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# Solving Systems of Non-Linear Volterra Integral Equations by Combined Sumudu Transform-Adomian Decomposition Method

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**Abstract**

This paper is used for solving component Volterra nonlinear systems by means of the combined Sumudu transform with Adomian decomposition process. We equate the numerical results with the exact solutions to demonstrate the high accuracy of the solution results. The results show that the approach is very straightforward and effective.

**Keywords:** Systems of non-linear Volterra integral equations; Sumudu Transform;

# حل نظام معادلات فولتيرا التكاملية اللاخطية بأستخدام جمع تحويل سومودو مع طريقة أنحلال أدوميان

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**الخلاصه**

هذا البحث أستخدم لحل مكون نظام فولتيرا اللاخطية بأستخدام جمع تحويل سومودو مع طريقة أنحلال أدوميان. نساوي النتائج العددية بالحلول الدقيقة لإثبات الدقة العالية لنتائج الحل. أظهرت النتائج أن التقارب واضح وفعال للغاية.

## 1. Introduction

Systems of Volterra integral equations of linear and non-linear types and their solutions play a pivotal role in the fields of science, industrial mathematics, control theory of financial mathematics, and engineering [1]. Physical systems such as those utilized in biological applications population dynamics and genetics, where impulses arise naturally or are caused by control, are modeled by a system of integral equations [2, 3]. It is not easy to find the exact solution because of the non-linear part of the equation.

In early 1990’s, Watugala [4] introduced a new integral transform, named the Sumudu transform, and applied it to the solution of ordinary differential equations in controlling engineering problems.

In the beginning of the 1980's, Adomian [5, 6] proposed a new and fruitful method, so-called the Adomian decomposition method (ADM), for solving linear and non-linear (algebraic, differential, partial differential, integral, etc.) equations [7-13, 15-17]. It has been shown that this method yields a rapid convergence of the solutions series. The non-linear term is decomposed into a series of special polynomials called Adomian’s polynomials.

The advantages of approximate methods over numerical methods are that they can solve the difficult nonlinear problems [8, 9]. Solving problems with approximate analytical methods often helps an

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engineer or scientist to better understand a physical problem, and may help improve future procedures and designs used to solve their problems [12, 13]. These methods solve high order FIVPs directly without reducing them into first order system [14-19]. The accuracy of the approximate solution can also be determined without needing the exact solution, especially in the nonlinear equations [7, 9, 11].
In recent years, many new methods, such as He's homotopy perturbation method [13, 20, 21], modified homotopy perturbation method [22], Adomian decomposition method [13, 23], variational iteration methods [13, 24, 25], and many more, were used to solve integral and integro–differential equations. Finally, Al-Hayani [26] combined Laplace transform-Adomian decomposition method to solve nth-order integro-differential equations.
In this paper, the main objective of this work is to use the Combined Sumudu Transform-Adomian Decomposition Method (CST-ADM) to solve systems of non-linear Volterra integral equations.

## 2. Preliminaries of Sumudu Transform

**Definition:** Assume that $f$ is a function of $x$. The Sumudu transform of $f$ is defined as

$$F(z) = S\{f(x)\} = \int_0^\infty f(zx)e^{-x}dx, \tag{2.1}$$

we shall refer to $f(x)$ as the original function of $F(z)$ and $F(z)$ as the Sumudu transform of the function $f(x)$. We also refer to $f(x)$ as the inverse Sumudu transform of $F(z)$.
The following are some basic properties of the Sumudu transform:

**Linearity**

$$S\{c_1 f(x) + c_2 g(x)\} = c_1 S\{f(x)\} + c_2 S\{g(x)\}, \tag{2.2}$$

**Convolution**

$$S\{(f * g)(x)\} = zS\{f(x)\} * S\{g(x)\}, \tag{2.3}$$

**Laplace–Sumudu Duality**

$$\mathcal{L}\{f(x)\} = S\left\{f\left(\frac{1}{x}\right)\right\}/z \ \ , S\{f(x)\} = \mathcal{L}\left\{f\left(\frac{1}{x}\right)\right\}/z, \tag{2.4}$$

More properties can be found in [27, 28, 29].

## 3. Applications and numerical results

In this section, we apply CST-ADM to obtain approximate-exact solutions for the systems of non-linear Volterra integral equations, as illustrated in the following examples. To show the high accuracy of the solution results compared with the exact solutions, we give the numerical results, maximum error of Padé approximation **(MPA)**, maximum absolute error **(MAE)**, maximum relative error **(MRE),** and maximum residual error **(MRR),** as follows

$\mathbf{MPA} = |\text{Padé Approximation} - (\text{CST} - \text{ADM})|$

$\mathbf{MAE} = |\text{Exact Solution} - (\text{CST} - \text{ADM})|$

$\mathbf{MRA} = \left|\frac{\text{Exact Solution} - (\text{CST} - \text{ADM})}{\text{Exact Solution}}\right|$

**MRR**: It is clear that the approximate solution in the given integral equation was substituted.
The computations associated with the examples were performed using Maple 18 package with a precision of 30 digits.

**Example 1:** We solve the following system of non-linear two Volterra integral equations by using CST-ADM [20]:

$$\begin{cases} f_1(t) = sint - t + \int_0^t \left(f_1^2(s) + f_2^2(s)\right) ds, \\ f_2(t) = cost - \frac{1}{2}sin^2 t + \int_0^t f_1(s)f_2(s)\, ds, \end{cases} \tag{3.1}$$

with the exact solutions $f_1(t) = \sin t$ and $f_2(t) = \cos t$.
Taking Sumudu transform of both sides of the system (3.1) gives

$$\begin{cases} S\{f_1(t)\}(r) = S\{sint - t\} + S\left\{\int_0^t \left(f_1^2(s) + f_2^2(s)\right) ds\right\}, \\ S\{f_2(t)\}(r) = S\left\{cost - \frac{1}{2}sin^2t\right\} + S\left\{\int_0^t f_1(s)f_2(s)\, ds\right\}, \end{cases} \quad (3.2)$$

so that

$$\begin{cases} F_1(r) = \frac{r}{1+r^2} - r + S\left\{\int_0^t \left(f_1^2(s) + f_2^2(s)\right) ds\right\}, \\ F_2(r) = \frac{1}{1+r^2} - \frac{1}{4} + \frac{1}{4(1+4r^2)} + S\left\{\int_0^t f_1(s)f_2(s)\, ds\right\}, \end{cases} \quad (3.3)$$

where $S\{f_i(t)\}(r) = F_i(r),\ i = 1,2.$
Firstly, we set the series

$$F_1(r) = \sum_{n=0}^{\infty} F_{1,n}(r), \quad F_2(r) = \sum_{n=0}^{\infty} F_{2,n}(r) \quad (3.4)$$

$$f_1^2(s) = \sum_{n=0}^{\infty} A_{1,n}, \quad f_2^2(s) = \sum_{n=0}^{\infty} A_{2,n}, \quad f_1(s)f_2(s) = \sum_{n=0}^{\infty} A_{3,n} \quad (3.5)$$

where $A_{1,n}$, $A_{2,n}$ and $A_{3,n}$ are the Adomian polynomials given by

$$A_{1,n} = \sum_{i=0}^{n} f_{1,i}\, f_{1,n-i}, \quad n \geq i, \quad n = 0,1,2,\cdots$$

$$A_{2,n} = \sum_{i=0}^{n} f_{2,i}\, f_{2,n-i}, \quad n \geq i, \quad n = 0,1,2,\cdots$$

$$A_{3,n} = \sum_{i=0}^{n} f_{1,i}\, f_{2,n-i}, \quad n \geq i, \quad n = 0,1,2,\cdots$$

