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Complete Classification of Degree 7 for Genus 1

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Abstract
Assume that $\mathcal{F}: X \to \mathbb{P}$ is a meromorphic fuction of degree n where X is compact Riemann surface of genus g. The meromorphic function gives a branched cover of the compact Riemann surface X. Classes of such covers are in one to one correspondence with conjugacy classes of r-tuples $(\alpha_1, \alpha_2, \dots, \alpha_r)$ of permutations in the symmetric group S_n, in which $\alpha_1 . \alpha_2 . \dots . \alpha_r = 1$ and α_i's generate a transitive subgroup G of S_n. This work is a contribution to the classification of all primitive groups of degree 7, where X is of genus one.
Keywords: Monodromy Groups, Genus one Systems and Primitive Groups.

1. INTRODUCTION

A function $\mathcal{F}: X \to \mathbb{P}$ is a non-constant meromorphic function from compact connected Riemann surface X of genus g to Riemann sphere $\mathbb{P}$, if it can locally be described as two holomorphic functions. Furthermore, the mesomorphic function $\mathcal{F}$ is of degree n if the order of the fiber $(|\mathcal{F}^{-1}(x)|)$ for general x equal to n. A point $\rho\, in\, \mathbb{P}$ is a branch point if $(|\mathcal{F}^{-1}(\rho)| < n$. We suppose that $Y = \{\rho_1, \rho_2, \dots \rho_r\}$ is the branch points of the meromorphic function $\mathcal{F}$. The fundamental group $\pi_1(\mathbb{P}\backslash Y, x)$ for any $x \in \mathbb{P}\backslash Y$ acts transitively on n elements of the fiber. This action gives us a homomorphism $\mu_F: \pi_1(\mathbb{P}\backslash Y, x) \to S_n$. The image of μ_F is called Monodromy group of $\mathcal{F}$ and denoted by Mon(X, $\mathcal{F}$). We denote β_i for $1 \le i \le r$ as the closed path winding once around the point ρ_i. So, the fundamental group $\pi_1(\mathbb{P}\backslash Y, x)$ is generated by the homotopy class of β_i. This generator satisfies the relation $\beta_1 . \beta_2 . \dots . \beta_r = 1$, which yields the generators $\{\alpha_1, \alpha_2, \dots, \alpha_r\}$ of monodromy group Mon(X, $\mathcal{F}$), where $\alpha_i = \mu_F(\beta_i)$ satisfies the following conditions

$$G = \langle \alpha_1, \alpha_2, \dots, \alpha_r \rangle \qquad \dots (1)$$

$$\alpha_1 . \alpha_2 . \dots . \alpha_r = 1 \qquad \dots (2)$$

$$\sum_{i=1}^{r} ind\alpha_i = 2(n + g - 1) \qquad \dots (3)$$

Equation 3 is called Riemann Hurwtiz formula where $ind\alpha_i = n - Number\ of\ orbit(\alpha_i)$ on $\mathcal{F}^{-1}(\rho_i)$. Let G be a transitive subgroup of S_n. A genus g-system is a tuple $(\alpha_1, \cdots, \alpha_r)$ satisfying equations (1), (2) and (3) [1]. A natural question that arises from this study is: what value can monodromy group Mon(X, $\mathcal{F}$) take for a fixed genus of X?

Guralnick and Thompson [2], in 1990, gave a conjecture; There is a finite set $\varsigma(g)$ of simple groups, for any positive integer g, such that if $\mathcal{F}: X \to \mathbb{P}$ is a cover , where X is compact connected Riemann surface of genus g and $\mathfrak{I}$ is a non-commutative composition factor of Mon(X, $\mathcal{F}$), then this composition factor is either an isomorphism to the finite set $\varsigma(g)$ or $\mathfrak{I}$ is

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equal to A_n. This fact was established in 2001 by Frohardt and Magaard [1]. As the set $\varsigma(g)$ is finite , one hopes to see the sets $\varsigma(0)$, $\varsigma(1)$ and $\varsigma(2)$ clearly. In 1990, Guralnick and Thompson[2] used Riemann's Existence theorem to prove that the finite set $\varsigma(g)$ takes place in a primitively monodromy action. That is, if G is a group in the set $\varsigma(g)$, then there exists (X, $\mathcal{F}$) in which G= Mon(X, $\mathcal{F}$), and this group acts primitively on the fibers. This result refers to the theorem of Aschbacher and Scott [3].

