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Certain Properties for Analytic Functions Associated with q-Ruscheweyh Differential Operator

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Abstract
In this paper, by making use of the q-Ruscheweyh differential operator $\mathfrak{R}_q^k f(z)$, and the notion of the Janowski function, we study some subclasses of holomorphic functions . Moreover, we obtain some geometric characteristic such as coefficient estimates, radii of starlikeness, distortion theorem, close-to-convexity, convexity, extreme points, neighborhoods, and the integral mean inequalities of functions affiliated to these classes.

Keywords: Analytic functions, Subordination, q-Ruscheweyh derivative, Hadamard product, Univalent functions.

بعض الخصائص لدوال التحليلية المرتبطة بالمؤثر التفاضلي q−Rusceweyh

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الخلاصه
في هذه البحث،من خلال باستخدام المؤثر التفاضلي q−Ruscheweh $\mathfrak{R}_q^k f(z)$, ومفهوم الدالة يانوفسكي . ندرس بعض الاصناف الفرعية من الدوال التحليلية. علاوة على ذلك ، نحصل على بعض الخصائص الهندسية مثل تقديرات المعامل ، نظرية التشويه ، نصف الاقطار النجمية ،المحدبة والقريبة من التحدب ، النقاط المتطرفة ، والجوارات ، متباينات قيم التكامل للدوال المنتمية إلى هذه الاصناف

Introduction

Let $\mathcal{A}$ represents the class of functions f which are holomorphic functions in the unit disc $E^* = \{z \in \mathbb{C} : |Z| < 1\}$ and of the form

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m, \quad (z \in E^*) \tag{1}$$

The subclass of A consisting of univalent functions is denoted by S . A function f in $\mathcal{A}$ is said to be starlike of order $\sigma(0 \le \sigma < 1)$ in E^* if this condition satisfies

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \sigma, \qquad (Z \in E^*).$$

$S^*(\sigma)$ symbolize this class . In certain, for$(\sigma = 0)$, we obtain $S^*(0) = S^*$, the class of starlike functions. The class $C(\sigma), (0 \le \sigma < 1)$ comprised of convex functions of order σ can be expressed by the relation $f \in C(\sigma)$ if and only if $zf' \in S^*(\sigma)$

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Let f and g be holomorphic functions such that both the subordination between f and g in E^* are written as $f \prec g$ or $f(z) \prec g(z)$. In addition to that, we say that f is subordinate to g if there is a Schwarz function w with $w(z) = 0, |w(z)| < 1, z \in E^*$,
such that $f(z) = g(w(z))$ for all $z \in E^*$. Furthermore, if $g(z)$ is univalent in E^*, then we have the following equivalence:
$f \prec g$ if and only if $f(0) = g(0)$ and $f(E^*) \subseteq g(E^*)$.
For some details, see earlier works [1,2,3] . By the application of the notion of subordination, Janowski provided the class$P[A, B]$. A given holomorphic function h with $h(0) = 1$ is said to be in the class $P[A, B]$, if and only if the following condition satisfies:

$$h(z) \prec \frac{1 + Az}{1 + Bz}, \qquad -1 \le B < A \le 1.$$

Geometrically, the function $h(z) \in P[A, B]$ maps the unit disk E^* onto the domain $\Xi[A, B]$ defined by

$$\Xi[A, B] = \left\{w: \left|w - \frac{1 - AB}{1 - B^2}\right| < \frac{A - B}{1 - B^2}\right\}. \tag{2}$$

This domain symbolizes an open circular disk centered on real axis with diameter end points $D_1 = \frac{1-A}{1-B}$ and $D_2 = \frac{1+A}{1+B}$ with $0 < D_1 < 1 < D_2$.
Consider $f, g \in \mathcal{A}$. Then, the convolution * or Hadamard product of f in $\mathcal{A}$ and $g(z) = z + \sum_{m=2}^{\infty} b_m z^m$ are defined as:

$$(f * g) = z + \sum_{m=2}^{\infty} a_m b_m z^m. \quad (z \in E^*).$$

Now, we define the Ruscheweyh derivative operator $\mathcal{R}^k$ as follows

$$\mathcal{R}^k f(z) = \frac{z}{(1 - z)^{k+1}} * f(z).$$

Hence

$$\mathcal{R}^k f(z) = \frac{z\left(z^{k-1} f(z)\right)^{(k)}}{k!}.$$

For $\in \mathbb{N}_0 = \{0,1,2,3 \dots\}, z \in E^*$,
as previously described [4]. We briefly recover here the concept of q-operators, i.e., q-difference operator that takes a vital role in hyper geometric series, quantum physics, and operator theories. The usage of q-calculus was initiated by Jackson [5] (also see [6, 7]). For the applications of q-calculus in geometric function theory, one may refer to the papers of Mohamad and Darus [8], Mohamad and Sokol [9], and Purohit and Raina [10].
Next , we provide some fundamental definitions and results of q -calculus which we shall apply in our results. For more information, see earlier reports [10,11,12]. The application of q-calculus was initiated by JacksOn [13] (also see [14,15]) in geOmetric function theory.
Now, if $q \in (0,1)$ is fixed, then JacksOn explained the q-derivative and the q-integral of $f \in \mathcal{A}$ as in the next step:

$$\partial_q f(z) = \frac{f(z) - f(zq)}{z(1 - q)}, \qquad (q \in (0,1), z \in E^*). \tag{3}$$

and

$$\int_0^z f(z)\, \partial_q t = z(1 - q) \sum_{m=0}^{\infty} q^m f(zq^m),$$

if that series converges.
It can simply be seen that for $m \in \mathbb{N}_0 = \{0,1,2,3, \dots\}$ and $z \in E^*$,

$$\partial_q \left\{ \sum_{m=1}^{\infty} a_m z^m \right\} = \sum_{m=1}^{\infty} [m, q] a_m z^{m-1},$$

where

$$[m, q] = \frac{1 - q^m}{1 - q} = \sum_{i=1}^{m-1} q^i + 1, \;\; [0, q] = 0. \tag{4}$$

