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Essential-small Projective Modules

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Abstract
In this paper, we introduce the concept of e-small Projective modules as a generlization of Projective modules.

Keywords: projective modules, e-small submodules, e-small projective modules.

مقاسات جوهرية صغيرة اسقاطية

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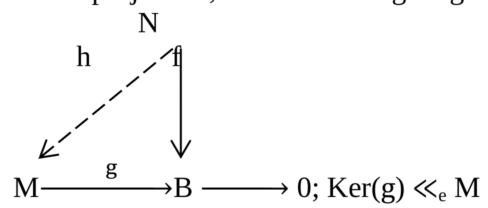
الخلاصة
في هذا البحث قدمنا مفهوم مقاسات جوهرية صغيرة اسقاطية كتعميم لمفهوم مقاسات اسقاطية .

1. Introduction
A. K. Tiwary and K. N. Chaubey studied the concept of small projective modules as a proper generalization of projective modules [1]. In this paper, we introduce the concept of e-small Projective modules as a generlization of Projective modules. In this paper, all rings are associative and all modules are right and unitary. For definitions and notations in this paper we refer to [2] and [3].

2. e-small Projective
Definition 2.1
A module N is called e-small projective, if the following diagram is commutative:



Where g is an e-small epimorphism and f is a homomorphism.
Clearly every projective modules is e-small projective.

Proposition 2.2 For a module Q, the following statements are equivalent:

a. Q is e-small projective module;
b. For each e-small epimorphism f: N ⟶ K , the functor Hom(I, f) : Hom(Q, N) ⟶ Hom(Q, K) is an epimmorphism;
c. For any e-small epimorphism g : B ⟶ A, g∘Hom(Q, B) = Hom(Q, A);

Proof (a) ⟹ (b) . Let f : N ⟶ K be an e-small epimorphism and
ψ ∈ Hom(Q, K). Since Q is e-small projective module there exists a homomorphism h: Q ⟶ N, such that f∘h = **ψ**. Thus Hom(I, f)∘h = **ψ,** where h ∈ Hom (Q, N).therefore Hom(I, f) is an epimorphism.

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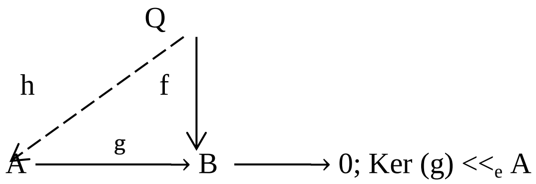
(b) $\Longrightarrow$ (c). Let g : B $\longrightarrow$ A be an e-small epimorphism, by (b) Hom(I, g) : Hom(Q, B) $\longrightarrow$ Hom(Q, A) is an epimorphism.
Now, to show that g∘Hom(Q,B) = Hom(Q, A).
Let f ∈ Hom(Q, A) so there exists f_1 ∈ Hom(Q,B) such that Hom(I, g)∘f_1=f .
i.e g∘f_1 = f . Thus f ∈ g∘Hom(Q,B); so Hom(Q, A) ≤ g∘Hom(Q, B).
It is clear that g∘Hom(Q, B) ≤ Hom(Q, A).
(c) $\Longrightarrow$ (a). Consider the following diagram:

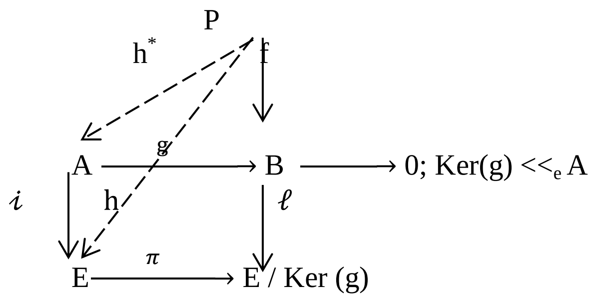


Where A, B are any modules and f is any homomorphism since g∘Hom(Q, A) = Hom(Q, B) and f ∈ Hom(Q, B).So there exists h ∈ Hom(Q, A), such that g∘h = f. Thus, Q is an e-small projective module.

Proposition 2.3 A module P is e-small projective if and only if for every homomorphism f : P $\longrightarrow$ B, and every e-small epimorphism g : A $\longrightarrow$ B from an injective module A, there exists a homomorphism h: P $\longrightarrow$ A such that g∘h = f.

Proof $\Longrightarrow$ ⟩ Clear.