Theorem1:(Aschbacher and Scott). Let G be a finite group, F(G) be a generated fitting subgroup of the group G, and L be a maximal subgroup of G such that $\bigcap_{g\in G} L^g = \{1\}$. Let K be a minimal normal subgroups of G and P be a minimal normal subgroup of K. Let $\nabla = \{P_1, \dots, P_t\}$ be the set of G conjugates of P. Then G=LK and exactly one of the following holds:

1- P has prime order.

2- F(G)=K$\times B$ where $K \cong B$ and $L \cap K = 1$.

3- F(G)=K is non-commutative and $L \cap K = 1$.

4- F(G)=K is non-commutative and $L \cap K \neq L \cap P = \{1\}$.

5- F(G)=K and $L \cap K = L \cap P_1 \times \dots \times P_t$ where $P_i = L \cap P \neq \{1\}$.

The first case in the above theorem was studied by Guralnick and Thompson [2]. They showed that there are only finitely many primitive affine groups which occur as a composition factor of a primitive genus zero. Furthermore, MohammedSalih [4] studied the same case of genus one and two. Cases 2 and 3 of the above theorem were considered by Shilh [5] and Guralnick and Thompson [2], respectively. They showed that there is no primitive genus zero system. Case 4 of theorem 1 was investigated by Aschbacher [6], who proved that the general fitting subgroup (F(G)) in case of genus zero system must be equal to $A_5 \times A_5$. MohammedSalih [7] classified the primitive group A_8 for genus zero. The final case was studied by Khudhur [8,9,10], who determined all sporadic simple groups of genus zero, one, and two. Our aim in this paper is to determine all primitive genus one groups of degree seven, except S_7. Now we give the following theorem.

Theorem 2: Let G denotes a group of degree 7. There exists non-constant meromorphic function $\mathcal{F}$ from compact connected Riemann surface X of genus one, such that G is a composition factor of Mon(X, $\mathcal{F}$), if and only if $G \cong, 7{:}3,\ AGL(1,7), L(3,2)$ and A_7. Next, we will prove theorem 2.

2- PRELIMINARY

In this section, we provide some backgrounds and results, while more are presented in, Haval , 2014, Peshawa,2016 and Volkein He.1996. From now onwards, we denote the space of cardinality r of subsets $\mathbb{C}$ by O_r.

Definition 2.1 [4,5]. Let $B \in O_r$ and $x \in \mathbb{P} \setminus Y$. We call a map $\mathcal{F}: \pi_1(\mathbb{P} \setminus Y, x) \longrightarrow G$ as admissible if it is a surjective homomorphism, and $\mathcal{F}(\sum_b) \neq 1$ for each $b \in B$. Here $\sum_b$, is the conjugacy class of $\pi_1(\mathbb{P} \setminus Y, x)$.

Definition 2.2 [4,5]. Let $B \in O_r$ and $\mathcal{F}: \pi_1(\mathbb{P} \setminus Y, x) \longrightarrow G$ is admissible . Then, we say that the two pairs $(B, \mathcal{F})$ and $(B^*, \mathcal{F}^*)$ are A -equivalent if and only if $\mathcal{F} = \mathcal{F}^*$ and $\mathcal{F}^* = a \circ \mathcal{F}$ for some $a \in A$.

Let $[B, \mathcal{F}]_A$ denote the A-equivalence class of $(B, \mathcal{F})$. The set of equivalence classes $[B, \mathcal{F}]_A$ is denoted by $H_r^A(G)$ and is called the *Hurwitz space* of G-covers.

We now introduce the **Nielsen classes** as follows. For a ramification type $C = \{C_1, \dots, C_r\}$, $N(C) = \{(x_1, x_2, \dots, x_r):\ G = \langle x_1, x_2, \dots, x_r \rangle\ ,\ \prod_i^r x_i = 1,\ \exists\sigma \in S_n$ such that $x_i \in C_{i\sigma}$ for all $i\}$.

