noise in the input data. The Crank-Nicolson finite difference method (FDM) is utilized to solve the direct problem, whilst the inverse problem is viewed as a nonlinear optimization problem. The latter problem is solved numerically using the routine optimization toolbox *lsqnonlin* from MATLAB. Consequently, the Tikhonov regularization method is used in order to gain stable solutions. The results were compared with their exact solution and tested via root mean squares error (RMSE). We found that the numerical results are accurate and stable.

Keywords: inverse problem; coefficient identification problem; nonlinear optimization; FDM

الحل العددي لاسترداد معامل معتمد على الزمن والحدود الحرة من شروط إضافية من نوع ستيفان وغير محلية في معادلة الحرارة

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الخلاصة

هذا البحث يبحث في استرداد معامل معتمد على الزمن والحدود الحرة لمعادلة الحرارة. تم دراسة هذه المسألة تحت شروط إضافية من نوع ستيفان وشروط الطاقة/ الكتلة. تكمن صعوبة هذه المسألة ان الحل يكون غير مستقر وحساس للضوضاء (الأخطاء) في بيانات الادخال. تم استخدام طريقة كرانك–نكيلسون للفروقات المنتهية لحل المسألة المباشرة بينما تم حل المسألة المعكوسة كمسالة أمثليه عددية غير خطية. المسالة الأخيرة تم حلها عدديا باستخدام الروتين *lsqnonlin* من MATLAB . وبالتالي، من اجل الحصول على حلول مستقرة تم استخدام طريقة Tikhonov. تمت مقارنه الحلول العددية مع الحلول المضبوطة وتم اختبارها عبر خطا جذر معدل التربيعات (rmse). وجد ان النتائج العددية دقيقة و مستقرة.

1. Introduction

The field of inverse problems for heat conduction has very wide applications and physical

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background. It has been applied in almost all fields of scientific engineering computations, such as power engineering, aerospace engineering, biomedical engineering, etc... The mathematical modeling used in many current applied problems in science and technology revealed the need for numerical solutions of inverse problems in mathematical physics, such as the numerical solution of the two-sided Stefan problem [1]. We consider a one-sided free domain problem in one-dimensional space for the parabolic heat equation, with non-homogenous Dirichlet boundary condition when the thermal conductivity is equal to unity. This problem contains free boundary depending on time only [2, 3]. Therefore, the shape of the problem varies with time step marching. In this problem, some unknown terms or coefficients are determined by using some additional specified information about their solution, like Stefan condition and zero and first order heat moment conditions. Inverse problems are usually difficult to be solved analytically and, therefore, the numerical approaches are created to overcome complexities of the analytical methods. In [4], the authors suggested an iterative method for solving the inverse problem to determine the right-hand side part (heat source problem) with separating variables on spatial variables and time. Determination of time-dependent lowest order coefficient in heat equation was investigated in [5]. The authors in [6 - 9] investigated the theoretical and numerical aspects of several types of parabolic heat equations, in fixed and free domains of one- and two-dimensions for various types of additional information.

The paper is organized as follows. Section 2 presents the mathematical formulation of the inverse problem with Stefan condition and zero-order heat moment as measurements data, while the unique solvability theorems are also stated. In Section 3, a numerical procedure based on the finite difference method for solving the direct problem, while the numerical approach for the inverse problem is described in Section 4. The results and discussion are addressed in Section 5. Finally, the conclusions are highlighted in Section 6.

2. Mathematical formulation

Consider the one-dimensional inverse time-dependent parabolic heat equation [10],

$$u_t = u_{xx} + b(t)u_x + c(x,t)u + f(x,t), \qquad (x,t) \in \Omega_T, \tag{1}$$

where the free domain $Q_T = \{(x,t): 0 < x < h(t),\ 0 < t < T\}$, subject to initial and Dirichlet boundary conditions, is written as

$$u(x,0) = \varphi(x), \qquad 0 \leq x \leq h(0) \tag{2}$$

$$u(0,t) = \mu_1(t) \qquad u(h(t),t) = \mu_2(t), \qquad 0 \leq t \leq T, \tag{3}$$

respectively, with the overspecified conditions of Stefan-type and heat moment of zero order,

$$h'(t) + u_x(h(t),t) = \mu_3(t), \qquad t \in [0,T], \tag{4}$$

$$\int_0^{h(t)} u(x,t)dx = \mu_4(t), \qquad t \in [0,T]. \tag{5}$$

In order to solve this problem, we change the variables by Landau transformation $y = \frac{x}{h(t)}, t = t$ to reduce the free domain problem (1)-(3) to the following fixed domain problem for the unknown solution $v(y,t)$. Assume that the transformed solution $v(y,t) = u(yh(t),t)$ is in the area with the fixed domain $Q_T = \{(y,t): 0 \leq y \leq 1, 0 < t < T\}$, where the continuous functions $\varphi(x), \mu_1(t)$ and $\mu_2(t)$ are given. This model was investigated theoretically in [10] and no numerical solution is undertaken, since the purpose of the paper is to find the numerical solution based on reliable algorithm. The unique solvability of the direct problem is guaranteed by the continuity of the coefficients b, c, f and h. Now, the problem in the fixed domain has the following form:

$$v_t = \frac{1}{h^2(t)} v_{yy} + \left(\frac{b(t) + h'(t)y}{h(t)}\right) v_y + c(yh(t),t)v + f(yh(t),t), \quad (y,t) \in Q_T \tag{6}$$

$$v(y,0) = \varphi(yh(0)), \qquad 0 \leq y \leq 1, \tag{7}$$

$$v(0,t) = \mu_1(t), \qquad v(1,t) = \mu_2(t), \quad 0 \leq t \leq T, \tag{8}$$

$$h'(t) + \frac{v_y(1,t)}{h(t)} = \mu_3(t), \qquad t \in [0,T] \tag{9}$$

$$h(t) \int_0^1 v(y,t)dy = \mu_4(t), \qquad t \in [0,T] \tag{10}$$

Definition 1. Consider a solution to the inverse problem (6)-(10), the triplet class $h(t), b(t), v(y,t) \in \left(C^1[0,T] \times C[0,T] \times C^{2,1}(\bar{Q}_T)\right)$ and $h(t) > 0$, for $t \in [0,T]$ that satisfies equations (1)-(5) [10].

Theorem 1: Assume that the following conditions hold, [10]

1. $\mu_j \in C^1[0,T]$, for $i = 1,2,3$ and $\mu_i(t) > 0, i = 1,2,3$, for $t \in [0,T]$, $f \in C([0,+\infty) \times [0,T]), f(x,t) \geq 0$, for $x \in [0,+\infty), t \in [0,T]$ also $\varphi \in c[0,+\infty), \varphi(x) \geq \varphi_0$, for $x \in [0,+\infty)$,

2. $\varphi \in C^2[0,h(0)]$, $f,c \in H^{\propto,0}([0,H_0] \times [0,T])$, where $\propto \in (0,1), \mu_1(t) \neq \mu_2(t)$, for $t \in [0,T]$; where

$$H_0 = \frac{max_{t\in[0,T]}\,\mu_3(t)}{C_1 \min\left[min_{(0,h(0))}\varphi(x), \; min_{t\in[0,T]}\mu_1(t), \; min_{t\in[0,T]}\mu_2(t)\right]},$$

3. The compatibility conditions are expressed as:
$$\varphi(0) = \mu_1(0), \quad \varphi(h(0)) = \mu_2(0),$$
$$\varphi''(0) + b(0)\varphi'(0) + c(0,0)\varphi(0) + f(0,0) = \mu'_1(0),$$
$$\varphi''(h(0)) + b(0)\varphi'(h(0)) + c(h(0),0)\varphi(h(0)) + f(h(0),0) = \mu'_2(0),$$
$$b(0) = \left[\mu'_4(0) + (\mu_2(0) - h'(0)) - \int_0^{h(0)} [\varphi''(x) + c(x,0)\varphi(x) + f(x,0)]dx\right](\mu_2(0) - \mu_1(0))^{-1}.$$

Then it is possible to identify a time $T_0 \in (0,T]$, which is determined by the input data such that there exists a (local) solution to problem (1)-(5) for $(x,t) \in \Omega_{T_o}$.

Theorem 2: Assume that the following conditions are satisfied;

1. $f,c \in C^{1,0}([0,+\infty) \times [0,T])$;
2. $\varphi(x) \geq \varphi_0 > 0$ for $x \in [0,+\infty)$, and $f(x,t) \geq 0$, for $x \in ([0,+\infty) \times [0,T])$
3. $\mu_i(t) > 0, i = 1,2,3$, for $t \in [0,T]$ and $\mu_1(t) \neq \mu_2(t)$ for $t \in [0,T]$

Then the solution of the problem (1)-(5) is unique.

3. Solution of the direct problem

In this section, let us consider the direct initial boundary value problem (IBVP) (6)-(8), where the functions $b(t), c(x,t), \varphi(x)$ and $\mu_i(t), i = 1,2$ are known and the solution $v(y,t)$ is to be computed. In addition to some required information (9)-(10) to solve the problem, we employ the Crank-Nicolson finite difference scheme.

3.1 Discretization of the problem

The discrete form of the problem (6)-(8) is as follows. We divided the domain $Q_T = (0,1) \times (0,1)$ into M and N intervals of equal step lengths of Δy and Δt, where the uniform space and time increments are $\Delta t = \frac{T}{N}$ and $\Delta y = \frac{1}{M}$, respectively. We denote the solution at the node point (i,j) as $v_{i,j} = v(y_i, t_j), b(t_j) = b_j, c(y_i, t_j) = c_{i,j}$ and $f(y_i, t_j) = f_{i,j}$ where $y_i = i\Delta y$, $t_j = j\Delta t$, $i = \overline{0,M}$, $j = \overline{0,N}$.