By substituting (3.4) and (3.5) into the system (3.3) and using the recursive relation, we get

$$\begin{cases} F_{1,0}(r) = \frac{r}{1+r^2} - r, \\ F_{2,0}(r) = \frac{1}{1+r^2} - \frac{1}{4} + \frac{1}{4(1+4r^2)}, \\ F_{1,n+1}(r) = S\left\{\int_0^t \left(A_{1,n} + A_{2,n}\right) ds\right\}, \\ F_{2,n+1}(r) = S\left\{\int_0^t A_{3,n}\, ds\right\}. \end{cases} \quad (3.6)$$

Taking the inverse Sumudu transform on both sides of the system (3.6) gives

$$\begin{cases} f_{1,0}(t) = sint - t, \\ f_{2,0}(t) = cost - \frac{1}{2}sin^2t, \end{cases}$$

$$\begin{cases} f_{1,n+1}(t) = S^{-1}\left\{S\left\{\int_0^t \left(A_{1,n} + A_{2,n}\right) ds\right\}\right\}, \\ f_{2,n+1}(t) = S^{-1}\left\{S\left\{\int_0^t A_{3,n}\, ds\right\}\right\}, \end{cases} \quad (3.7)$$

where $S^{-1}\{F_{i,n+1}(r)\}(t) = f_{i,n+1}(t),\ i = 1,2$. Then, from the system (3.7), the iterations are

$$f_{1,1}(t) = S^{-1}\left\{S\left\{\int_0^t \left(A_{1,0} + A_{2,0}\right)ds\right\}\right\}$$

$$= \frac{35}{32}t + \frac{t^3}{3} + 2tcost - \frac{5}{32}\sin t\cos t - \frac{7}{3}\sin t + \frac{1}{3}\sin t\cos^2 t + \frac{1}{16}\sin t\cos^3 t\,,$$

$$f_{2,1}(t) = S^{-1}\left\{S\left\{\int_0^t A_{3,0}\, ds\right\}\right\}$$

$$= \frac{31}{24} + \frac{1}{8}t^2 - \frac{1}{2}\cos t - t\sin t - \frac{1}{4}t\sin t\cos t - \frac{5}{8}\cos^2 t - \frac{1}{6}\cos^3 t,$$

$$f_{1,2}(t) = S^{-1}\left\{S\left\{\int_0^t \left(A_{1,1} + A_{2,1}\right)ds\right\}\right\}$$

$$= -\frac{2203}{768}t - \frac{3}{4}t^3 - \frac{2}{15}t^5 + \frac{6871}{540}\sin t - \frac{7}{4}t^2\sin t - \frac{545}{48}t\cos t - \frac{2}{3}t^3\cos t$$
$$-\frac{39}{32}t\cos^2 t + \frac{13}{18}t\cos^3 t + \frac{3}{32}t\cos^4 t + \frac{791}{256}\sin t\cos t - \frac{1513}{2160}\sin t\cos^2 t$$
$$-\frac{55}{128}\sin t\cos^3 t - \frac{7}{120}\sin t\cos^4 t + \frac{1}{16}t^2\sin t\cos t,$$

$$f_{2,2}(t) = S^{-1}\left\{S\left\{\int_0^t A_{3,1}\, ds\right\}\right\}$$

$$= -\frac{19849}{8640} - \frac{289}{768}t^2 - \frac{5}{96}t^4 - \frac{1}{192}\cos^6 t - \frac{11}{240}\cos^5 t - \frac{11}{768}\cos^4 t + \frac{745}{864}\cos^3 t$$
$$+\frac{1859}{768}\cos^2 t - \frac{265}{288}\cos t - \frac{1}{8}t^2\cos t + \frac{443}{288}t\sin t + \frac{1}{3}t^3\sin t$$
$$+\frac{267}{128}t\sin t\cos t + \frac{17}{36}t\sin t\cos^2 t + \frac{1}{12}t^3\sin t\cos t,$$

and so on. In this manner, the rest of the iterations can be obtained. Thus, the approximate solution in a series form is given by

$$f_1(t) = \sum_{i=0}^{n-1} f_{1,i}(t) = f_{1,0}(t) + f_{1,1}(t) + f_{1,2}(t) + \cdots$$

$$= \frac{6151}{540}\sin t - \frac{7}{4}t^2\sin t - \frac{2}{3}t^3\cos t + \frac{3}{32}t\cos t^4 + \frac{13}{18}t\cos t^3 - \frac{39}{32}t\cos t^2$$
$$-\frac{449}{48}t\cos t + \frac{1}{16}t^2\sin t\cos t - \frac{7}{120}\sin t\cos t^4 - \frac{47}{128}\sin t\cos t^3$$
$$-\frac{793}{2160}\sin t\cos t^2 + \frac{751}{256}\sin t\cos t - \frac{2}{15}t^5 - \frac{5}{12}t^3 - \frac{2131}{768}t + \cdots,$$

$$f_2(t) = \sum_{i=0}^{n-1} f_{2,i}(t) = f_{2,0}(t) + f_{2,1}(t) + f_{2,2}(t) + \cdots$$

$$= -\frac{121}{288}\cos t + \frac{1763}{768}\cos t^2 + \frac{601}{864}\cos t^3 - \frac{11}{768}\cos t^4 - \frac{11}{240}\cos t^5 - \frac{1}{192}\cos t^6$$
$$-\frac{1}{8}t^2\cos t + \frac{155}{288}t\sin t + \frac{1}{3}t^3\sin t + \frac{235}{128}t\sin t\cos t + + \frac{17}{36}t\sin t\cos t^2$$
$$+\frac{1}{12}t^3\sin t\cos t - \frac{5}{96}t^4 - \frac{193}{768}t^2 - \frac{13009}{8640} + \cdots.$$

Notice that the noise terms that appear between various components vanish. These series have the closed form as $n \to \infty$ gives $f_1(t) = sint$ and $f_2(t) = cost$, which is the exact solutions of the system (3.1).

Table-1 shows a comparison of the numerical results, applying the CST-ADM ($n = 5$) and the Padé approximation (PA) of order [4/4] about the point $t = 0$, with the exact solutions and HPM [16]. It can be noticed that the results obtained by the present method are compatible with those of the HPM [16]. In Table-2, we list the MPA, MAE, MRE and MRR obtained by the CST-ADM on the interval $[-0.5,0.5]$. Fig. 1 represents both the exact solutions and our approximations by the CST-ADM ($n = 4$) in the interval $-0.5 \leq x \leq 0.5$.