Lemma 2.3 [5]. The map $\psi_A: H_r^A(G) \longrightarrow O_r$, $\psi_A([P, \mathcal{F}]) = P$ is a covering.

Lemma2.4 [5]. We obtain a bijection $\psi_A^{-1}(P_0) \longrightarrow \epsilon_r^A(G)$ by sending $[P_0, \mathcal{F}]_A$ to the generators $(x_1, x_2, \dots, x_r)$ where $x_i = \mathcal{F}([\gamma_i])$ for $i = 1, \dots, r$.

The group A also acts on $N(C)$ via sending $(x_1, x_2, \dots, x_r)$ to $(x_1^a, x_2^a, \dots, x_r^a)$, for $a \in A$. Let $N^A(C) = N(C)/A$.

Proposition 2.5 [5]. Let C be a fixed ramification type in G, and the subset $H_r^A(C)$ of $H_r^A(G)$ consists of all $[B, \mathcal{F}]_A$ with $Y = \{\rho_1, \rho_2, \dots \rho_r\}$, $\phi: \pi_1(\mathbb{P}^1 \setminus Y, \infty) \longrightarrow G$ and $\phi(\sum_{b_i}) \in C_i$ for $i = 1, \dots, r$. Then, $H_r^A(C)$ is a union of connected components in $H_r^A(G)$. Under the bijection from Lemma 2.4, the fiber in $H_r^A(C)$ over P_0 corresponds to the set $N^A(C)$. This yields a one to one correspondence between components of $H_r^A(C)$ and the braid orbits on $N^A(C)$. In particular, $H_r^A(C)$ is connected if and only if B_r acts transitively on $N^{in}(C) = N(C)$.

Definition 2.6 [1,8]. Let G be a group that acts on the set Ʊ. Then, the fixed point ratio of a point $x \in G$ is defined by the set $\{\left\{\frac{f(x)}{n}\right\}$, where $f(x)$ is the number of fixed points of x on Ʊ and n is the size of the set Ʊ.
The least upper bound for all fixed point ratios of x is denoted by b(G) and defined by $b(G) = Max\{\frac{f(x)}{n}$, n=[K:M] ; $M \geq G\}$ where K is a group of degree 7 with F(K)=G, and $x \in K$ acts by right translation on the right cosets of some maximal subgroups M of M. Now we will explain by the following example.
Example 2.7: The group A_7 has the following conjugacy classes of maximal subgroups: $A_6, L_2(7), S_5,$ and $(A_4 \times 3){:}\,2$ **.** Since index $[A_7{:}\,A_6] = 7$, then the number of fixed points of $g_i \in C_i$ in A_7 action by translation on the right coset of A_6 is {3,4,1,1,2,0,0,0}. Therefore, the maximal fixed point ratio of g_i is equal to $\frac{4}{7}$. Similarly, the maximal fixed point ratio of the conjugacy classes of maximal subgroups $L_2(7), S_5$ and $(A_4 \times 3){:}\,2$ are equal to $\frac{1}{5}, \frac{6}{21},$ and $\frac{1}{5}$, respectively. Hence, the maximal fixed point ratio is $\frac{4}{7}$=b(A_7).

For any gnus system $(x_1, x_2, \ldots, x_r)$ we denote $\hat{x} = (x_1, x_2, \ldots, x_r)$, $A(\hat{x}) = \sum_{i=1}^{r} \frac{d_i - 1}{d_i}$, where $|x_i| = d_i$. Next, we present a lemma which was proved by Khudhur [4] .
Lemma 2.8 [8]. Let G be a finite group and M be a Maximal subgroup of G. Then, the following holds

1- If $\frac{1}{[G:M]} \sum_{i=1}^{r} \sum_{j}^{d_i - 1} \frac{f(x_i^j)}{d_i} < A(\hat{x}) - 2$, then $\hat{x}$ does not have a genus zero system.