In order to apply the CN-scheme for equation (6), we simply assume the right-hand side as follows:

$$Q(t,y,v,v_y,v_{yy}) = \frac{v_{yy}}{h^2(t)} + \left(\frac{b(t) + h'(t)y}{h(t)}\right)v_y + c(yh(t),t)v + f(yh(t),t), (y,t) \in Q_T \qquad (11)$$

Therefore, equation (6) can be approximated, in which the right hand-side of the heat conduction equation is expressed in one-half by the temperature function $v_{i,j}$ and, in the second-half, by the temperature function $v_{i,j+1}$ [8],

$$\frac{v_{i,j+1} - v_{i,j}}{\Delta t} = \frac{1}{2}(Q_{i,j} + Q_{i,j+1}) \qquad (12)$$

with the initial and boundary conditions

$$v(y_i, 0) = \varphi(y_i h(0)), \qquad i = \overline{0,M}, \qquad (13)$$
$$v(0, t_j) = \mu_1(t_j), \quad v(1, t_j) = \mu_2(t_j), \qquad j = \overline{0,N}, \qquad (14)$$

where,

$$Q_{i,j} = \frac{1}{h^2(t_j)}\frac{v_{i+1j} - 2v_{i,j} + v_{i-1j}}{\Delta y^2} + \left(\frac{b(t_j) + h'(t_j)y_i}{h(t_j)}\right)\frac{v_{i+1j} - v_{i-1j}}{2\Delta y} + c_{i,j}v_{i,j} + f_{i,j}, \; i = \overline{0,M}, j = \overline{0,N} \quad (15)$$

Now we substitute $Q_{i,j}$ and $Q_{i,j+1}$ in equation (12) and get

$$-A_{i,j+1}v_{i-1j+1} + [1 - B_{i,j} + 1]v_{i,j} - C_{i,j+1}v_{i+1j+1}$$
$$= A_{i,j}v_{i-1j} + [1 + B_{i,j+1}]v_{i,j} + C_{i,j}v_{i+1j} + \frac{\Delta t}{2}(f_{i,j} + f_{i,j+1}), \qquad (16)$$

where

$$A_{i,j} = \frac{\Delta t}{2\Delta y^2} - \frac{\Delta t}{4\Delta y} b_j, \quad B_{i,j} = \frac{\Delta t}{2} c_{i,j} - \frac{\Delta t}{\Delta y^2}, \quad C_{i,j} = \frac{\Delta t}{2\Delta y^2} + \frac{\Delta t}{4\Delta y} b_j. \tag{17}$$

Generally, the three values on the right hand of (16) are known, whereas values on the left hand are unknown. If we divide the y-interval into M quad intervals, we have $M - 1$ interval mesh points per time step, due to the available data from boundary conditions. Then, for $j = 0$, i.e. at initial time and $i = 1, \ldots \ldots, M - 1$, equation (17) gives a linear system of the $M - 1$ unknown values $v_{1,1}, v_{2,1} \ldots \ldots, v_{M-1,1}$ in the first time step in terms of the initial value $v_{0,0}, v_{0,1}, \ldots, v_{0,n}$, as the Dirichlet boundary values $v_{0,1}(= \mu_1(t(0)), v_{M,1}(= \mu_2(t(0))$. Similarly for the next time step, $j = 1,2, \ldots$, and so on, that is, for each time step t_j. For $j = \overline{0, N-1}$, the above difference equation (10) can be reformulated as $(M - 1) \times (M - 1)$ system of a linear equation of the from

$$Av^{n+1} = Bv^n + d \tag{18}$$

where $v^{n+1} = (v_{1,j+1}, v_{2,j+1}, \ldots \ldots, v_{M-1,j+1})^{tr}$, $v^n = (v_{1,j}, v_{2,j}, \ldots \ldots \ldots, v_{M-1,j})^{,tr}$ and A and B are $(M - 1) \times (M - 1)$ as follows:

$$A = \begin{bmatrix} 1 - B_{i,j+1} & -C_{i,j+1} & 0 & 0 & 0 \\ -A_{2,j+1} & 1 - B_{2,j+1} & -C_{2,j+1} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & -A_{M-2,j+1} & 1 - B_{M-2,j+1} & -C_{M-2,j+1} \\ 0 & 0 & 0 & -A_{M-1,j+1} & 1 - B_{M-1,j+1} \end{bmatrix}$$

$$B = \begin{bmatrix} 1 + B_{i,j+1} & C_{i,j} & 0 & 0 & 0 \\ A_{2,j} & 1 + B_{2,j} & C_{2,j} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & A_{M-2,j} & 1 + B_{M-2,j} & -C_{M-2,j+1} \\ 0 & 0 & 0 & A_{M-1,j} & 1 + B_{M-1,j} \end{bmatrix}$$

$$d = \begin{bmatrix} A_{1,j+1}v_{0,j+1} + \frac{\Delta t}{2}(f_{1,j} + f_{1,j+1}) \\ \frac{\Delta t}{2}(f_{2,j} + f_{2,j+1}) \\ 0 \\ \vdots \\ 0 \\ \frac{\Delta t}{2}(f_{M-2,j} + f_{M-2,j+1}) \\ \frac{\Delta t}{2}(f_{i,j} + f_{i,j+1}) + C_{M-1,j+1}v_{M,j+1} \end{bmatrix}$$