**Table 1**-Numerical results for example 1

| $t$ | $i$ | Exact: $f_i(x)$ | CST-ADM | $PA_i[4/4]$ | HPM [16] |
|---|---|---|---|---|---|
| 0.0 | 1 | 0.000000000 | 0.000000000 | 0.000000000 | 0.000000000 |
| | 2 | 1.000000000 | 1.000000000 | 1.000000000 | 1.000000000 |
| 0.1 | 1 | 0.099833416 | 0.099833389 | 0.099833384 | 0.0998332766 |
| | 2 | 0.995004165 | 0.995003990 | 0.995003990 | 0.9950038426 |
| 0.2 | 1 | 0.198669330 | 0.198665915 | 0.198660916 | 0.1986657895 |
| | 2 | 0.980066577 | 0.980055681 | 0.980055670 | 0.9800556685 |
| 0.3 | 1 | 0.295520206 | 0.295465084 | 0.295718170 | 0.2954652197 |
| | 2 | 0.955336489 | 0.955217761 | 0.955217161 | 0.9552177828 |
| 0.4 | 1 | 0.389418342 | 0.389037000 | 0.389754023 | 0.3890372160 |
| | 2 | 0.921060994 | 0.920434324 | 0.920424651 | 0.9204345016 |
| 0.5 | 1 | 0.479425538 | 0.477783222 | 0.480213982 | 0.4777833813 |
| | 2 | 0.877582561 | 0.875377759 | 0.875297805 | 0.8753778568 |

**Table 2**-MPA, MAE, MRE and MRR for example 1

| $t$ | $i$ | MPA | MAE | MRE | MRR |
|---|---|---|---|---|---|
| 0.0 | 1 | 0.0 | 0.0 | -- | 0.0 |
| | 2 | 0.0 | 0.0 | -- | 0.0 |
| 0.1 | 1 | 3.258E-08 | 2.761E-08 | 2.766E-07 | 2.257E-08 |
| | 2 | 1.748E-07 | 1.748E-07 | 1.757E-07 | 1.742E-07 |
| 0.2 | 1 | 8.414E-06 | 3.415E-06 | 1.719E-05 | 2.767E-06 |
| | 2 | 1.090E-05 | 1.089E-05 | 1.111E-04 | 1.075E-05 |
| 0.3 | 1 | 1.979E-04 | 5.512E-05 | 1.865E-04 | 4.401E-05 |
| | 2 | 1.193E-04 | 1.187E-04 | 1.242E-04 | 1.153E-04 |
| 0.4 | 1 | 3.356E-04 | 3.813E-04 | 9.792E-04 | 2.982E-04 |
| | 2 | 6.363E-04 | 6.266E-04 | 6.803E-04 | 5.955E-04 |
| 0.5 | 1 | 7.884E-04 | 1.642E-03 | 3.425E-03 | 1.249E-03 |
| | 2 | 2.284E-03 | 2.204E-03 | 2.512E-03 | 2.037E-03 |

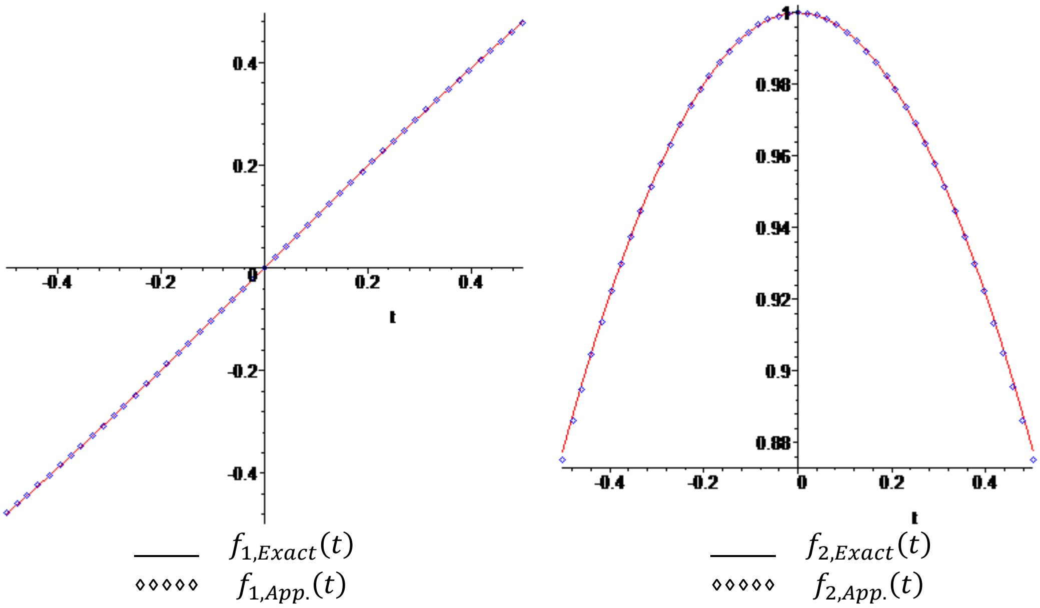



**Figure 1**-Plots of the exact solutions and the approximations of CST-ADM.

**Example 2:** The following system of non-linear three Volterra integral equations is solved by using CST-ADM [20]:

$$\begin{cases} f_1(t) = \ln t - 2t^2 \ln t + t^2 + 4\int_0^t f_1(s)f_2(s)ds\,, \\ f_2(t) = t - \frac{1}{6}t^6 \ln t + \frac{1}{36}t^6 + \int_0^t sf_1(s)f_3^2 ds, \\ f_3(t) = t^2 + \frac{1}{15}t^5 - \frac{1}{3}\int_0^t sf_2(s)f_3(s)ds. \end{cases} \quad (3.8)$$

with the exact solutions $f_1(t) = \ln t$, $f_2(t) = t$ and $f_3(t) = t^2$.
Taking Sumudu transform of both sides of the system (3.8) gives

$$\begin{cases} S\{f_1(t)\}(r) = S\{\ln t - 2t^2 \ln t + t^2\} + S\left\{4\int_0^t f_1(s)f_2(s)ds\right\}, \\ S\{f_2(t)\}(r) = \left\{t - \frac{1}{6}t^6 \ln t + \frac{1}{36}t^6\right\} + S\left\{\int_0^t sf_1(s)f_3^2 ds\right\}, \\ S\{f_3(t)\}(r) = S\left\{t^2 + \frac{1}{15}t^5\right\} - S\left\{\frac{1}{3}\int_0^t sf_2(s)f_3(s)ds\right\}. \end{cases} \quad (3.9)$$

$$\begin{cases} F_1(r) = S\{\ln t\} - S\{2t^2 \ln t\} + 2r^2 + S\left\{4\int_0^t f_1(s)f_2(s)ds\right\}, \\ F_2(r) = r - S\left\{\frac{1}{6}t^6 \ln t\right\} + 20r^6 + S\left\{\int_0^t sf_1(s)f_3^2 ds\right\}, \\ F_3(r) = 2r^2 + 8r^5 - S\left\{\frac{1}{3}\int_0^t sf_2(s)f_3(s)ds\right\}. \end{cases} \quad (3.10)$$

where $S\{f_i(t)\}(r) = F_i(r), i = 1,2,3$.
Firstly, we set the series

$$F_i(r) = \sum_{n=0}^{\infty} F_{i,n}(r), \quad i = 1,2,3 \tag{3.11}$$

$$f_1(s)f_2(s) = \sum_{n=0}^{\infty} A_{1,n}, \quad f_1(s)f_3^2(s) = \sum_{n=0}^{\infty} A_{2,n}, \quad f_2(s)f_3(s) = \sum_{n=0}^{\infty} A_{3,n} \tag{3.12}$$

where $A_{1,n}$, $A_{2,n}$ and $A_{3,n}$ are the Adomian polynomials given by

$$A_{1,n} = \sum_{i=0}^{n} f_{1,i}\, f_{2,n-i}, \quad n \geq i, \quad n = 0,1,2,\cdots$$

$$A_{2,n} = \left(\sum_{i=0}^{n} f_{1,i}\right)\left(\sum_{i=0}^{n} f_{2,i}\, f_{2,n-i}\right), \quad n \geq i, \quad n = 0,1,2,\cdots$$

$$A_{3,n} = \sum_{i=0}^{n} f_{2,i}\, f_{3,n-i}, \quad n \geq i, \quad n = 0,1,2,\cdots$$