2- If $\frac{1}{[G:M]} \sum_{i=1}^{r} \sum_{j}^{d_i - 1} \frac{f(x_i^j)}{d_i} \neq A(\hat{x}) - 2$, then $\hat{x}$ does not have a genus one system.

3- If $\frac{1}{[G:M]} \sum_{i=1}^{r} \sum_{j}^{d_i - 1} \frac{f(x_i^j)}{d_i} > A(\hat{x}) - 2$, then $\hat{x}$ does not have a genus two system.

In the next lemma, we generalize the above lemma for genus k -system for k>2.
Lemma 2.9: Let G be a finite group and M is a Maximal subgroup of G. Then
If $\frac{1}{[G:M]} \sum_{i=1}^{r} \sum_{j}^{d_i - 1} \frac{f(x_i^j)}{d_i} < A(\hat{x}) - 2 - \frac{2k}{n}$, then $\hat{x}$ has not a genus k- system.
Proof: Let $\hat{x}$ be a genus k- system, then $\sum_{i=1}^{r} indx_i = 2n + 2k - 2$. Since

$$indx_i = n - \sum_{j=1}^{d_i} \frac{f(x_i^j)}{d_i} = n - \left(\sum_{j=1}^{d_{i-1}} \frac{f\left(x_i^j\right)}{d_i} + \frac{f\left(x_i^{d_i}\right)}{d_i} \right) = n - \frac{n}{d_i} - \sum_{j=1}^{d_{i-1}} \frac{f\left(x_i^j\right)}{d_i} = n\left(\frac{d_i - 1}{d_i}\right) - \sum_{j=1}^{d_{i-1}} \frac{f\left(x_i^j\right)}{d_i}.$$

then we get $\sum_{i=1}^{r} indx_i = n \sum_{i=1}^{r} \frac{d_i - 1}{d_i} - \sum_{i=1}^{r} \sum_{j=1}^{d_{i-1}} \frac{f\left(x_i^j\right)}{d_i}$.
Thus $\frac{1}{n} \sum_{i=1}^{r} \sum_{j=1}^{d_{i-1}} \frac{f\left(x_i^j\right)}{d_i} = \sum_{i=1}^{r} \frac{d_i - 1}{d_i} - \frac{1}{n} \sum_{i=1}^{r} indx_i = A(\hat{x}) - 2 - \frac{2k}{n} - \frac{2}{n}$.
which implies that $\frac{1}{[G:M]} \sum_{i=1}^{r} \sum_{j}^{d_i - 1} \frac{f(x_i^j)}{d_i} = A(\hat{x}) - 2 - \frac{2k}{n} - \frac{2}{n}$. Since $[G{:}\,M] = n > 1$, then if $\frac{1}{[G:M]} \sum_{i=1}^{r} \sum_{j}^{d_i - 1} \frac{f(x_i^j)}{d_i} < A(\hat{x}) - 2 - \frac{2k}{n}$ then the Riemann Hurwitz formula failed. Hence the claim.
Proposition 2.10 [1]. Assume that a group G acts transitively and faithfully on Ω and $|\Omega| = n$. Let $r \geq 2$, $G = \langle x_1, x_2, \ldots, x_r \rangle$, $\prod_{i}^{r} x_i = 1$, and $|x_i| = d_i > 1$, $i = 1, \ldots, r$. Then one of the following holds:
(1) $\sum_{i=1}^{r} \frac{d_i - 1}{d_i} \geq \frac{85}{42}$.
(2) G is solvable group and G is of type (3,3,3), (2,3,6) , $(2,2,d)$, (2,4,4) or (2,2,2,2).
(3) $r = 3$ and (up to permutation) $(d_1, d_2, d_3) =$
a) (2,3,3) and $G \cong A_4$.
b) (2,3,4) and $G \cong S_4$.
c) (2,3,5) and $G \cong A_5$.
(4) $r = 2$ and G is cyclic.

3- POSSIBLE RAMIFICATION TYPES

In this section, we find all possible ramification types and use filers to eliminate signature as much as we can. Firstly, we start by using equation 3 to remove the groups $C(7), D(2*7)$.