3.2 Example for the direct problem

Consider the case where the unknown coefficients are given, so we have a direct problem with the input data being as follows;

$$b(t) = e^{-t}, \quad c(x,t) = t^2 + x^2, \quad h(t) = 1 + t,$$
$$u(x,t) = e^t + \cos^2(x) \tag{19}$$
$$f(x,t) = e^t + 2\cos^2(x) - (t^2 + x^2)(e^t + \cos^2(x)) + 2 e^{-t} \cos(x) \sin(x) - 2 \sin^2(x)$$

using Landau transformation,

$$y = \frac{x}{h(t)} = \frac{x}{1+t},$$

we have the transformed quantities

$$b(t) = e^{-t},$$

$$c(y, t) = t^2 + (1 + t)^2 y^2,$$
$$f(y, t) = e^t + 2\cos^2((1+t)y) - (t^2 + (1+t)^2 y^2)\left(e^t + \cos^2((1+t)y)\right)$$
$$+ 2e^{-t}\cos((1+t)y)\sin((1+t)y) - 2\sin^2(((1+t)y))$$
$$v(y, t) = e^t + \cos^2((1+t)y), \tag{20}$$

Therefore, the input data will be
$$\varphi(y) = e^t + \cos^2(y), \quad \mu_1(t) = v(0, t) = e^t + 1, \quad \mu_2(t) = v(1, t) = e^t + \cos^2(1 + t),$$
$$\mu_3(t) = h'(t) + \frac{1}{h(t)} v_y(1, t) = 1 - 2\cos(1 + t)\sin(1 + t), \tag{21}$$
$$\mu_4(t) = h(t)\int_0^1 v(y, t)dy = (1 + t)(\frac{1}{2} + e^t + \frac{\sin(2(1 + t))}{4(1 + t)}), \tag{22}$$

Equations (21) and (22) can be computed numerically using the following finite difference approximation formulas and the trapezoidal value for integral.

$$\mu_3(t_j) = \frac{h(t_j) - h(t_{j-1})}{\Delta t} + \frac{4v_{M-1,j} - v_{M-2,j} - 3v_{M,j}}{2(\Delta y)h(t_j)}, \quad j = \overline{1, N}, \tag{23}$$

$$\mu_4(t_j) = \frac{h(t_j)}{2N}\left(v_{0,j} + v_{M,j} + 2\sum_{i=1}^{M-1} v_{i,j}\right), \quad j = \overline{1, N} \tag{24}$$
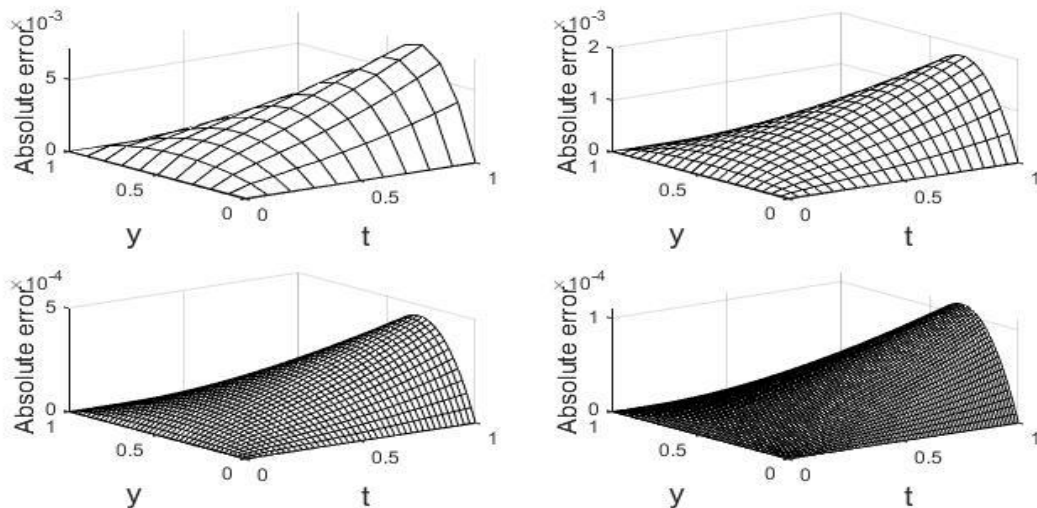

Figure 1-The absolute error graphs for the solution of the direct problem (6)-(8) for various mesh sizes $M = N \in \{10, 20, 40, 80\}$.

Table 1-The exact and numerical values for the desired output $\mu_3(t)$ compared at various time nodes and mesh sizes $M = N \in \{10, 20, 40, 80\}$.

T	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
M=N=10	0.1722	0.3052	0.4641	0.6470	0.8441	1.0469	1.2545	1.4516	1.6330
M=N=20	0.1872	0.3199	0.4800	0.6611	0.8561	1.0575	1.2566	1.4463	1.6188
M=N=40	0.1904	0.3234	0.4834	0.6641	0.8583	1.0582	1.2560	1.4437	1.6138
M=N=80	0.1912	0.3243	0.4842	0.6648	0.8587	1.0584	1.2557	1.4428	1.6124
exact	0.1915	0.3245	0.4845	0.6650	0.8589	1.0584	1.2555	1.4425	1.6119

Table 2-The exact and numerical values for the desired output μ_4(t) compared at various time nodes and mesh sizes $M = N \in \{10,20,40,80\}$.