For every non-negative integer n, the q-number shift factorial is defined by

$$[m,q]! = \begin{cases} 1, & m = 0 \\ [1,q][2,q][3,q]\dots[m,q], & m \in \mathbb{N}. \end{cases}$$

In addition, the q-generalized Pochhamer symbol for $y > 0$ is defined as

$$[m,q]! = \begin{cases} 1, & m = 0 \\ [y,q][y+1,q]\dots[y+m-1,q], & m \in \mathbb{N}. \end{cases}$$

Let F be the function given as

$$F_{k+1,q} = z + \sum_{m=2}^{\infty} \frac{[k+1,q]_{m-1}}{[m-1,q]!} z^m. \tag{5}$$

Now, the differential q-Ruscheweyh operator $\mathfrak{R}_q^k: \mathcal{A} \longrightarrow \mathcal{A}$ of order $k \in \mathbb{N}_0 = \{0,1,2,\dots\}, q \in (0,1)$ and for f given by (1) is defined as

$$\mathfrak{R}_q^k f(z) = F_{k+1,q}(z) * f(z)$$
$$= z + \sum_{m=2}^{\infty} \vartheta_q^k[m] a_m z^m, \tag{6}$$

where $\vartheta_q^k[m] = \frac{[k+1,q]_{m-1}}{[m-1,q]!}$ (for more details see a previous report [16]),
and $\mathfrak{R}_q^0 f(z) = f(z)$, also $\mathfrak{R}_q^1 f(z) = z\partial_q f(z)$.
Equation (6) can be expressed as

$$\mathcal{R}_q^k f(z) = \frac{z\partial_q^k\left(z^{k-1} f(z)\right)}{[k,q]!}, \quad k \in \mathbb{N}_0.$$

Since

$$\lim_{q \to 1^-} F_{k+1,q}(z) = \frac{z}{(1-z)^{k+1}},$$

it follows that

$$\lim_{q \to 1} \mathcal{R}_q^k f(z) = \frac{z}{(1-z)^{k+1}} * f(z) = \mathcal{R}^k.$$

Definition 1. Let $\mathcal{J}_{k,j}(q,\beta,A,B)$ indicate the subclass of $\mathcal{A}$ consisting of functions f of the form (1) and satisfy the following subordination condition,

$$\frac{\mathcal{R}_q^k f(z)}{\mathcal{R}_q^j f(z)} - \beta\left|\frac{\mathcal{R}_q^k f(z)}{\mathcal{R}_q^j f(z)} - 1\right| \prec \frac{1+Az}{1+Bz},$$

where $-1 \le B < A \le 1, \beta \ge 0, k \in \mathbb{N}_0, j \in \mathbb{N}_0, k > j, q \in (0,1), z \in E^*$
We note the following:
(i) For $A = 1-2\alpha; B = -1; \beta = 0; k = 1$ and $j = 0$, the class $\mathcal{J}_{k,j}(q,\beta,A,B)$ reduces to the class $S_q^*(\alpha)$ discussed by Agrawal and Sahoo [17].
(ii) For $A = 1; B = -1; \beta = 0; k = 1$ and $j = 0$, the class $\mathcal{J}_{k,j}(q,\beta,A,B)$ reduces to the class S_q^* discussed by Ismail *et al*. [18].
(iii) For $q \to 1; \beta = 0; k = 1;$ and $j = 0$, the class $\mathcal{J}_{k,j}(q,\beta,A,B)$ reduces to the class $S_q^*(A,B)$ discussed by Libera [19].
(iv) For $q \to 1; \beta = 0; k = 1$and $j = 0$, the class $\mathcal{J}_{k,j}(q,\beta,A,B)$ reduces to class $S^*(A,B)$ discussed by Janowski [20].
(v) For $q \to 1; \beta = 0; k = 2$ and $j = 1$, the class $\mathcal{J}_{k,j}(q,\beta,A,B)$ reduces to the class $\mathcal{K}(A,B)$ discussed by Padmanabhan and Ganesan [21] .
(vi) For $q \to 1;$ $B = -1$and $A = 1-2\alpha$, the class $\mathcal{J}_{k,j}(q,\beta,A,B)$ reduces to class $N_{k,j}(\alpha,\beta), (0 \le \alpha < 1)$ discussed by Eker and Owa [22].
(vii) For $k = 1; q \to 1; A = 1-2\alpha; B = -1$ and $j = 0$, the class $\mathcal{J}_{k,j}(q,\beta,A,B)$ reduces to the class $\mathcal{US}(\alpha,\beta)$ $(0 \le \alpha < 1)$, $(0 \le \alpha < 1)$discussed by Shams *et al*.[23].
Definition 2. Let $\mathcal{T}$ represents the subclass of functions of $\mathcal{A}$ of the form:

$$f(z) = z - \sum_{m=2}^{\infty} a_m z^m, \ a_m \ge 0. \tag{7}$$

Further, we define the class $\mathcal{TJ}_{k,j}(q,\beta,A,B) = \mathcal{J}_{k,j}(q,\beta,A,B) \cap \mathcal{T}$.
For more details refer to an earlier work [24].