Lemma3.1: Let G be a group $C(7)$ or $D(2*7)$. Then G possesses no primitive genus one system.

Proof: The group $C(7)$ has six conjugacy classes of order 7 but in different types, which are (7A,7B,7C,7D,7E,7F). By using equation 3, we obtain that the group $C(7)$ has not possible ramification types. In other words, we do not have ramification types $(\alpha_1, \cdots, \alpha_r)$ such that $\sum_{i=1}^{r} ind\alpha_i = 14 = 2(7+1-1)$. Similarly, by using equation 3, we gain that the group $D(2*7)$ has not possible ramification types.

Lemma3.2: Let G be a group $7:3$ and $\hat{x} \in G^r$, where $x_i, i = 1, \dots, n$ acts on the right cosets of maximal subgroup of the group $7:3$. Then, $r \leq 3$, and if $r = 3$ then $x_1, x_2 \in \{3A, 3B\}$ and $x_3 \in \{7A, 7B\}$.

Proof: Suppose that $\hat{x}$ is genus one system. Then, by Riemann Hurwitz formula, $\sum_{i=1}^{r} indx_i = 2(7+1-1) = 14$. Since the indexes of x_i in $7:3$ are 0, 4, 4, 6, and 6, which are indexes of 1A,3A,3B,7A, and 7B, respectively, then $r \leq 3$, and if $r = 3$ then $x_1, x_2 \in \{3A, 3B\}$ and $x_3 \in \{7A, 7B\}$.

Note that the possible ramification types are (3A,3A,7A), (3A,3A,7B), (3A,3B,7A), (3A,3B,7B), (3B,3B,7A), and (3B,3B,7B). The next lemma allow us to eliminate some of them.

Lemma 3.3: Let G be a group $7:3$ and $\hat{x} \in G^r$, where $x_i, i = 1, \dots, n$ acts on the right cosets of maximal subgroup of the group $7:3$. Then, G is not of types (3A,3A,7A), (3A,3A,7B), (3B,3B,7A), (3B,3B,7B).

Proof: By using the GAP program, the types $(x, y, z) = (3A, 3A, 7A), (3A, 3A, 7B), (3B, 3B, 7A), (3B, 3B, 7B)$ do not satisfy $xyz = 1$. It follows that all of them will be eliminated.

Next, we present the possible ramification type of the group AGL(1,7). Since the degree operation of AGL(1,7) is equal to seven, then Riemann Hurwitz formula implies that $\sum_{i=1}^{r} indx_i = 14$.

Lemma 3.4: Let G be a group AGL(1,7) and $\hat{x} \in G^r$, where $x_i, i = 1, \dots, n$ acts on the right cosets of the maximal subgroup of the group AGL(1,7). Then, $r \leq 4$ and if $r = 4$, then $x_1, x_2 \in \{2A\}$ $x_3 \in \{2A, 3A, 3B, 6A, 6B\}$ and $x_4 \in \{3A, 3B, 6A, 6B\}$.

Proof: Suppose that $\hat{x}$ is genus one system. Then Riemann Hurwitz formula is equivalent to 14. Since the minimal index x_i is equal to 3, so if r=5, then $\sum_{i=1}^{5} indx_i = 15 > 14$, which is impossible, then $r \leq 4$. So, if r=4 then $x_1, x_2 \notin \{3A, 3B, 6A, 6B\}$, otherwise Riemann Hurwitz formula has failed. On the other hand, by $x_4 \notin \{2A\}$. Hence, $x_1, x_2 \in \{2A\}$, $x_3 \in \{2A, 3A, 3B, 6A, 6B\}$, and $x_4 \in \{3A, 3B, 6A, 6B\}$.

By the above lemma, the possible ramification types are (3A,6A,6A), (3B,6A,6A), (3A,6A,6B), (3B,6A,6B),(3B,6B,6B) (3A,3A,7A), (3B,3B,7A), (3A,3B,7A), (3A,6B,6B), (2A,6A,7A),(2A,6B,7A),(2A,2A,3A,3A),(2A,2A,2A,6A) ,(2A,2A,3A,3B), (2A,2A,2A,6B) , (2A,2A,3B,3B).