T	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
M=N=10	1.9676	2.2352	2.5350	2.8745	2.2615	3.7051	4.2150	4.8022	5.4784
M=N=20	1.9678	2.2347	2.5340	2.8728	2.2592	3.7019	4.2109	4.7971	5.4723
M=N=40	1.9678	2.2346	2.5338	2.8724	2.2586	3.7011	4.2098	4.7958	5.4708
M=N=80	1.9678	2.2346	2.5337	2.8723	2.2584	3.7009	4.2096	4.7955	5.4704
exact	1.9364	2.2345	2.5337	2.8723	2.2584	3.7008	4.2095	4.7953	5.4703

Figure-1 presents the absolute graph of interior points for various mesh sizes $M = N \in \{10,20,40,80\}$. From this figure, it can be seen that the mesh independence is achieved. In addition, the convergence of the numerical solution toward an exact solution and the excellent agreement were obtained. From Tables- 1 and 2, one can clearly notice the convergence of numerical results for μ_3 and μ_4 for exact ones as the number of discretization increased.

4. Numerical approach for the inverse problem

In this section, we aim to find the numerical solution for the inverse problem described in section 2. We focus on finding stable reconstructions for unknown coefficient $b(t)$ and unknown free boundary $h(t) > 0$ of the one-dimensional heat equation, together with temperature distribution $u(x,t)$ or $v(y,t)$, satisfying the problem given by equations (1)-(5) or the transform inverse problem (6)-(10). At initial time, i.e, $t = 0$, we can use input data to obtain values for $b(0)$ and $h'(0)$, where $h(0)$ is given by the problem setting. These values will be considered as initial guesses in the process of solving the problem. In order to solve this problem, we recast the inverse problem as a nonlinear minimization problem. In other words, we minimize the gap between the measured data and the computed solution. Since the problem is ill-posed, we adapted Tikhonov regularization method to find a stable and smooth solution. The zero-order Tikhonov regularization functional can be constructed from the over determination conditions (9)-(10) as:

$$F(h,b) = \left\| h'(t) + \frac{v_y(1,t)}{h(t)} - \mu_3(t) \right\|^2 + \left\| h(t)\int_0^1 v(y,t)dy - \mu_4(t) \right\|^2 + \beta_1\|h(t)\|^2$$
$$+ \beta_2\|b(t)\|^2, \tag{25}$$

or, in a discredited form as

$$F(h,b) = \sum_{j=1}^N \left(h'(t_j) + \frac{v_y(1,t_j)}{h(t_j)} - \mu_3(t_j) \right)^2 + \sum_{j=1}^N \left(h(t_j)\int_0^1 v(y,t)dy - \mu_4(t_j) \right)^2 + \beta_1 \sum_{j=1}^N h^2{}_j$$
$$+ \beta_2 \sum_{j=1}^N b^2{}_j, \tag{26}$$

where $\beta_i \geq 0, i = 1,2$ are regularization parameters and should be determined according to a suitable selection strategy. The norm is taken in the space $L^2[0,T]$. Also, $v(y,t)$ solves (6)-(10) for given h and b. The minimization of the objective functional (26), subject to simple physical bound constrain $h > 0$, is accomplished using *lsqnonlin* routine from MATLAB optimization toolbox. This routine does not require providing the gradient of the objective function [11]. This routine aims to find the minimum of a sum of squares by starting from initial guesses. Moreover, within *lsqnonlin*, we use the trust region reflective (TRR) algorithm that is described in [12,13, 14], which is also based on the interior-reflective- Newton method. Each iteration includes a large linear system of equations whose solution, based on preconditioned Conjugate Gradient method (PCG), allows a regular and sufficiently smooth decrease of the objective functional (26). During the numerical simulation, we use the parameters of the routine *lsqnonlin,* as follows;

- Maximum number of iterations = $10^5 \times$ (number of variables).
- Maximum number of objective function evaluation= $10^8 \times$ (number of variables).
- Solution tolerance (SolTOL)= 10^{-10}.

- Objective function tolerance (FunTOL)$= 10^{-10}$.

The inverse problem (1)-(5) is solved subject to both exact and noisy measurements (4) and (5), respectively. The noisy data is numerically simulated by adding random errors as follows;

$$\mu_3^{\epsilon_1}(tj) = \mu_3(t_j) + \epsilon_{1,j}, \qquad j = \overline{1,N}, \tag{27}$$

$$\mu_4^{\epsilon_1}(t_j) = \mu_4(t_j) + \epsilon_{2,j}, \qquad j = \overline{1,N}, \tag{28}$$

where ϵ_1, ϵ_2 are random vectors generated from a Gaussian normal distribution with a zero mean and standard deviations of σ_1 and σ_2, respectively, given by

$$\sigma_1 = p \times max_{t\in[0,T]}|\mu_3(t)|; \quad \sigma_2 = p \times max_{t\in[0,T]}|\mu_4(t)|, \tag{29}$$

where p is the percentage of noise. We use the MATLAB builtin function *normrnd* to generate the random variables $\underline{\epsilon}_1 = (\epsilon_{1,j}), j = \overline{0,N}$ and $\underline{\epsilon}_2 = (\epsilon_{2,j}), j = \overline{0,N}$ as follows:, $\underline{\epsilon}_1 = normrnd\,(0,\sigma_1,N)$, $\underline{\epsilon}_2 = normrnd\,(0,\sigma_1,N)$.