Main Results

In this part, we will prove our main results.

Theorem 1. A function f of the form (1) belongs to the class $\mathcal{J}_{k,j}(q,\beta,A,B)$ if:

$$\sum_{m=2}^{\infty}\left\{(\beta(|B|+1)\left(\vartheta_q^k[m]-\vartheta_q^j[m]\right)+\left|B\vartheta_q^k[m]-A\vartheta_q^j[m]\right|\right\}a_m \le A-B. \tag{8}$$

where $-1 \le B < A \le 1, \beta \ge 0, k \in \mathbb{N}_0, j \in \mathbb{N}_0, k > j, q \in (0,1)$

Proof. It is sufficient to prove that

$$\left|\frac{p(z)-1}{A-Bp(z)}\right| < 1,$$

where

$$p(z) = \frac{\mathcal{R}_q^k f(z)}{\mathcal{R}_q^j f(z)} - \beta\left|\frac{\mathcal{R}_q^k f(z)}{\mathcal{R}_q^j f(z)} - 1\right|.$$

We obtain

$$\left|\frac{p(z)-1}{A-Bp(z)}\right| = \left|\frac{\mathcal{R}_q^k f(z) - \mathcal{R}_q^j f(z) - \beta e^{i\theta}\left|\mathcal{R}_q^k f(z) - \mathcal{R}_q^j f(z)\right|}{A\mathcal{R}_q^j f(z) - B\left[\mathcal{R}_q^k f(z) - \beta e^{i\theta}\left|\mathcal{R}_q^k f(z) - \mathcal{R}_q^j f(z)\right|\right]}\right|$$

$$= \left|\frac{\sum_{m=2}^{\infty}\left\{\left(\vartheta_q^k[m]-\vartheta_q^j[m]\right)a_m z^m - \beta e^{i\theta}\left|\sum_{m=2}^{\infty}\left(\vartheta_q^k[m]-\vartheta_q^j[m]\right)a_m z^m\right|\right\}}{(A-B)z - \left[\sum_{m=2}^{\infty}\left(B\vartheta_q^k[m]-A\vartheta_q^j[m]\right)a_m z^m - Be^{i\theta}\left|\sum_{m=2}^{\infty}\left(\vartheta_q^k[m]-\vartheta_q^j[m]\right)a_m z^m\right|\right]}\right|$$

$$\le \frac{\sum_{m=2}^{\infty}\left(\vartheta_q^k[m]-\vartheta_q^j[m]\right)|a_m||z^m| + \beta\sum_{m=2}^{\infty}\left(\vartheta_q^k[m]-\vartheta_q^j[m]\right)|a_m||z^m|}{(A-B)|z| - \left[\sum_{m=2}^{\infty}\left|B\vartheta_q^k[m]-A\vartheta_q^j[m]\right||a_m||z^m| + \beta|B|\sum_{m=2}^{\infty}\left(\vartheta_q^k[m]-\vartheta_q^j[m]\right)|a_m||z^m|\right]}$$

$$\le \frac{\sum_{m=2}^{\infty}\left(\vartheta_q^k[m]-\vartheta_q^j[m]\right)(1+\beta)|a_m|}{(A-B) - \sum_{m=2}^{\infty}\left|B\vartheta_q^k[m]-A\vartheta_q^j[m]\right||a_m| - \beta|B|\sum_{m=2}^{\infty}\left(\vartheta_q^k[m]-\vartheta_q^j[m]\right)|a_m|}.$$

This final statement is bounded above by one if

$$\sum_{m=2}^{\infty}\left\{(\beta(|B|+1)\left(\vartheta_q^k[m]-\vartheta_q^j[m]\right)+\left|B\vartheta_q^k[m]-A\vartheta_q^j[m]\right|\right\}a_m \le A-B.$$

hence, the proof is completed. □

Theorem 2. Consider that $f \in \mathcal{T}$. Then, $f \in \mathcal{TJ}_{k,j}(q,\beta,A,B)$ if and only if:

$$\sum_{m=2}^{\infty}\left\{(\beta(|B|+1)\left(\vartheta_q^k[m]-\vartheta_q^j[m]\right)+\left|B\vartheta_q^k[m]-A\vartheta_q^j[m]\right|\right\}a_m \le A-B.$$

Proof. Since $\mathcal{TJ}_{k,j}(q,\beta,A,B) \subset \mathcal{J}_{k,j}(q,\beta,A,B)$ for functions $f \in \mathcal{TJ}_{k,j}(q,\beta,A,B)$ we can put: $\left|\frac{p(z)-1}{A-Bp(z)}\right| < 1$,where $p(z) = \frac{\mathcal{R}_q^k f(z)}{\mathcal{R}_q^j f(z)} - \beta\left|\frac{\mathcal{R}_q^k f(z)}{\mathcal{R}_q^j f(z)} - 1\right|$.