By substituting (3.11) and (3.12) into the system (3.10) and using the recursive relation, we get

$$\begin{cases} F_{1,0}(r) = S\{\ln t\} - S\{2t^2 \ln t + t^2\} + 2r^2, \\ F_{2,0}(r) = r - S\left\{\frac{1}{6} t^6 \ln t\right\} + 20r^6, \\ F_{3,0}(r) = 2r^2 + 8r^5, \\ F_{1,n+1}(r) = S\left\{4 \int_0^t A_{1,n} ds\right\}, \\ F_{2,n+1}(r) = S\left\{\int_0^t sA_{2,n} ds\right\}, \\ F_{3,n+1}(r) = -S\left\{\frac{1}{3}\int_0^t sA_{3,n} ds\right\}, \end{cases} \tag{3.13}$$

Taking the inverse Sumudu transform on both sides of the system (3.13) gives

$$\begin{cases} f_{1,0}(t) = \ln t - 2t^2 \ln t + t^2, \\ f_{2,0}(t) = t - \frac{1}{6} t^6 \ln t + \frac{1}{36} t^6, \\ f_{3,0}(t) = t^2 + \frac{1}{15} t^5, \end{cases}$$

$$\begin{cases} f_{1,n+1}(t) = S^{-1}\left\{S\left\{4 \int_0^t A_{1,n} ds\right\}\right\}, \\ f_{2,n+1}(t) = S^{-1}\left\{S\left\{\int_0^t sA_{2,n} ds\right\}\right\}, \\ f_{3,n+1}(t) = S^{-1}\left\{-S\left\{\frac{1}{3}\int_0^t sA_{3,n} ds\right\}\right\}. \end{cases} \tag{3.14}$$

where $S^{-1}\{F_{i,n+1}(r)\}(t) = f_{i,n+1}(t),\ i = 1,2,3$. Then from the system (3.14), the iterations are

$$f_{1,1}(t) = S^{-1}\left\{S\left\{4 \int_0^t A_{1,0} ds\right\}\right\}$$

$$= \frac{59}{2187}t^9 - \frac{19}{3087}t^7 + \frac{3}{2}t^4 - t^2 - \frac{32}{243}t^9 \ln t + \frac{19}{441}t^7 \ln t$$
$$-2t^4 \ln t + 2t^2 \ln t + \frac{4}{27}t^9 \ln t^2 - \frac{2}{21}t^7 \ln t^2,$$
$$f_{2,1}(t) = S^{-1}\left\{S\left\{\int_0^t sA_{2,0}ds\right\}\right\}$$
$$= \frac{4}{11025}t^{14} - \frac{1}{32400}t^{12} + \frac{26}{1815}t^{11} - \frac{2}{1215}t^9 + \frac{5}{32}t^8 - \frac{1}{36}t^6$$
$$-\frac{1}{1575}t^{14} \ln t + \frac{1}{2700}t^{12} \ln t - \frac{4}{165}t^{11} \ln t + \frac{2}{135}t^9 \ln t - \frac{1}{4}\ t^8 \ln t + \frac{1}{6}t^6 \ln t,$$
$$f_{3,1}(t) = S^{-1}\left\{-S\left\{\frac{1}{3}\int_0^t sA_{3,0}ds\right\}\right\}$$
$$= -\frac{19}{273780}t^{13} - \frac{1}{675}t^{10} - \frac{1}{360}t^8 - \frac{1}{15}t^5 + \frac{1}{3510}t^{13} \ln t + \frac{1}{180}t^{10} \ln t.$$

and so on. In this manner, the rest of the iterations can be obtained. Thus, the approximate solution in a series form is given by

$$f_1(t) = \sum_{i=0}^{n-1} f_{1,i}(t) = f_{1,0}(t) + f_{1,1}(t) + \cdots$$
$$= \ln t - 2t^4 \ln t + \frac{19}{441}t^7 \ln t - \frac{32}{243}t^9 \ln t - \frac{2}{21}t^7 \ln t^2 + \frac{4}{27}t^9 \ln t^2$$
$$+\frac{3}{2}t^4 - \frac{19}{3087}t^7 + \frac{59}{2187}t^9 + \cdots,$$
$$f_2(t) = \sum_{i=0}^{n-1} f_{2,i}(t) = f_{2,0}(t) + f_{2,1}(t) + \cdots$$
$$= t + \frac{5}{32}t^8 - \frac{2}{1215}t^9 + \frac{26}{1815}t^{11} - \frac{1}{32400}t^{12} + \frac{4}{11025}t^{14} - \frac{1}{4}t^8 \ln t$$
$$+\frac{2}{135}t^9 \ln t - \frac{4}{165}t^{11} \ln t + \frac{1}{2700}t^{12} \ln t - \frac{1}{1575}t^{14} \ln t + \cdots,$$
$$f_3(t) = \sum_{i=0}^{n-1} f_{3,i}(t) = f_{3,0}(t) + f_{3,1}(t) + \cdots$$
$$= t^2 - \frac{1}{360}t^8 - \frac{1}{675}t^{10} - \frac{19}{273780}t^{13} + \frac{1}{180}t^{10} \ln t + \frac{1}{3510}t^{13} \ln t + \cdots.$$

Notice that the noise terms that appear between various components vanish. This series has the closed form as $n \to \infty$ which gives$f_1(t) = \ln t$, $f_2(t) = t$ and $f_3(t) = t^2$, which is the exact solution of the system (3.8).

Table-3 shows a comparison of the numerical results applying the CST-ADM ($n = 5$) and the Padé approximation (PA) of order [4/4] about the point $t = 0.2$, with the exact solutions and HPM [16]. It can be noticed that the results obtained by the present method are compatible with those of the HPM [16]. In Table-4, we list the MPA, MAE, MRE, and MRR obtained by the CST-ADM on the interval [0,0.6]. Fig. 2 presents both the exact solutions and our approximations by the CST-ADM ($n = 4$) in the interval$0 \le x \le 0.6$.

**Table 3-**Numerical results for example 2

| $t$ | $i$ | Exact: $f_i(x)$ | CST-ADM | $PA_i$ [4/4] | HPM [16] |
|---|---|---|---|---|---|
| 0.1 | 1 | -2.302585092 | -2.302585092 | -2.302584204 | -2.302585093 |
| | 2 | 0.100000000 | 0.100000000 | 0.099999999 | 0.100000000 |
| | 3 | 0.010000000 | 0.009999999 | 0.009999999 | 0.010000000 |
| 0.2 | 1 | -1.609437912 | -1.609437911 | -1.609437911 | -1.609437911 |
| | 2 | 0.200000000 | 0.200000000 | 0.200000000 | 0.200000000 |
| | 3 | 0.040000000 | 0.039999999 | 0.039999999 | 0.040000000 |
| 0.3 | 1 | -1.203972804 | -1.203972712 | -1.203972722 | -1.203972712 |
| | 2 | 0.300000000 | 0.300000000 | 0.300000000 | 0.3000000002 |
| | 3 | 0.090000000 | 0.089999999 | 0.089999999 | 0.09000000001 |
| 0.4 | 1 | -0.916290731 | -0.916288712 | -0.916290123 | -0.9162887123 |
| | 2 | 0.400000000 | 0.400000011 | 0.399999999 | 0.4000000120 |
| | 3 | 0.160000000 | 0.159999999 | 0.159999999 | 0.1599999998 |
| 0.5 | 1 | -0.693147180 | -0.693128726 | -0.693148409 | -0.6931287262 |
| | 2 | 0.500000000 | 0.500000343 | 0.499999999 | 0.5000003436 |
| | 3 | 0.250000000 | 0.249999995 | 0.249999999 | 0.2499999953 |
| 0.6 | 1 | -0.510825623 | -0.510735929 | -0.510843381 | -0.5107359296 |
| | 2 | 0.600000000 | 0.600005256 | 0.599999999 | 0.6000052560 |
| | 3 | 0.360000000 | 0.359999886 | 0.359999999 | 0.3599998863 |