5. Results and discussion

In this section, we present numerical results for the recovery of time-dependent coefficients $b(t)$ and $h(t) > 0$ and the temperature (y,t) , in the case of noisy and exact data (9)-(10). To assess the accuracy of the numerical solution, we utilize the RMSE which is defined as:

$$rmse(b) = \left[\frac{T}{N}\sum_{j=1}^{N}\left(b^{numerical}(t_j) - b^{exact}(t_j)\right)^2\right]^{\frac{1}{2}}, \tag{30}$$

$$rmse(h) = \left[\frac{T}{N}\sum_{j=1}^{N}\left(h^{numerical}(t_j) - h^{exact}(t_j)\right)^2\right]^{\frac{1}{2}}, \tag{31}$$

In our simulation, we fix $T = 1$.

5.1 Example for the inverse problem

Consider the inverse problem (1)-(5) with unknown coefficient *b(t)* and unknown free boundary $h(t)$, with the following given input data:

$$\varphi(x) = (1+x)^2, \qquad \mu_1(t) = 1 + 10t, \qquad \mu_2(t) = 10t + (2+t)^2$$
$$f(x,t) = 8 + 3(1+t)(1+x), \qquad \mu_3(t) = 1 + 2(2+t),$$

One can easily remark that the conditions of Theorems 1 and 2 are hold and, therefore, the local existence and uniqueness of the solution are guaranteed. In fact, the exact solution for this problem is given by:

$$u(x,t) = 10t + (1+x)^2, \quad b(t) = -1-t, \quad h(t) = 1+t, \tag{32}$$

and the transformed solution is given by

$$v(y,t) = 10t + (1+(1+t)y)^2, \quad b(t) = -1-t, \quad h(t) = 1+t \tag{33}$$

Let us fix the mesh size to be $M = N = 40$, which gives us reasonable and accurate results (see direct problem output $\mu_3(t)$ and $\mu_4(t)$, Tables 1 and 2, respectively).

In the beginning of our investigation, we start with a noise free case, i. e. $p = 0$ in equation (29). Figure 2 shows the objective function (26) values as a function of the number of iterations for the no regularization applied; $\beta_1 = \beta_2 = 0$. From this figure, the speed convergence minimization toward local minima can be observed with a very low value of order $O(10^{-22})$ in 11 iteration. Whilst, the associated numerical results are presented in Figures 3 and 4. From these figures, one can clearly see the overlap between the exact and numerical solution for unknown functions $h(t)$ and *b(t),* which indicates an excellent agreement with $rmse(h) = 4.9 - 5$ and $rmse(b) = 7.9 - 4$, respectively.
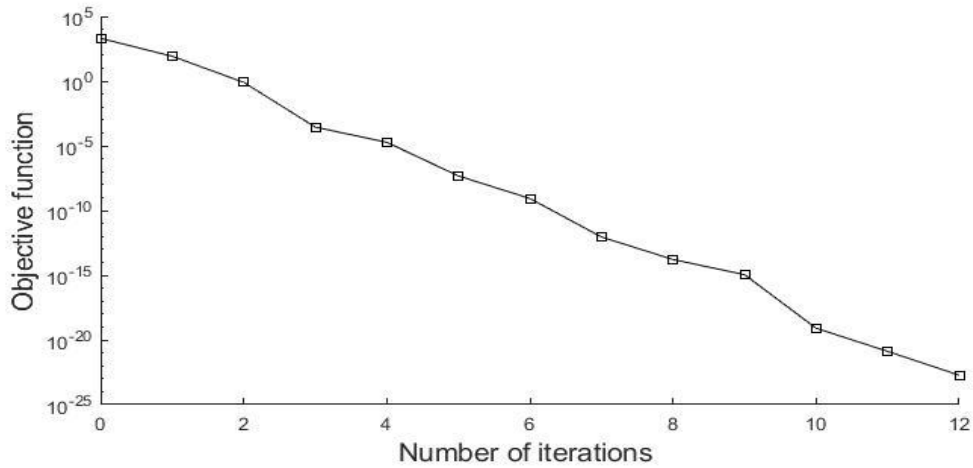
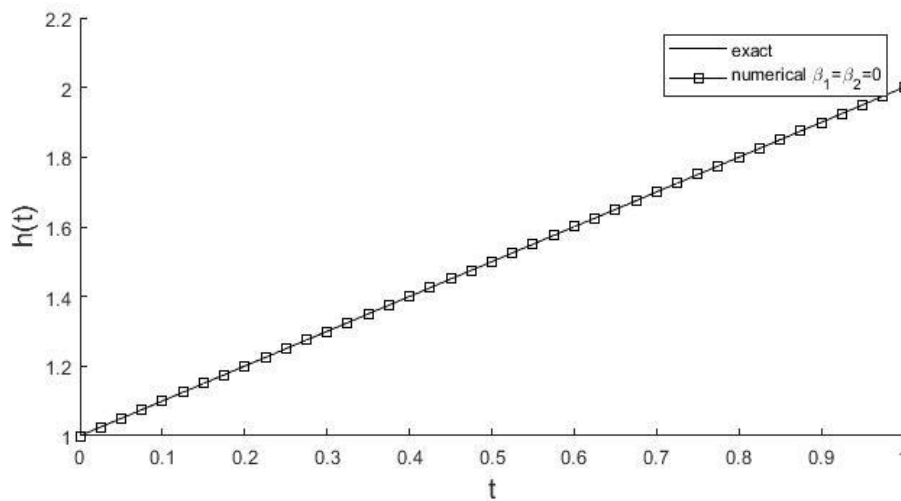**Figure 2**-The objective function (26), where no noise and no regularization are applied.