Then

$$\left|\sum_{m=2}^{\infty}\left\{\left(\vartheta_q^k[m]-\vartheta_q^j[m]\right)a_m z^m - \beta e^{i\theta}\left|\sum_{m=2}^{\infty}\left(\vartheta_q^k[m]-\vartheta_q^j[m]\right)a_m z^m\right|\right\} \times \left\{\begin{array}{c}(A-B)z + \sum_{m=2}^{\infty}\left(B\vartheta_q^k[m]-A\vartheta_q^j[m]\right)a_m z^m \\ +Be^{i\theta}\left|\sum_{m=2}^{\infty}\left(\vartheta_q^k[m]-\vartheta_q^j[m]\right)a_m z^m\right|\end{array}\right\}^{-1}\right| < 1.$$

Since $Re(z) \leq |z|$, then we get

$$Re\left\{\begin{array}{c}\sum_{m=2}^{\infty}\left\{\left(\vartheta_q^k[m]-\vartheta_q^j[m]\right)a_m z^m-\beta e^{i\theta}\left|\sum_{m=2}^{\infty}\left(\vartheta_q^k[m]-\vartheta_q^j[m]\right)a_m z^m\right|\right\} \\ \times\left\{\begin{array}{c}(A-B)z+\sum_{m=2}^{\infty}\left(B\vartheta_q^k[m]-A\vartheta_q^j[m]\right)a_m z^m \\ +Be^{i\theta}\left|\sum_{m=2}^{\infty}\left(\vartheta_q^k[m]-\vartheta_q^j[m]\right)a_m z^m\right|\end{array}\right\}^{-1}\end{array}\right\}<1.$$

Now taking z to be real and letting $z \to 1^-$, we have

$$\sum_{m=2}^{\infty}\left\{(1+\beta(1-B)\left(\vartheta_q^k[m]-\vartheta_q^j[m]\right)-\left|B\vartheta_q^k[m]-A\vartheta_q^j[m]\right|\right\}a_m \leq A-B.$$

or equivalently

$$\sum_{m=2}^{\infty}\left\{(1+\beta(1+|B|)\left(\vartheta_q^k[m]-\vartheta_q^j[m]\right)+\left|B\vartheta_q^k[m]-A\vartheta_q^j[m]\right|\right\}a_m \leq A-B.$$

By this the proof is finished.

Corollary 1. A function $f \in \mathcal{T}$ is in the class $\mathcal{TJ}_{k,j}(q,\beta,A,B)$. Then:

$$a_m \leq \frac{A-B}{\left\{(1+\beta(1+|B|)\left(\vartheta_q^k[m]-\vartheta_q^j[m]\right)+\left|B\vartheta_q^k[m]-A\vartheta_q^j[m]\right|\right\}}, m \geq 2. \quad (9)$$

The result of the function is sharp, as follows:

$$f(z)=z-\frac{A-B}{\left\{(1+\beta(1+|B|)\left(\vartheta_q^k[m]-\vartheta_q^j[m]\right)+\left|B\vartheta_q^k[m]-A\vartheta_q^j[m]\right|\right\}}z^2, \quad m \geq 2. \quad (10)$$

That is, the function defined in (10) can achieve the equality.

Distortion theorems

Theorem 3. Consider the function f defined by (7) in the class $\mathcal{TJ}_{k,j}(q,\beta,A,B)$. Then:

$$|f(z)| \geq |z|-\frac{A-B}{\left\{(\beta(1+|B|)+1\left(\vartheta_q^k[2]-\vartheta_q^j[2]\right)+\left|B\vartheta_q^k[2]-A\vartheta_q^j[2]\right|\right\}}|z|^2,$$

and

$$|f(z)| \leq |z|+\frac{A-B}{\left\{(\beta(1+|B|)+1\left(\vartheta_q^k[2]-\vartheta_q^j[2]\right)+\left|B\vartheta_q^k[2]-A\vartheta_q^j[2]\right|\right\}}|z|^2,$$

The result is sharp.

Proof . From Theorem 2, let the function:

$$\Omega(\mathrm{m})=\left\{(\beta(1+|B|)+1\left(\vartheta_q^k[m]-\vartheta_q^j[m]\right)+\left|B\vartheta_q^k[m]-A\vartheta_q^j[m]\right|\right\}.$$

Then, it is obvious that it is an increasing function of $m(m \geq 2)$, therefore:

$$\Omega(2)\sum_{m=2}^{\infty}|a_m| \leq \sum_{m=2}^{\infty}\Omega(\mathrm{m})\,|a_m| \leq A-B.$$

That is

$$\sum_{m=2}^{\infty}|a_m| \leq \frac{A-B}{\Omega(2)}.$$

Thus, we get

$$|f(z)| \leq |z|+|z|^2\sum_{m=2}^{\infty}|a_m|,$$

$$|f(z)| \leq |z|+\frac{A-B}{\left\{(\beta(1+|B|)+1\left(\vartheta_q^k[2]-\vartheta_q^j[2]\right)+\left|B\vartheta_q^k[2]-A\vartheta_q^j[2]\right|\right\}}|z|^2.$$

Likewise, we get

$$|f(z)| \geq |z| - \sum_{m=2}^{\infty} |a_m| |z|^M \geq |f(z)| \geq |z| - |z|^2 \sum_{m=2}^{\infty} |a_m|,$$

$$\geq |z| - \frac{A-B}{\left\{(\beta(1+|B|)+1\left(\vartheta_q^k[2]-\vartheta_q^j[2]\right)+\left|B\vartheta_q^k[2]-A\vartheta_q^j[2]\right|\right\}} |z|^2.$$

Lastly, we can achieve the equality for the function, as follows:

$$f(z) = z - \frac{A-B}{\left\{(1+\beta(1+|B|)\left(\vartheta_q^k[2]-\vartheta_q^j[2]\right)+\left|B\vartheta_q^k[2]-A\vartheta_q^j[2]\right|\right\}} z^2. \quad (11)$$

At $|z| = r$ and $z = re^{i(2k+1)\pi} (k \in \mathbb{Z})$. This ended the result.