**Table 4-**MPA, MAE, MRE and MRR for example 2

| $t$ | $i$ | MPA | MAE | MRE | MRR |
|---|---|---|---|---|---|
| 0.1 | 1 | 8.883E-07 | 3.094E-13 | 1.343E-13 | 3.085E-13 |
| | 2 | 2.164E-14 | 5.644E-18 | 5.644E-17 | 5.627E-18 |
| | 3 | 4.077E-16 | 2.932E-21 | 2.932E-19 | 2.923E-21 |
| 0.2 | 1 | 9.528E-10 | 9.528E-10 | 5.920E-10 | 9.415E-10 |
| | 2 | 2.820E-13 | 2.820E-13 | 1.410E-12 | 2.786E-13 |
| | 3 | 5.873E-16 | 5.873E-16 | 1.468E-14 | 5.797E-16 |
| 0.3 | 1 | 8.193E-08 | 9.189E-08 | 7.632E-08 | 8.939E-08 |
| | 2 | 1.939E-10 | 1.477E-10 | 4.925E-10 | 1.438E-10 |
| | 3 | 5.771E-13 | 7.052E-13 | 7.836E-12 | 6.849E-13 |
| 0.4 | 1 | 6.079E-07 | 2.019E-06 | 2.204E-06 | 1.917E-06 |
| | 2 | 1.983E-10 | 1.189E-08 | 2.973E-08 | 1.139E-08 |
| | 3 | 3.489E-12 | 1.044E-10 | 6.528E-10 | 9.926E-11 |
| 0.5 | 1 | 1.228E-06 | 1.845E-05 | 2.662E-05 | 1.691E-05 |
| | 2 | 1.738E-10 | 3.436E-07 | 6.873E-07 | 3.252E-07 |
| | 3 | 7.430E-12 | 4.934E-09 | 1.973E-08 | 4.568E-09 |
| 0.6 | 1 | 1.775E-05 | 8.969E-05 | 1.755E-04 | 7.798E-05 |
| | 2 | 1.926E-10 | 5.256E-06 | 8.760E-06 | 4.965E-06 |
| | 3 | 1.514E-11 | 1.138E-07 | 3.163E-07 | 1.023E-07 |

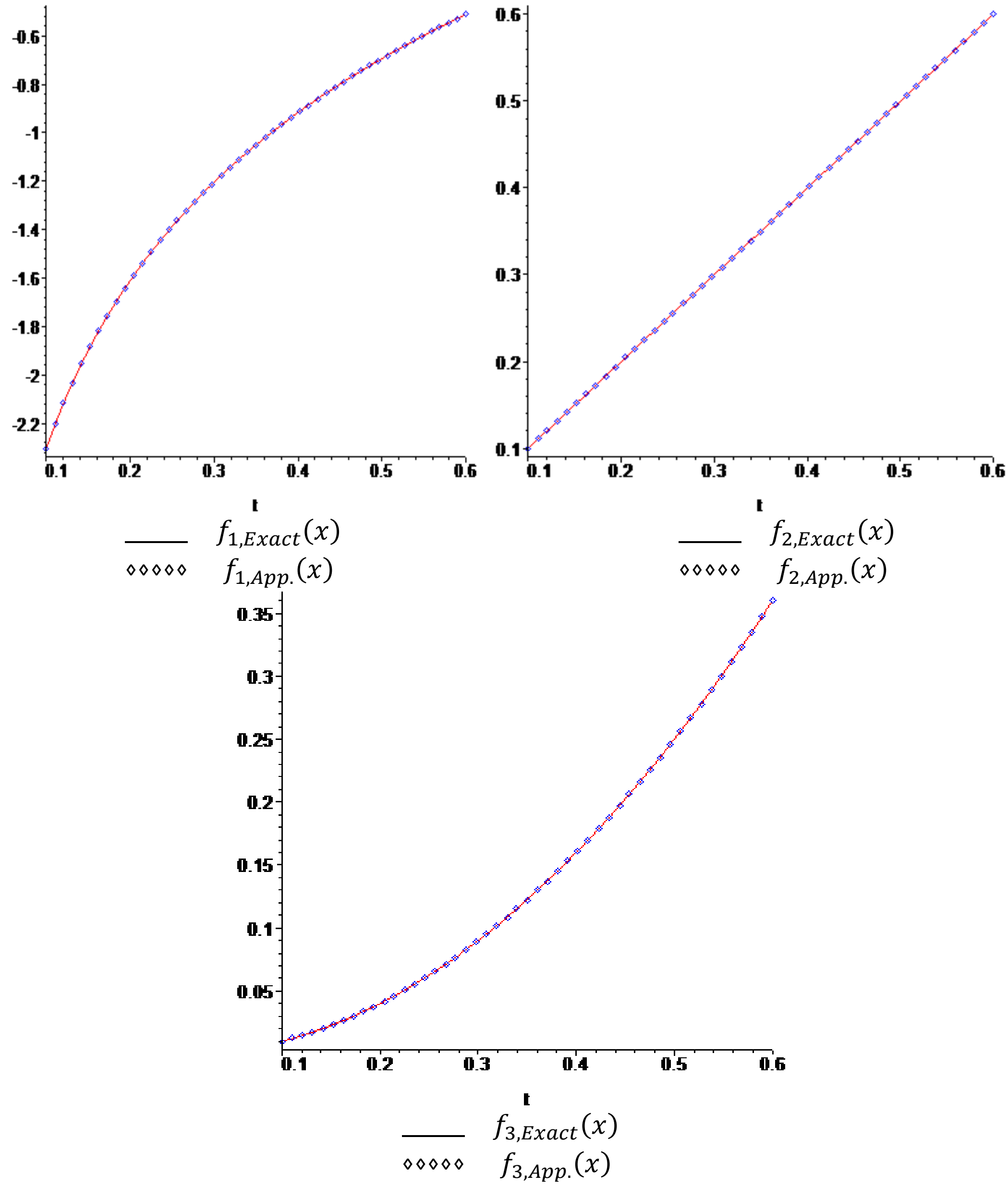



**Figure 2-**Plots of the exact solutions and the approximations of CST-ADM.

**Example 3:** The following system of non-linear four Volterra integral equations is solved by using CST-ADM [20]:

$$\begin{cases} f_1(t) = t - t^4 + 4\int_0^t f_1(s) f_2(s) ds, \\ f_2(t) = t^2 - t^8 + 8\int_0^t f_3(s) f_4(s) ds, \\ f_3(t) = \frac{2}{3}t^3 - \frac{1}{2}t^2 - \frac{1}{5}t^5 + \int_0^t \big(f_1(s) + f_2(s) + f_4(s)\big) ds, \\ f_4(t) = t^4 - t^6 + 6\int_0^t f_2(s) f_3(s) ds. \end{cases} \tag{3.15}$$