Figure 3-The exact and numerical solutions for $h(t)$, where no noise and no regularization are applied.
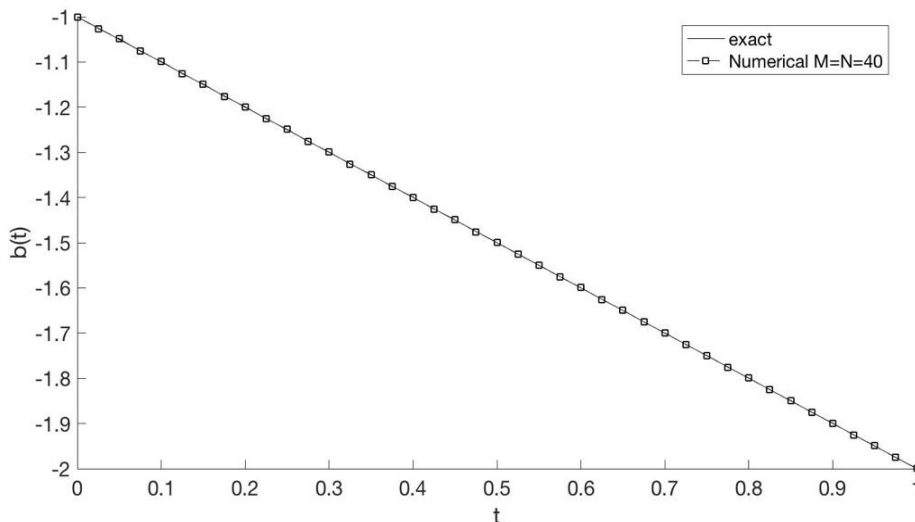

Figure 4-The exact, initial guess, and numerical solutions for b(t), where no noise and no regularization are applied.

Next, we perturb the measured data with $p = 0.01\%$ noise, added as in equations (27) and (28) for μ_3 and μ_4, respectively. In the case where no regularization is applied, i.e. $\beta_1 = \beta_2 = 0$, we obtain an

accurate and stable solution for $h(t)$, while the solution for $b(t)$ is unstable. This is expected since the problem under investigation is ill-posed and any slight error in the input data can lead to enormous errors in the output solutions. Consequently, we will not present results for this case. However, in order to overcome this difficulty, a kind of regularization / stabilization should be applied. We employ Tikhonov regularization method by adding a penalty term $(\beta_1\|h\|^2 + \beta_2\|b\|^2)$ to the objective functional (25). We fix $\beta_1 = 0$ because it seems that the inverse problem is ill-posed only in $b(t)$. Various regularization parameters $\beta_2 \in \{10^{-4}, 10^{-3}, 10^{-2}\}$ are applied in order to gain a stable solution. Trial and error strategy of parameter selection is adapted. We deduce that the appropriate selection for β_3 is 10^{-3}, which meets the lowest *rmse* values for $b(t)$, as demonstrated in Figures 5 - 7 and Table 3. Also, it is reported here, but not listed, that the numerical solution for the transformed solution $v(y,t)$ has an excellent agreement with exact one.
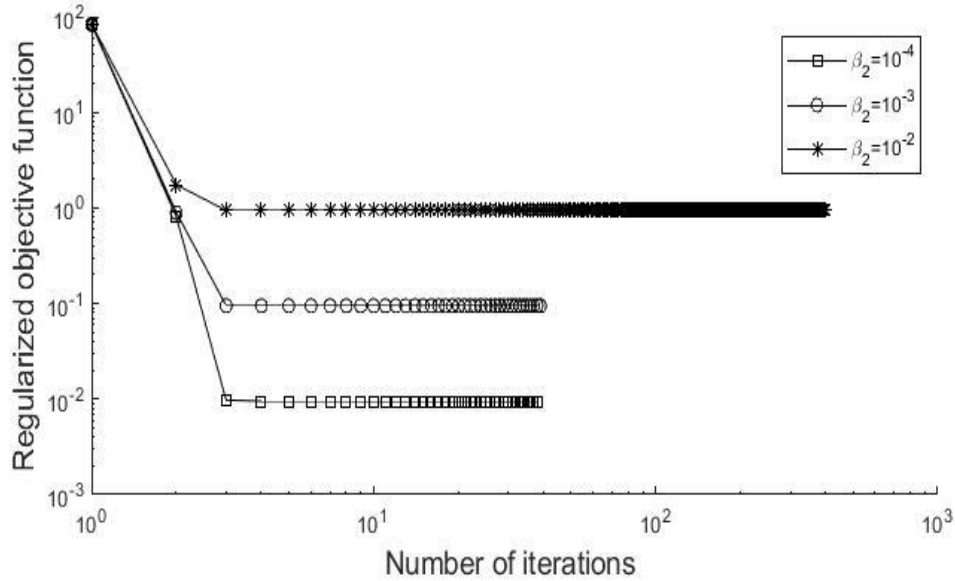