Theorem 4. Let the function f be defined by (7) in the class $\mathcal{TJ}_{k,j}(q, \beta, A, B)$. Then:

$$\left|f'(z)\right| \geq 1 - \frac{2(A-B)}{\left\{(\beta(1+|B|)+1\left(\vartheta_q^k[2]-\vartheta_q^j[2]\right)+\left|B\vartheta_q^k[2]-A\vartheta_q^j[2]\right|\right\}} |z|,$$

and:

$$\left|f'(z)\right| \leq 1 + \frac{2(A-B)}{\left\{(\beta(1+|B|)+1\left(\vartheta_q^k[2]-\vartheta_q^j[2]\right)+\left|B\vartheta_q^k[2]-A\vartheta_q^j[2]\right|\right\}} |z|.$$

The result is sharp.

Proof. Since $\frac{\Omega(m)}{m}$ is an increasing function for $m(m \geq 2)$, then from Theorem 2, we get

$$\frac{\Omega(2)}{2} \sum_{m=2}^{\infty} |a_m| \leq \sum_{m=2}^{\infty} \frac{\Omega(m)}{m} |a_m| = \sum_{m=2}^{\infty} \Omega(m) |a_m| = \leq A - B,$$

that is:

$$\sum_{m=2}^{\infty} |a_m| \leq \frac{2(A-B)}{\Omega(2)}.$$

Thus, we obtain

$$\left|f'(z)\right| \leq 1 + |z| \sum_{m=2}^{\infty} m |a_m|,$$

$$\left|f'(z)\right| \leq 1 + \frac{2(A-B)}{\left\{(\beta(1+|B|)+1\left(\vartheta_q^k[2]-\vartheta_q^j[2]\right)+\left|B\vartheta_q^k[2]-A\vartheta_q^j[2]\right|\right\}} |z|.$$

Likewise, we obtain:

$$\left|f'(z)\right| \geq 1 - |z| \sum_{m=2}^{\infty} m |a_m|$$

$$\geq 1 - \frac{2(A-B)}{\left\{(\beta(1+|B|)+1\left(\vartheta_q^k[2]-\vartheta_q^j[2]\right)+\left|B\vartheta_q^k[2]-A\vartheta_q^j[2]\right|\right\}} |z|.$$

Lastly , we can notice that the affirmations of the theorem are sharp for the function $f(z)$ defined by (11). This finishes the proof. □

Convexity, Radii of starlikeness and close-to-convexity

Theorem 5. A function f of the from (7) belongs to the class $\mathcal{TJ}_{k,j}(q, \beta, A, B)$. Then:

(i) f is starlike of order $\zeta (0 \leq \zeta < 1)$ in $|z| < r_1$ where:

$$r_1 = \inf_{m \geq 2} \left\{ \frac{\left\{(1+\beta(1+|B|)\left(\vartheta_q^k[m]-\vartheta_q^j[m]\right)+\left|B\vartheta_q^k[m]-A\vartheta_q^j[m]\right|\right\}}{(A-B)} \times \left(\frac{1-\zeta}{m-\zeta}\right) \right\}^{\frac{1}{m-1}}. \quad (12)$$

(ii) f is convex of order in $\zeta (0 \leq \zeta < 1)$ in $|z| < r_2$, where:

$$r_2 = \inf_{m \geq 2} \left\{ \frac{\left\{(1+\beta(1+|B|)\left(\vartheta_q^k[m]-\vartheta_q^j[m]\right)+\left|B\vartheta_q^k[m]-A\vartheta_q^j[m]\right|\right\}}{(A-B)} \times \left(\frac{1-\zeta}{m(m-\zeta)}\right) \right\}^{\frac{1}{m-1}}. \quad (13)$$

(iii) f is close to convex of order $(0 \leq \zeta < 1)$ in $|z| < r_3$, where:

$$r_3 = \inf_{m\geq 2}\left\{\frac{\left\{(1+\beta(1+|B|)\left(\vartheta_q^k[m]-\vartheta_q^j[m]\right)+\left|B\vartheta_q^k[m]-A\vartheta_q^j[m]\right|\right\}}{(A-B)} \times \left(\frac{1-\zeta}{m}\right)\right\}^{\frac{1}{m-1}}. \quad (14)$$

The function f is provided by (10). All of these results are sharp.

Proof. We need to show that

$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1-\zeta$ where $|z| < r_1$

where r_1 is specified by(12) . Indeed, we get from (7) that:

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{\sum_{m=2}^{\infty}(m-1)a_m|z|^{m-1}}{1-\sum_{m=2}^{\infty}a_m|z|^{m-1}}.$$

Hence, we obtain:

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1-\zeta,$$

if and only if:

$$\frac{\sum_{m=2}^{\infty}(m-\zeta)a_m|z|^{m-1}}{(1-\zeta)} \leq 1. \quad (15)$$

By Theorem 2, the relation (15) is true if:

$$\left(\frac{m-\zeta}{1-\zeta}\right)|z|^{m-1} \leq \frac{\left\{(1+\beta(1+|B|)\left(\vartheta_q^k[m]-\vartheta_q^j[m]\right)+\left|B\vartheta_q^k[m]-A\vartheta_q^j[m]\right|\right\}}{(A-B)}.$$