$\Longleftarrow$ ⟩ Let g be any e-small epimorphism from A onto B, where A, B are any modules, and f: P $\longrightarrow$ B be any homomorphism. Consider the following diagram:

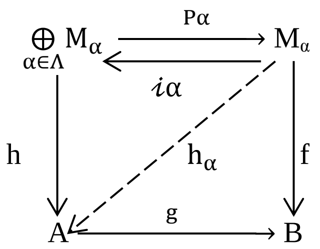


Where E is injective module, $\imath$: A $\longrightarrow$E is the inclusion homomorphism and π : E $\longrightarrow$ E/Ker(g) is the nature epimorphism. E exists, since every module can be embedded in an injective module, [1].
Define ℓ: B $\longrightarrow$ E/ Ker(g) by ℓ(b) = a + Ker(g), for all b ∈ B, where g(a) = b.
Let b,$\acute{b}$ ∈ B, where g(a) = b and g($\acute{a}$) = $\acute{b}$.
If $b = \acute{b}$ this implies g(a)=g($\acute{a}$), which means that $a - \acute{a}$ ∈ Ker(g), so a + Ker(g) = $\acute{a}$ + Ker(g). So ℓ is well define. Clearly ℓ is a homomorphism.
By hypothesis, there exists a homomorphism h: P $\longrightarrow$ E,such that π∘h = ℓ∘f.
We claim that h(P) ≤ A. To see this, let w ∈ h(P),
So there exists m∈ P, with w=h(m). Now, π∘h(m) = ℓ∘f(m), where f(m)= g(a). This implies that h(m) – a ∈ Ker(g) and hence h(m) ∈ A.
Let h^* : P $\longrightarrow$ A defined by $h^*(x)$ = h(x), for all x ∈ P.
Now, ℓ∘f = π∘h = π∘$\imath$∘h^* = ℓ∘g∘h^* .
T. P. that ℓ is monomorphism. Let $\ell(b) = \ell(\acute{b})$, where $b, \acute{b}$ ∈ B
So a + Ker(g) = $\acute{a}$ + Ker(g) where g(a) = b and g($\acute{a}$) = $\acute{b}$. Thus a − $\acute{a}$ ∈ Ker(g) this implies that g(a) = g($\acute{a}$) and so $b = \acute{b}$, thus ℓ is monomorphism.
Hence P is an e-small projective module.

3 Some properties of e-small projective Modules

Proposition 3.1 $\bigoplus_{\alpha\in\Lambda} M_\alpha$ is an e-small projective module if and only if M_α is an e-small projective module for each $\alpha \in \Lambda$.

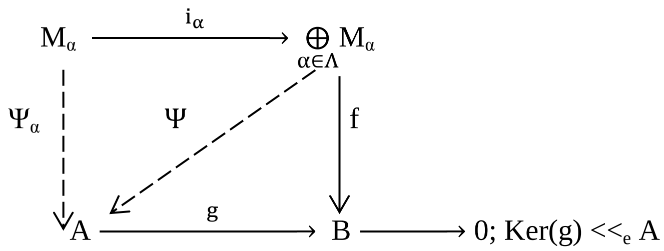
Proof ⟹⟩ Suppose that $\bigoplus_{\alpha\in\Lambda} M_\alpha$ is an e-small projective and let $\alpha \in \Lambda$, consider the following diagram:



Where $g : A \longrightarrow B$ is an e-small epimorphism, $f: M_\alpha \longrightarrow B$ is a homomorphism, $p_\alpha: \bigoplus_{\alpha\in\Lambda} M_\alpha \longrightarrow M_\alpha$ is the projection homomorpism and $i_\alpha : M_\alpha \longrightarrow \bigoplus_{\alpha\in\Lambda} M_\alpha$ is the injective homomorphism. Since $\bigoplus_{\alpha\in\Lambda} M_\alpha$ is ane-small projective module, there exists a homomorphism $h : \bigoplus_{\alpha\in\Lambda} M_\alpha \longrightarrow A$, such that $g\circ h=f\circ p_\alpha$. Define $h_\alpha : M\alpha \longrightarrow A$ by $h_\alpha = h\circ i_\alpha$. Now, $g\circ h_\alpha= g\circ h\circ i_\alpha = f\circ p_\alpha\circ i_\alpha = f\circ I = f$.

Hence M_α is an e-small projective module.

⟸⟩ suppose that $M\alpha$ is an e-small projective module, for each $\alpha \in \Lambda$, and consider the following diagram:



Where $g:A \longrightarrow B$ is an e-small epimorphism, $f : \bigoplus_{\alpha\in\Lambda} M_\alpha \longrightarrow B$ is a homomorphism and $i_\alpha: M_\alpha \longrightarrow \bigoplus_{\alpha\in\Lambda} M_\alpha$ is the injective homomorphism since M_α is an e-small projective module for all $\alpha \in \Lambda$, there exists a homomorphism $\Psi\alpha : M\alpha \longrightarrow A$ for all $\alpha \in \Lambda$ such that $g\circ\Psi_\alpha = f\circ i_\alpha$, for all $\alpha \in \Lambda$.