Lemma 3.5: Let G be a group AGL(1,7) and $\hat{x} \in G^r$, where $x_i, i = 1, \dots, n$ acts on the right cosets of the maximal subgroup of the group AGL(1,7). Then G is not of the type (3A,6A,6A), (3A,6A,6B), (3B,6A,6B),(3B,6B,6B) (3A,3A,7A), (3B,3B,7A), (3A,3B,7A), (2A,6A,7A),(2A,6B,7A),(2A,2A,3A,3A),(2A,2A,2A,6A) , (2A,2A,2A,6B) ,(2A,2A,3B,3B).

Proof: Suppose that z is representative of conjugacy classes of order seven of type A. Then, there are 6 conjugacy classes of the pair (x, y), such that the product of x and y is equal to inverse z, where x and y are representatives of the conjugacy classes of order three of types A and B, respectively. However, the order of the group generated by the pair (x, y) is not equal to the group AGL(1,7). Hence, the triple (3A,3B,7A) will be ruled out. A GAP calculation shows that the group algebra structure constants [4] of (3A,6A,6A), (3A,6A,6B), (3B,6A,6B),(3B,6B,6B) (3A,3A,7A), (3B,3B,7A), (2A,6A,7A),(2A,6B,7A),(2A,2A,3A,3A),(2A,2A,2A,6A) , (2A,2A,2A,6B) ,(2A,2A,3B,3B) are equal to zero. Thus, they will be cancelled.

The above arguments imply that the ramification types (3B,6A,6A) ,(3A,6B,6B),(2A,2A,3A,3B) will remain.

Note that the group $L(3,2)$ is of order 168 and it has elements of order 1,2,3,4 and 7 [3]. The group

$L(3,2)$ acts 2-transitively on seven and eight points. Furthermore, it has two classes of maximal subgroups, which are the symmetric group S_4 and 7:3. To determine the list of possible ramification types, we first use the GAP program to find the action of representative x_i of conjugacy classes of $L(3,2)$ on the right cosets of the maximal subgroups S_4 and 7:3.

Lemma 3.6: Let G be a group $L(3,2)$ and$\hat{x} \in G^r$, where $x_i, i = 1, \dots, n$, acts on the right cosets of the maximal subgroup S_4. Then, $r \leq 7$ and, if $r = 7$, then $x_1, x_2 \dots x_7 \in \{2A\}$.

Proof: Similar to proof 3.4, by using Riemann Hurwitz formula, we obtain the result.

By the above lemma, the group $L(3,2)$ has 25 possible ramification types when it acts on the right coset of the maximal subgroup S_4. One of these ramification types will be cancelled, which is (2A,7A,7B), because its group algebra structure constant is equal to zero. So, the 24 ramification type cannot be ruled out , as presented in table 1.

Lemma 3.7: Let G be a group $L(3,2)$ and $\hat{x} \in G^r$, where $x_i, i = 1, \dots, n$ acts on the right cosets of the maximal subgroup $7{:}3$. Then, $r \leq 4$ and, if $r = 4$, then $x_1, x_2 \dots x_4 \in \{2A, 3A\}$.

Proof: Similar to proof 3.4, by using Riemann Hurwitz formula, we obtain the result.

Group $L(3,2)$ has 17 possible ramification types when it acts on the right coset of subgroup 7:3. However, four of these ramification types will be cancelled, which are (2A,4A,4A), (2A,7A,7B), (2A,2A,2A,3A) , and (2A,2A,2A,2A). By proposition 2.10, types (2A,4A,4A) and (2A,2A,2A,2A) are solvable groups, then they will be ruled out. Finally, the group algebra structure constants of (2A,2A,2A,3A) and (2A,7A,7B) are equal to zero, hence they will be removed. So, 13 ramification types cannot be ignored , which are presented in table 2.