That is, if:

$$|z| \leq \left\{\frac{\left\{(1+\beta(1+|B|)\left(\vartheta_q^k[m]-\vartheta_q^j[m]\right)+\left|B\vartheta_q^k[m]-A\vartheta_q^j[m]\right|\right\}}{(A-B)} \times \left(\frac{1-\zeta}{m-\zeta}\right)\right\}^{\frac{1}{m-1}}, \quad \text{for } m \geq 2.$$

Implies

$$r_1 = \inf_{m\geq 2}\left\{\frac{\left\{(1+\beta(1+|B|))\left(\vartheta_q^k[m]-\vartheta_q^j[m]\right)+\left|B\vartheta_q^k[m]-A\vartheta_q^j[m]\right|\right\}}{(A-B)} \times \left(\frac{1-\zeta}{m-\zeta}\right)\right\}^{\frac{1}{m-1}}, \ m \geq 2.$$

This completes the proof (12).

For proving (13) and (14), we need only to show that:

$$\left|1+\frac{zf''}{f'(z)} - 1\right| 1-\zeta \quad (\zeta(0 \leq \zeta < 1);\ |z| < r_2),$$

and

$$|f'(z)-1| 1-\zeta \quad (\zeta(0 \leq \zeta < 1);\ |z| < r_3),$$

respectively.

Extreme points

Theorem 6. Consider that $f_1(z) = z$, and:

$$f_m(z) = z - \frac{A-B}{\left\{(\beta(1+|B|)+1)\left(\vartheta_q^k[2]-\vartheta_q^j[2]\right)+\left|B\vartheta_q^k[2]-A\vartheta_q^j[2]\right|\right\}} z^m, m = 2,3,..$$

Then, $f \text{in} \mathcal{TJ}_{k,j}(q,\beta,A,B)$ if and only if it can be written in the following form:

$$f(z) = \sum_{m=1}^{\infty} \omega_m \ f_m(z),$$

where $\omega_m \geq 0, \sum_{m=1}^{\infty}\omega_m = 1.$

Proof. Assume that:

$$f(z) = \sum_{m=1}^{\infty} \omega_m \ f_m(z)$$

$$= z - \sum_{m=2}^{\infty} \omega_m \frac{(A-B)}{\left\{(\beta(1+|B|)+1)\left(\vartheta_q^k[2]-\vartheta_q^j[2]\right)+\left|B\vartheta_q^k[2]-A\vartheta_q^j[2]\right|\right\}} z^m.$$

Then, by Theorem 2, we get:

$$\sum_{m=2}^{\infty}\left[\frac{\left\{(\beta(1+|B|)+1)\left(\vartheta_q^k[2]-\vartheta_q^j[2]\right)+\left|B\vartheta_q^k[2]-A\vartheta_q^j[2]\right|\right\}(A-B)}{\left\{(\beta(1+|B|)+1)\left(\vartheta_q^k[2]-\vartheta_q^j[2]\right)+\left|B\vartheta_q^k[2]-A\vartheta_q^j[2]\right|\right\}}\omega_m\right]$$

$$=(A-B)\sum_{m=2}^{\infty}\omega_m=(A-B)(1-\omega_1)\le(A-B).$$

Thus, in view of Theorem 2, we obtain $f\in\mathcal{TJ}_{k,j}(q,\beta,A,B)$.
Contrariwise, let us assume that, f in $\mathcal{TJ}_{k,j}(q,\beta,A,B)$ then

$$a_m\le\frac{(A-B)}{\left\{(\beta(1+|B|)+1)\left(\vartheta_q^k[2]-\vartheta_q^j[2]\right)+\left|B\vartheta_q^k[2]-A\vartheta_q^j[2]\right|\right\}}.$$

By setting

$$\omega_m=\frac{\left\{(\beta(1+|B|)+1)\left(\vartheta_q^k[2]-\vartheta_q^j[2]\right)+\left|B\vartheta_q^k[2]-A\vartheta_q^j[2]\right|\right\}}{(A-B)}a_m$$

for

$$\omega_1=1-\sum_{m=2}^{\infty}\omega_m,$$

we obtain

$$f(z)=\sum_{m=1}^{\infty}\omega_m\ f_m(z).$$

This completes the proof.

Corollary 2. The extreme points of the class $\mathcal{TJ}_{k,j}(q,\beta,A,B)$ are given by

$$f_1(z)=z,$$

and

$$f_m(z)=z-\frac{A-B}{\left\{(\beta(1+|B|)+1)\left(\vartheta_q^k[m]-\vartheta_q^j[m]\right)+\left|B\vartheta_q^k[m]-A\vartheta_q^j[m]\right|\right\}}z^m,m=2,3,..$$

Integral mean inequalities

Lemma 1.[25] Let f and g be holomorphic functions in E^* with:

$$f(z)\prec g(z),$$

then for $p>0$ and $z=re^{i\theta},(0<r<1)$,

$$\int_0^{2\pi}|f(z)|^p d\theta\le\int_0^{2\pi}|g(z)|^p d\theta. \tag{16}$$

Now, we find the following result by taking Lemma 1.