with the exact solutions$f_1(t) = t$, $f_2(t) = t^2$, $f_3(t) = t^3$ and $f_3(t) = t^4$.
Taking Sumudu transform of both sides of the system (3.15) gives

$$\begin{cases} S\{f_1(t)\}(r) = S\{t - t^4\} + S\left\{4\int\limits_0^t f_1(s)f_2(s)ds\right\}, \\ S\{f_2(t)\}(r) = S\{t^2 - t^8\} + S\left\{8\int\limits_0^t f_3(s)f_4(s)ds\right\}, \\ S\{f_3(t)\}(r) = S\left\{\frac{2}{3}t^3 - \frac{1}{2}t^2 - \frac{1}{5}t^5\right\} + S\left\{\int\limits_0^t \big(f_1(s) + f_2(s) + f_4(s)\big)ds\right\}, \\ S\{f_4(t)\}(r) = S\{t^4 - t^6\} + S\left\{6\int\limits_0^t f_2(s)f_3(s)ds\right\}. \end{cases} \tag{3.16}$$

$$\begin{cases} F_1(r) = r - 24\,r^4 + S\left\{4\int\limits_0^t f_1(s)f_2(s)ds\right\}, \\ F_2(r) = 2r^2 - 40320\,r^8 + S\left\{8\int\limits_0^t f_3(s)f_4(s)ds\right\}, \\ F_3(r) = 4r^3 - r^2 - 24\,r^5 + S\left\{\int\limits_0^t \big(f_1(s) + f_2(s) + f_4(s)\big)ds\right\}, \\ F_4(r) = 24r^4 - 720\,r^6 + S\left\{6\int\limits_0^t f_2(s)f_3(s)ds\right\}. \end{cases} \tag{3.17}$$

where $S\{f_i(t)\}(r) = F_i(r), i = 1,2,3,4.$
Firstly, we set the series

$$F_i(r) = \sum\nolimits_{n=0}^{\infty} F_{i,n}(r)\,, \quad i = 1,2,3,4 \tag{3.18}$$

$$f_1(s)f_2(s) = \sum\nolimits_{n=0}^{\infty} A_{1,n}\,, \quad f_3(s)f_4(s) = \sum_{n=0}^{\infty} A_{2,n}\,, \quad f_2(s)f_3(s) = \sum_{n=0}^{\infty} A_{3,n} \tag{3.19}$$

where $A_{1,n}$, $A_{2,n}$ and $A_{3,n}$ are the Adomian polynomials given by

$$A_{1,n} = \sum\nolimits_{i=0}^{n} f_{1,i}\, f_{2,n-i}, \quad n \geq i, \quad n = 0,1,2,\cdots$$
$$A_{2,n} = \sum\nolimits_{i=0}^{n} f_{3,i}\, f_{4,n-i}, \quad n \geq i, \quad n = 0,1,2,\cdots$$
$$A_{3,n} = \sum\nolimits_{i=0}^{n} f_{2,i}\, f_{3,n-i}, \quad n \geq i, \quad n = 0,1,2,\cdots$$

By substituting (3.18) and (3.19) into the system (3.17) and using the recursive relation, we get

$$\begin{cases} F_{1,0}(r) = r - 24\,r^4, \\ F_{2,0}(r) = 2r^2 - 40320\,r^8, \\ F_{3,0}(r) = 4r^3 - r^2 - 24\,r^5, \\ F_{4,0}(r) = 24r^4 - 720r^6, \\ F_{1,n+1}(r) = S\left\{4\int_0^t A_{1,n}ds\right\}, \\ F_{2,n+1}(r) = S\left\{8\int_0^t A_{2,n}ds\right\}, \\ F_{3,n+1}(r) = S\left\{\int_0^t \left(f_{1,n}(s) + f_{2,n}(s) + f_{4,n}(s)\right)ds\right\}, \\ F_{4,n+1}(r) = S\left\{6\int_0^t A_{3,n}ds\right\}. \end{cases} \quad (3.20)$$

Taking the inverse Sumudu transform on both sides of the system (3.20) gives

$$\begin{cases} f_{1,0}(t) = t - t^4, \\ f_{2,0}(t) = t^2 - t^8, \\ f_{3,0}(t) = -\frac{1}{2}t^2 + \frac{2}{3}t^3 - \frac{1}{5}t^5, \\ f_{4,0}(t) = t^4 - t^6. \end{cases}$$

$$\begin{cases} f_{1,n+1}(t) = S^{-1}\left\{S\left\{4\int_0^t A_{1,n}ds\right\}\right\}, \\ f_{2,n+1}(t) = S^{-1}\left\{S\left\{8\int_0^t A_{2,n}ds\right\}\right\}, \\ f_{3,n+1}(t) = S^{-1}\left\{S\left\{\int_0^t \left(f_{1,n}(s) + f_{2,n}(s) + f_{4,n}(s)\right)ds\right\}\right\}, \\ f_{4,n+1}(t) = S^{-1}\left\{S\left\{6\int_0^t A_{3,n}ds\right\}\right\}. \end{cases} \quad (3.21)$$

where $S^{-1}\{F_{i,n+1}(r)\}(t) = f_{i,n+1}(t),\ i = 1,2,3,4$. Then from the system (3.21), the iterations are

$$f_{1,1}(t) = S^{-1}\left\{S\left\{4\int_0^t A_{1,0}ds\right\}\right\} = t^4 - \frac{4}{7}t^7 - \frac{2}{5}t^{10} + \frac{4}{13}t^{13},$$

$$f_{2,1}(t) = S^{-1}\left\{S\left\{8\int_0^t A_{2,0}ds\right\}\right\} = -\frac{4}{7}t^7 + \frac{2}{3}t^8 + \frac{4}{9}t^9 - \frac{52}{75}t^{10} + \frac{2}{15}t^{12},$$

$$f_{3,1}(t) = S^{-1}\left\{S\left\{\int_0^t \left(f_{1,0}(s) + f_{2,0}(s) + f_{4,0}(s)\right)ds\right\}\right\} = \frac{1}{2}t^2 + \frac{1}{3}t^3 - \frac{1}{7}t^7 - \frac{1}{9}t^9,$$

$$f_{4,1}(t)=S^{-1}\left\{S\left\{6\int_0^t A_{3,0}ds\right\}\right\}=-\frac{3}{5}t^5+\frac{2}{3}t^6-\frac{3}{20}t^8+\frac{3}{11}t^{11}-\frac{1}{3}t^{12}+\frac{3}{35}t^{14}.$$

$$f_{1,2}(t)=S^{-1}\left\{S\left\{4\int_0^t A_{1,1}ds\right\}\right\}$$

$$=\frac{4}{7}t^7-\frac{16}{63}t^9+\frac{4}{105}t^{10}+\frac{16}{99}t^{11}-\frac{64}{1575}t^{12}-\frac{124}{195}t^{13}-\frac{4}{45}t^{14}$$

$$+\frac{208}{1125}t^{15}+\frac{20}{91}t^{16}-\frac{8}{255}t^{17}+\frac{8}{95}t^{19}-\frac{8}{143}t^{22},$$

$$f_{2,2}(t)=S^{-1}\left\{S\left\{8\int_0^t A_{2,1}ds\right\}\right\}$$

$$=\frac{4}{7}t^7+\frac{19}{30}t^8-\frac{148}{135}t^9+\frac{4}{45}t^{10}+\frac{39}{275}t^{11}-\frac{79}{315}t^{12}-\frac{5171}{121275}t^{14}$$