Define $\Psi : \bigoplus_{\alpha\in\Lambda} M\alpha \longrightarrow A$ by $\Psi(a) = \sum_{\alpha\in\Lambda} \Psi_\alpha \circ p_\alpha(a_\alpha)$ for each $a \in \bigoplus_{\alpha\in\Lambda} M_\alpha$,

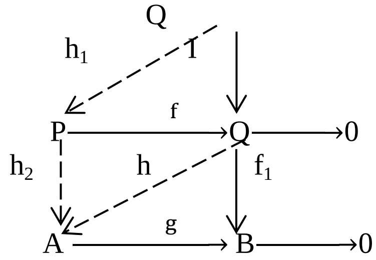
$g\circ\Psi(a)=g(\Psi(a)) = g(\sum_{\alpha\in\Lambda} \Psi_\alpha \circ p_\alpha(a_\alpha)) = \sum_{\alpha\in\Lambda} g \circ \Psi_\alpha \circ p_\alpha(a_\alpha)$

$= \sum_{\alpha\in\Lambda} f \circ i_\alpha \circ p_\alpha(a_\alpha)= f(\sum_{\alpha\in\Lambda} i_\alpha \circ p_\alpha(a_\alpha)) = f(I(a))=f(a)$ for each $a \in \bigoplus_{\alpha\in\Lambda} M\alpha$

Hence $\bigoplus_{\alpha\in\Lambda} M_\alpha$ is an e-small projective module.

Proposition 3.2 An e-small projective module which has a projective cover is projective.

Proof Let Q be an e-small projective module. Let (P, f) be a protective cover for Q. consider the following diagram



Where g: A ⟶ B is an epimorphism, f_1: Q ⟶ B is a homomorphism and I: Q ⟶ Q is the identity. Since Q is an e-small projective module, there exists a homomorphism h_1 : Q ⟶ P such that $f∘h_1 = I$. But P is a projective module so, there exists a homomorphism h_2 : P ⟶ A, such that $g∘h_2 = f_1∘f$. Definition h: Q ⟶ A by $h=h_2∘h_1$. Now, $g∘h=g∘h_2∘h_1 = f_1∘f∘h_1 = f_1∘I= f_1$. Thus, Q is a projective module.

Recall that A submodule L of P is called P-cyclic submodule if it is the image of an element of End(P) [4].A module L is called N-principally injective if for any endomorphism Ψ of N, and every homomorphism from Ψ(N) into L, can be extended to a homomorphism from N to L[4].

Definition 3.3

A module P is called e-small factor of a module L, if there exists an e-small epimorphism from L to P.

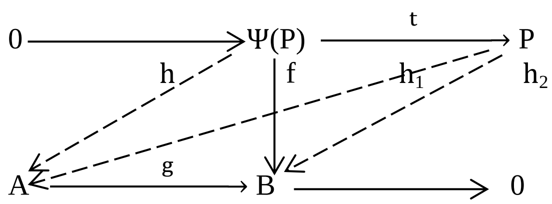
Proposition 3.4 Let P be an e-small projective module. The following are equivalent:

1- Every P-cyclic submodule of P is an e-small projective;

2- Every e-small factor module of an P-principally injective module is P- principally injective;

3- Every e-small factor module of an injective module is P-principally injective.

Proof 1⟶ 2⟩ Let g: A ⟶ B be an e-small epimorphism, where A is P- principally injective module. Consider the following diagram:



Where f: Ψ(P) ⟶B is a homomorphism, Ψ ∈ End(P) and t: Ψ(P) ⟶ P is inclusion homomorphism. By(1), Ψ(P) is an e-small projective module so, there exists a hmomorphism h: Ψ(P) ⟶ A such that g∘h = f. Now, since A is P-principally injective, there exists a homomorphism h_1: P ⟶ A such that $h_1∘t=h$. Define h_2: P ⟶ B by $h_2=g∘h_1$. Now, $h_2∘t= g∘h_1∘t = g∘h=f$.

2⟶ 3⟩ Clear.

3⟶ 1⟩ By propositions (2.3).

Proposition 3.5 Let Q be a module and C is adirect summand of Q, such that A∩C ≪ A, where A ≤ Q, if A+C is an e-small projective module, then A∩C=(0).

Proof Consider the following natural epimorphism: π_1 : A ⟶ A /A∩C ; π_2 :A+C ⟶A+C/ C by second isomorphism theorem A /A∩C ≃ A+C/ C. Since C is a direct summand of Q so, Q=C⊕K_1 , Where K_1 ≤ Q, by modular law Q∩(C+A)=(C⊕K_1)∩(A+C) So, A+C =C⊕(K_1∩(A+C)), so C is a direct summand of A+C. By (3.1) (K_1∩(A+C)) is an e-small projective module and hence A+C/ C is an e-small projective module and so is A /A∩C.

Thus π_1: A ⟶ A/A∩C splits, so A= Ker(π_1) ⊕ L, where L ≤ A, but A∩C≪A, therefore A∩C= (0).

The converse of (3.5) is no true in general as the following example.

Example 3.6 In Z_2 as Z-module, clearly $\{\bar{0}\}$ is a direct summand of Z_2 and $\{\bar{0}\}$∩ Z_2= (0)≪ Z_2, but Z_2 is not e-small projective as Z-module.

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