Theorem 7. Suppose that $f\in\mathcal{TJ}_{k,j}(q,\beta,A,B),p>0,k>j,k\in\mathbb{N},j\in\mathbb{N}_0,-1\le B<A\le 1,\beta\ge 0$, and $f_2(z)$ is defined by

$$f_2(z)=z-\frac{A-B}{\left\{(\beta(1+|B|)+1)\left(\vartheta_q^k[2]-\vartheta_q^j[2]\right)+\left|B\vartheta_q^k[2]-A\vartheta_q^j[2]\right|\right\}}z^2,$$

for $z=re^{i\theta},(0<r<1)$, we get

$$\int_0^{2\pi}|f(z)|^p d\theta\le\int_0^{2\pi}|f_2(z)|^p d\theta.$$

Proof . For $f(z)=z-\sum_{m=2}^{\infty}a_m z^m,\ a_m\ge 0$,
the relation (16)is equivalent to prove that

$$\int_0^{2\pi}\left|1-\sum_{m=2}^{\infty}a_m z^{m-1}\right|^p d\theta\le$$

$$\int_0^{2\pi}\left|1-\frac{A-B}{\left\{(\beta(1+|B|)+1)\left(\vartheta_q^k[2]-\vartheta_q^j[2]\right)+\left|B\vartheta_q^k[2]-A\vartheta_q^j[2]\right|\right\}}z\right|^p d\theta.$$

By using Lemma 1, it suffices to show that

$$1-\sum_{m=2}^{\infty} a_m z^{m-1} \prec 1-\frac{A-B}{\left\{(\beta(1+|B|)+1)\ \left(\vartheta_q^k[2]-\ \vartheta_q^j[2]\ \right)+\left|B\vartheta_q^k[2]-A\vartheta_q^j[2]\ \right|\right\}}z.$$

By setting

$$1-\sum_{m=2}^{\infty} a_m z^{m-1} = 1-\frac{A-B}{\left\{(\beta(1+|B|)+1)\ \left(\vartheta_q^k[2]-\ \vartheta_q^j[2]\ \right)+\left|B\vartheta_q^k[2]-A\vartheta_q^j[2]\ \right|\right\}}w(z).$$

and using (8), we get

$$|w(z)|=\left|\sum_{m=2}^{\infty}\frac{\left\{(\beta(1+|B|)+1)\ \left(\vartheta_q^k[2]-\ \vartheta_q^j[2]\ \right)+\left|B\vartheta_q^k[2]-A\vartheta_q^j[2]\ \right|\right\}}{A-B}\ a_m z^{m-1}\right|$$

$$\leq |z|\sum_{m=2}^{\infty}\frac{\left\{(\beta(1+|B|)+1)\ \left(\vartheta_q^k[2]-\ \vartheta_q^j[2]\ \right)+\left|B\vartheta_q^k[2]-A\vartheta_q^j[2]\ \right|\right\}}{A-B}a_m$$

$$\leq |z|\sum_{m=2}^{\infty}\frac{\left\{(\beta(1+|B|)+1)\ \left(\vartheta_q^k[m]-\ \vartheta_q^j[m]\ \right)+\left|B\vartheta_q^k[m]-A\vartheta_q^j[m]\ \right|\right\}}{A-B}a_m$$

$$\leq |z|<1.$$

This completes the proof.

Neighborhoods for the class $\mathcal{TJ}_{k,j}(q,\beta,A,B)$

We define the δ_1-neighborhood of a function f in $\mathcal{T}$ by

$$N_{\delta_1}(f)=\left\{g\in\mathcal{T};g(z)=z-\sum_{m=2}^{\infty}b_m z^m\ :\ \sum_{m=2}^{\infty}m|a_m-b_m|\leq\ \delta_1\right\}. \qquad (16)$$

In particular, for $e(z)=z$,

$$N_{\delta_1}(e)=\left\{g\in\mathcal{T};g(z)=z-\sum_{m=2}^{\infty}b_m z^m\ and\ \sum_{m=2}^{\infty}m|b_m|\leq\ \delta_1\right\}. \qquad (17)$$

On the other hand, a function $f(z)$ defined by (7) is said to be in the class $\mathcal{TJ}_{k,j}(q,\beta\ A,B)$ if there exists a function $g\in\mathcal{TJ}_{k,j}(q,\beta,A,B)$ such that

$$\left|\frac{f(z)}{g(z)}-1\right|<1-\Upsilon,\quad 0\leq\Upsilon<1 \qquad (18)$$

Theorem 8. If

$$\left\{(\beta(1+|B|)+1)\ \left(\vartheta_q^k[m]-\ \vartheta_q^j[m]\ \right)+\left|B\vartheta_q^k[m]-A\vartheta_q^j[m]\ \right|\right\}$$
$$\geq\left\{(\beta(1+|B|)+1)\ \left(\vartheta_q^k[2]-\ \vartheta_q^j[2]\ \right)+\left|B\vartheta_q^k[2]-A\vartheta_q^j[2]\ \right|\right\},\quad m\geq 2.$$

and

$$\delta_1=\frac{2(A-B)}{\left\{(\beta(1+|B|)+1)\ \left(\vartheta_q^k[2]-\ \vartheta_q^j[2]\ \right)+\left|B\vartheta_q^k[2]-A\vartheta_q^j[2]\ \right|\right\}},$$

then

$\mathcal{TJ}_{k,j}(q,\beta,A,B)\subset N_{\delta_1}(e)$

Proof. Let $f\in\mathcal{TJ}_{k,j}(q,\beta,A,B)$.Then from Theorem 2 and the condition

$$\left\{(\beta(1+|B|)+1)\ \left(\vartheta_q^k[m]-\ \vartheta_q^j[m]\ \right)+\left|B\vartheta_q^k[m]-A\vartheta_q^j[m]\ \right|\right\}$$
$$\geq\left\{(\beta(1+|B|)+1)\ \left(\vartheta_q^k[2]-\ \vartheta_q^j[2]\ \right)+\left|B\vartheta_q^k[2]-A\vartheta_q^j[2]\ \right|\right\},\quad m\geq 2.$$