$$+\frac{92}{495}t^{15}-\frac{1}{18}t^{16}-\frac{60}{1309}t^{17}+\frac{52}{945}t^{18}-\frac{6}{875}t^{20},$$

$$f_{3,2}(t)=S^{-1}\left\{S\left\{\int_0^t \left(f_{1,1}(s)+f_{2,1}(s)+f_{4,1}(s)\right)ds\right\}\right\}$$

$$=\frac{1}{5}t^5-\frac{1}{10}t^6+\frac{2}{21}t^7-\frac{1}{7}t^8+\frac{31}{540}t^9+\frac{2}{45}t^{10}-\frac{82}{825}t^{11}$$

$$+\frac{1}{44}t^{12}-\frac{1}{65}t^{13}+\frac{2}{91}t^{14}+\frac{1}{175}t^{15},$$

$$f_{4,2}(t)=S^{-1}\left\{S\left\{6\int_0^t A_{3,1}ds\right\}\right\}$$

$$=\frac{3}{5}t^5+\frac{1}{3}t^6+\frac{3}{35}t^{10}-\frac{51}{77}t^{11}-\frac{1}{9}t^{12}+\frac{7156}{20475}t^{13}-\frac{134}{525}t^{14}$$

$$-\frac{14}{255}t^{15}+\frac{2917}{21000}t^{16}+\frac{19}{675}t^{18},$$

and so on. In this manner, the rest of the iterations can be obtained. Thus, the approximate solution in a series form is given by

$$f_1(t)=\sum_{i=0}^{n-1}f_{1,i}(t)=f_{1,0}(t)+f_{1,1}(t)+f_{1,2}(t)+\cdots$$

$$=t-\frac{16}{63}t^9-\frac{38}{105}t^{10}+\frac{16}{99}t^{11}-\frac{64}{1575}t^{12}-\frac{64}{195}t^{13}-\frac{4}{45}t^{14}+\frac{208}{1125}t^{15}$$

$$+\frac{20}{91}t^{16}-\frac{8}{255}t^{17}+\frac{8}{95}t^{19}-\frac{8}{143}t^{22}+\cdots,$$

$$f_2(t)=\sum_{i=0}^{n-1}f_{2,i}(t)=f_{2,0}(t)+f_{2,1}(t)+f_{2,2}(t)+\cdots$$

$$=t^2+\frac{3}{10}t^8-\frac{88}{135}t^9-\frac{136}{225}t^{10}+\frac{39}{275}t^{11}-\frac{37}{315}t^{12}-\frac{5171}{121275}t^{14}+\frac{92}{495}t^{15}$$

$$-\frac{1}{18}t^{16}-\frac{60}{1309}t^{17}+\frac{52}{945}t^{18}-\frac{6}{875}t^{20}+\cdots,$$

$$f_3(t)=\sum_{i=0}^{n-1}f_{3,i}(t)=f_{3,0}(t)+f_{3,1}(t)+f_{3,2}(t)+\cdots$$

$$= t^3 - \frac{1}{10}t^6 - \frac{1}{21}t^7 - \frac{1}{7}t^8 - \frac{29}{540}t^9 + \frac{2}{45}t^{10} - \frac{82}{825}t^{11} + \frac{1}{44}t^{12}$$
$$- \frac{1}{65}t^{13} + \frac{2}{91}t^{14} + \frac{1}{175}t^{15} + \cdots,$$
$$f_4(t) = \sum_{i=0}^{n-1} f_{4,i}(t) = f_{4,0}(t) + f_{4,1}(t) + f_{4,2}(t) + \cdots$$
$$= t^4 - \frac{3}{20}t^8 + \frac{3}{35}t^{10} - \frac{30}{77}t^{11} - \frac{4}{9}t^{12} + \frac{7156}{20475}t^{13} - \frac{89}{525}t^{14} - \frac{14}{225}t^{15}$$
$$+ \frac{2917}{21000}t^{16} + \frac{19}{675}t^{18} + \cdots,$$

Notice that the noise terms that appear between various components vanish. These series have the closed form as $n \to \infty$ which gives$f_1(t) = t$,$f_2(t) = t^2$ $f_3(t) = t^3$ and $f_3(t) = t^4$, which is the exact solution of the system (3.15).

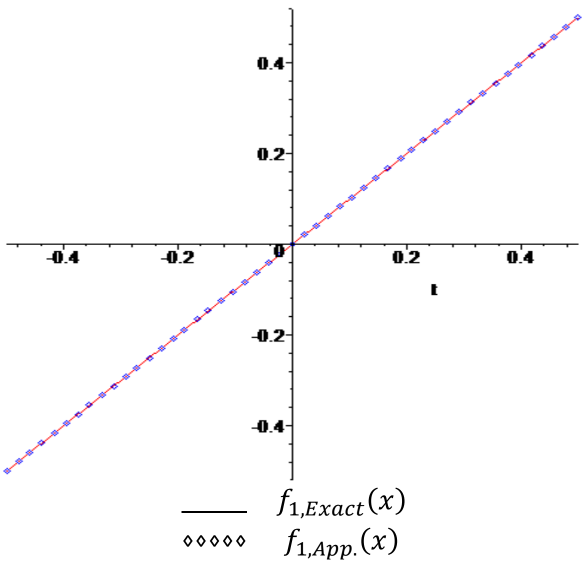
Table-5 shows a comparison of the numerical results applying the CST-ADM $(n = 5)$ and the Padé approximants (PA) of order [4/4] about the point$t = 0.02$, with the exact solutions and HPM [16]. It can be noticed that the results obtained by the present method are compatible with those of the HPM [16]. In Table 6, we list the MPA, MAE, MRE, and MRR obtained by the CST-ADM on the interval $[-0.5,0.5]$. Fig. 3 presents both the exact solutions and our approximations by the CST-ADM $(n = 4)$ in the interval$-0.5 \le x \le 0.5$.

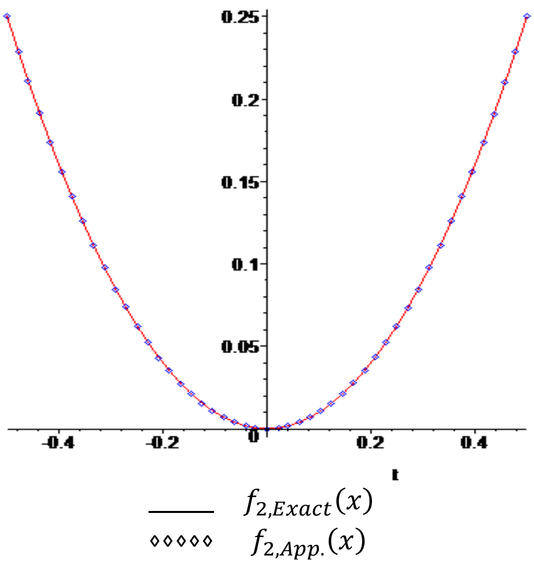
**Table 5-**Numerical results for example 3