Figure 5-The regularized objective function (26), where $p = 0.01\%$ noise is included in the input data, with a regularization $\beta_2 \in \{10^{-4}, 10^{-3}, 10^{-2}\}$.
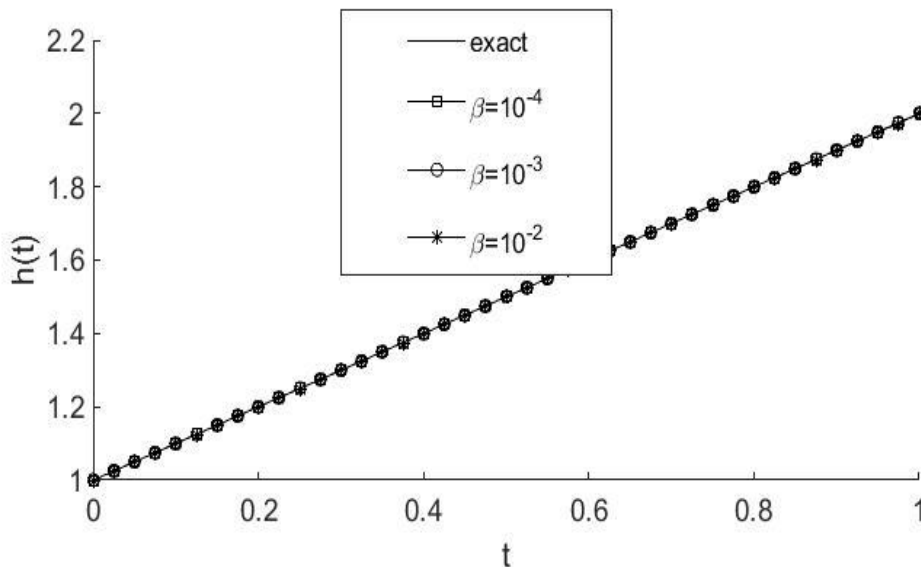

Figure 6-The exact and numerical solutions for h(t), where $p = 0.01\%$ noise is included in the input data, with a regularization $\beta_2 \in \{10^{-4}, 10^{-3}, 10^{-2}\}$.

Figure 7-The exact and numerical solutions for $b(t)$, where $p = 0.01\%$ noise is included in the input data, with a regularization $\beta_2 \in \{10^{-4}, 10^{-3}, 10^{-2}\}$.
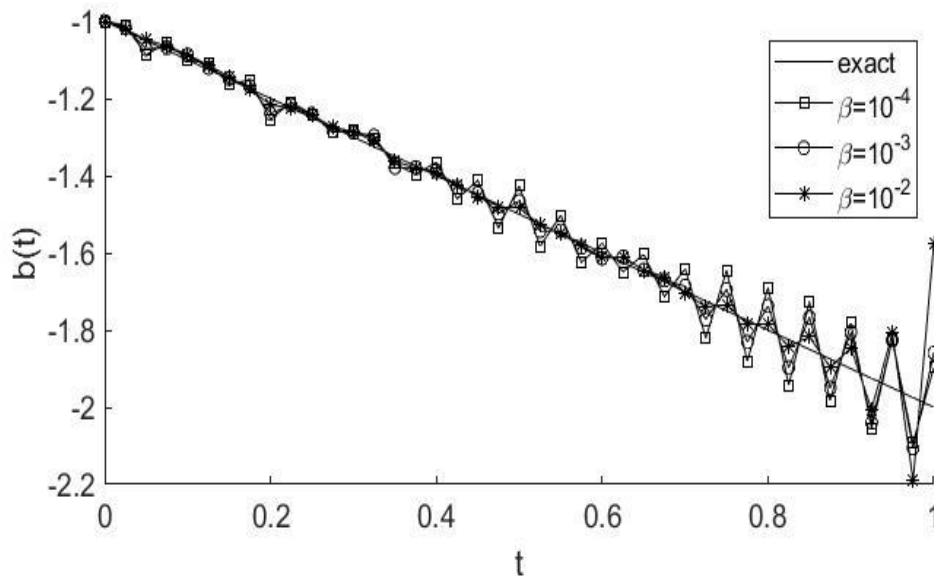
Table 3-Numerical information for various regularization parameters with $p = 0.01\%$ noise

Regularization parameters, $\beta_1 = 0$							
β_2	0	10^{-6}	10^{-5}	10^{-4}	10^{-3}	10^{-2}	10^{-1}
No. of iterations	12	41	38	38	39	401	401
No. of function evaluations	1079	3486	3237	3237	3320	33366	33366
Function value (26) at final iteration	1.8E-23	9.5E-5	9.5E-4	9.4E-3	9.4E-2	9.4E-1	9.09873
rmse(h)	2.5E-4	2.54E-4	2.46E-4	2.3E-4	2.3E-4	9.5E-4	0.0050
rmse(b)	0.1737	0.1674	0.1307	0.0694	0.0528	0.0804	0.1782
Computational time in sec.	439	414.74	1334.91	1359.72	1350	16083.5	13234.3

6. Conclusions

The problem of the determination of time-dependent coefficients in one-dimensional heat equation was investigated. In order to find the solution of the inverse problem, a quasi-solution was sought, which looked at the minimization of the gap between the measured data and numerically simulated data. However, the problem was still ill-posed and sensitive for error (noise) inclusion in the input data. This implied that we need to apply a regularization method to stabilize the solution, which was ensured by using Tikhonov regularization method. The presented results are accurate and stable.

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