We get,

$$\left\{(\beta(1+|B|)+1)\ \left(\vartheta_q^k[2]-\ \vartheta_q^j[2]\ \right)+\left|B\vartheta_q^k[2]-A\vartheta_q^j[2]\ \right|\right\}\sum_{m=2}^{\infty}a_m\leq$$

$$\left\{(\beta(1+|B|)+1)\ \left(\vartheta_q^k[m]-\ \vartheta_q^j[m]\ \right)+\left|B\vartheta_q^k[m]-A\vartheta_q^j[m]\ \right|\right\}\sum_{m=2}^{\infty}a_m$$
$\leq (A-B)$,

which implies

$$\sum_{m=2}^{\infty}a_m\leq\frac{(A-B)}{\left\{(\beta(1+|B|)+1)\ \left(\vartheta_q^k[2]-\ \vartheta_q^j[2]\ \right)+\left|B\vartheta_q^k[2]-A\vartheta_q^j[2]\ \right|\right\}}. \quad (19)$$

By using Theorem 2 with (19), we obtain

$$\left\{(\beta(1+|B|)+1)\ \left(\vartheta_q^k[2]-\ \vartheta_q^j[2]\ \right)+\left|B\vartheta_q^k[2]-A\vartheta_q^j[2]\ \right|\right\}\sum_{m=2}^{\infty}a_m$$
$\leq (A-B)$

$$2\left\{(\beta(1+|B|)+1)\ \left(\vartheta_q^k[2]-\ \vartheta_q^j[2]\ \right)+\left|B\vartheta_q^k[2]-A\vartheta_q^j[2]\ \right|\right\}\sum_{m=2}^{\infty}a_m$$
$\leq 2(A-B)$

$$\sum_{m=2}^{\infty}ma_m\leq\frac{2(A-B)}{\left\{(\beta(1+|B|)+1)\ \left(\vartheta_q^k[2]-\ \vartheta_q^j[2]\ \right)+\left|B\vartheta_q^k[2]-A\vartheta_q^j[2]\ \right|\right\}},$$

By (16) we get $f\in N_{\delta_1}(e)$.

This completes the proof of Theorem 8 .

Theorem 9. If $g\in\mathcal{TJ}_{k,j}(q,\beta,A,B)$ and

$$\Upsilon=1-\frac{\delta_1}{2}\frac{\left\{(\beta(1+|B|)+1)\ \left(\vartheta_q^k[2]-\ \vartheta_q^j[2]\ \right)+\left|B\vartheta_q^k[2]-A\vartheta_q^j[2]\ \right|\right\}}{\left\{(\beta(1+|B|)+1)\ \left(\vartheta_q^k[2]-\ \vartheta_q^j[2]\ \right)+\left|B\vartheta_q^k[2]-A\vartheta_q^j[2]\ \right|\right\}-(A-B)}. \quad (20)$$

Then $N_{\delta_1}(e)\subset\mathcal{TJ}_{k,j}(q,\beta,A,B)$.

Proof. Let f be in $N_{\delta_1}(e)$. We find by (16)that

$$\sum_{m=2}^{\infty}m|a_m-b_m|\leq\ \delta_1,$$

which means the coefficient of inequality

$$\sum_{m=2}^{\infty}|a_m-b_m|\leq\frac{\delta_1}{2}. \quad (21)$$

It follows that, since $g\in\mathcal{TJ}_{k,j}(q,\beta,A,B)$, then from (19) we obtain

$$\sum_{m=2}^{\infty}b_m\leq\frac{(A-B)}{\left\{(\beta(1+|B|)+1)\ \left(\vartheta_q^k[2]-\ \vartheta_q^j[2]\ \right)+\left|B\vartheta_q^k[2]-A\vartheta_q^j[2]\ \right|\right\}}, \quad (22)$$

Using (21) and (22), we get

$$\left|\frac{f(z)}{g(z)}-1\right|\leq\frac{\sum_{m=2}^{\infty}|a_m-b_m|}{1-\sum_{m=2}^{\infty}b_m}$$

$$\leq\frac{\delta_1}{2\left(1-\frac{(A-B)}{\left\{(\beta(1+|B|)+1)\ \left(\vartheta_q^k[2]-\ \vartheta_q^j[2]\ \right)+\left|B\vartheta_q^k[2]-A\vartheta_q^j[2]\ \right|\right\}}\right)}$$

$$\leq\frac{\delta_1}{2}\frac{\left\{(\beta(1+|B|)+1)\ \left(\vartheta_q^k[2]-\ \vartheta_q^j[2]\ \right)+\left|B\vartheta_q^k[2]-A\vartheta_q^j[2]\ \right|\right\}}{\left\{(\beta(1+|B|)+1)\ \left(\vartheta_q^k[2]-\ \vartheta_q^j[2]\ \right)+\left|B\vartheta_q^k[2]-A\vartheta_q^j[2]\ \right|\right\}-(A-B)}=1-\Upsilon,$$

provided that Υ is given by (20),
hence, by condition (18), f in $\mathcal{TJ}_{k,j}(q,\beta,A,B)$ is given by (9) .

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