| $t$ | $i$ | Exact: $f_i(x)$ | CST-ADM | $PA_i$ [4/4] | HPM [16] |
|---|---|---|---|---|---|
| 0.0 | 1 | 0.000000000 | 0.000000000 | 0.000000000 | 0.000000000 |
| | 2 | 0.000000000 | 0.000000000 | 0.000000000 | 0.000000000 |
| | 3 | 0.000000000 | 0.000000000 | 0.000000000 | 0.000000000 |
| | 4 | 0.000000000 | 0.000000000 | 0.000000000 | 0.000000000 |
| 0.1 | 1 | 0.100000000 | 0.099999999 | 0.100000000 | 0.100000000 |
| | 2 | 0.010000000 | 0.009999999 | 0.009999999 | 0.010000000 |
| | 3 | 0.001000000 | 0.000999999 | 0.001000000 | 0.000999999 |
| | 4 | 0.000100000 | 0.000099999 | 0.000099999 | 0.000099999 |
| 0.2 | 1 | 0.200000000 | 0.199999999 | 0.200000000 | 0.199999999 |
| | 2 | 0.040000000 | 0.039999999 | 0.039999999 | 0.039999999 |
| | 3 | 0.008000000 | 0.007999999 | 0.008000000 | 0.007999999 |
| | 4 | 0.001600000 | 0.001599997 | 0.001599999 | 0.001599997 |
| 0.3 | 1 | 0.300000000 | 0.299999965 | 0.300000000 | 0.299999965 |
| | 2 | 0.090000000 | 0.089999936 | 0.089999999 | 0.089999936 |
| | 3 | 0.027000000 | 0.026999972 | 0.027000000 | 0.026999972 |
| | 4 | 0.008100000 | 0.008099758 | 0.008099994 | 0.008099758 |
| 0.4 | 1 | 0.400000000 | 0.399998446 | 0.400000000 | 0.399998447 |
| | 2 | 0.160000000 | 0.159997478 | 0.159999999 | 0.159997478 |
| | 3 | 0.064000000 | 0.063999226 | 0.064000000 | 0.063999226 |
| | 4 | 0.025600000 | 0.025593236 | 0.025599941 | 0.025593236 |
| 0.5 | 1 | 0.500000000 | 0.499969255 | 0.500000000 | 0.499969255 |
| | 2 | 0.250000000 | 0.249954410 | 0.249999999 | 0.249954410 |
| | 3 | 0.125000000 | 0.124988570 | 0.125000000 | 0.124988570 |
| | 4 | 0.062500000 | 0.062405973 | 0.062499632 | 0.062405973 |

**Table 6**-MPA, MAE, MRE and MRR for example 3

| *t* | *i* | MPA | MAE | MRE | MRR |
|---|---|---|---|---|---|
| 0.0 | 1 | 0.0 | 0.0 | -- | 0.0 |
| | 2 | 0.0 | 0.0 | -- | 0.0 |
| | 3 | 0.0 | 0.0 | -- | 0.0 |
| | 4 | 0.0 | 0.0 | -- | 0.0 |
| 0.1 | 1 | 1.994E-20 | | 1.862E-13 | 1.840E-14 |
| | 2 | 2.714E-17 | 7.631E-14 | 7.631E-12 | 7.625E-14 |
| | 3 | 6.632E-15 | 1.186E-13 | 1.186E-10 | 1.099E-13 |
| | 4 | 3.443E-13 | 9.754E-13 | 9.754E-09 | 9.754E-13 |
| 0.2 | 1 | 6.222E-20 | 1.638E-10 | 8.194E-10 | 1.592E-10 |
| | 2 | 3.889E-16 | 3.920E-10 | 9.801E-09 | 3.900E-10 |
| | 3 | 3.992E-13 | 2.709E-10 | 3.386E-08 | 2.243E-10 |
| | 4 | 1.661E-10 | 2.362E-09 | 1.476E-06 | 2.361E-09 |
| 0.3 | 1 | 1.311E-19 | 3.422E-08 | 1.140E-07 | 3.242E-08 |
| | 2 | 1.936E-15 | 6.391E-08 | 7.102E-07 | 6.287E-08 |
| | 3 | 4.677E-12 | 2.705E-08 | 1.001E-06 | 1.911E-08 |
| | 4 | 5.239E-09 | 2.411E-07 | 2.977E-05 | 2.406E-07 |
| 0.4 | 1 | 2.262E-19 | 1.553E-06 | 3.882E-06 | 1.419E-06 |
| | 2 | 6.086E-15 | 2.521E-06 | 1.575E-05 | 2.429E-06 |
| | 3 | 2.680E-11 | 7.734E-07 | 1.208E-05 | 4.428E-07 |
| | 4 | 5.801E-08 | 6.763E-06 | 2.641E-04 | 6.719E-06 |
| 0.5 | 1 | 3.475E-19 | 3.074E-05 | 6.148E-05 | 2.677E-05 |
| | 2 | 1.482E-14 | 4.558E-05 | 1.823E-04 | 4.245E-05 |
| | 3 | 1.035E-10 | 1.142E-05 | 9.143E-05 | 5.071E-06 |
| | 4 | 3.677E-07 | 9.402E-05 | 1.504E-03 | 9.245E-05 |



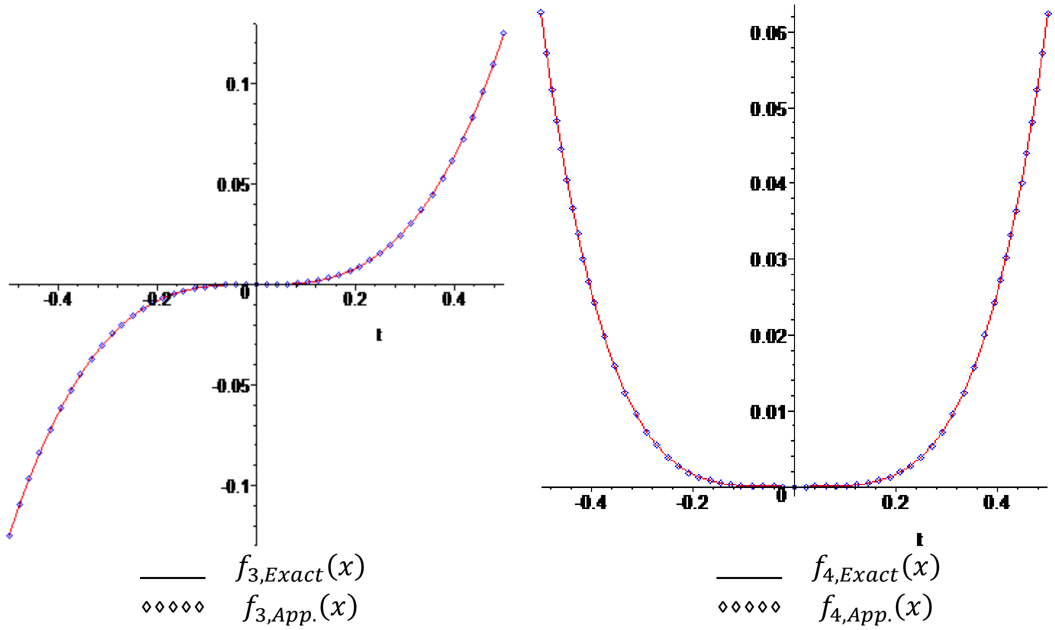



**Figure 3**-Plots the exact solutions and the approximations of CST-ADM.

**Conclusions**

The Combined Sumudu Transform-Adomian Decomposition Method has been applied for solving systems of non-linear Volterra integral equations. The solution process of CST-ADM is compatible with the method in the literature, providing analytical approximations such as HPM. The approach of CST-ADM has been tested by employing the method to obtain approximate-exact solutions of three examples. The results obtained in all cases demonstrate the reliability and the efficiency of this method. It has been shown that the error is monotonically reduced with the increment of the integer *n*.

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