The final group with which we work in this paper is the alternating group A_7. The group A_7 is of order 2520 and has elements of order 1,2,3,4,5,6 and 7 [3]. It acts 5-transitively on 7 points. The group A_7 has four classes of maximal subgroups, which are $A_6, L_2(7), S_5$ and $(A_4 \times 3){:}2$ [3]. To determine the list of possible ramification types, similar to the group $L(3,2)$, we first find, using GAP program, the action of representative x_i of conjugacy classes of A_7 on the right cosets of the maximal subgroups $A_6, L_2(7), S_5$ and $(A_4 \times 3){:}2$.

Lemma3.8: Let G be a group A_7 and $\hat{x} \in G^r$, where $x_i, i = 1, \dots, n$ acts on the right cosets of the maximal subgroup A_6. Then, $r \leq 7$ and, if $r = 7$, then $x_1, x_2 \dots x_7 \in \{2A, 3A\}$.

Proof: Similar to proof 3.4, by using Riemann Hurwitz formula, we obtain the result.

Lemma3.9: Let G be a group A_7 and $\hat{x} \in G^r$, where $x_i, i = 1, \dots, n$ acts on the right cosets of the maximal subgroup $L_2(7)$ and S_5. Then, $r \leq 4$.

Proof: Similar to proof 3.4, by using Riemann Hurwitz formula, we obtain the result.

Lemma3.10: Let G be a group A_7 and $\hat{x} \in G^r$, where $x_i, i = 1, \dots, n$ acts on the right cosets of the maximal subgroup $(A_4 \times 3){:}2$. . Then, $r \leq 3$.

Proof: Similar to proof 3.4, by using Riemann Hurwitz formula, we obtain the result.

By Riemann Hurwitz formula, group A_7, when acts on the right cosets of the maximal subgroup A_6, has 172 possible ramification types, such that two of them will be removed by using the filter. In table 3, all the generating ramification types are presented. Similarly, by using Riemann Hurwitz formula, the group A_7 has 30, 26, and 12 possible ramification types when it acts on the right cosets of the maximal subgroups $L_2(7), S_5$ and $(A_4 \times 3){:}2$, respectively. So, the numbers of the ramification types of A_7 which are passing the filter are 18, 18, and 4, when acting on the maximal subgroups $L_2(7), S_5$ and $(A_4 \times 3){:}2$, respectively.

4- Proof of Theorem 2

The purpose of this section is to prove theorem 2. For the groups of degree 7, we have a complete list of admissible genus one tuples. The groups of degree 7 are $C(7)$, $D(2*7)$, $7{:}3$, $AGL(1,7)$, $L(3,2)$, A_7 and S_7 [3]. In this step, we do not study the group S_7 because its tuples are too large so that our computers cannot compute the braid orbits. By lemma 3.1, we removed the groups $C(7)$, $D(2*7)$. So, the four groups of $7{:}3$, $AGL(1,7)$, $L(3,2)$ and A_7 remained.

The first step in this process is to find a list of representative conjugacy classes. In the next step, we give for each group the set of permutation indices. This set determines the tuples which satisfy Riemann Hurwitz formula. The filters presented in section two provided us with the generated ramification types. Next, by using MapClass, we determined the number of components of the Hurwitz spaces, $H_r^{in}(C)$. Now, we give the following lemmas.

Lemma 4.1: For the group $7:3$, the Hurwitz spaces, $H_r^{in}(C)$, are connected and $r = 3$.
Proof: By Lemma 3.2, r=3. So, all ramification types C of the group 7:3 of length 3. Furthermore, all types C which is presented in Table1 up to permutation has only one braid orbit on Nielsen classes $N\,(7:3, C)$. From Proposition 2.5, we obtain that the Hurwitz spaces $H_r^{in}(C)$ are connected.
Lemma 4.2: For the group AGL(1,7) , the Hurwitz spaces, $H_r^{in}(C)$, are connected if $r = 3$ or $r = 4$.
Proof : By Lemma 3.4, $r \leq 4$ and, if $r \leq 2$, then by Proposition 2.10, G is a cyclic group, hence it will be eliminated. So, all ramification types C are of the length three or four. From Table 2, we observe that all ramification types C up to permutation has only one braid orbit on the Nielsen classes N (AGL(1,7) $, C$). So, the Hurwitz spaces $H_r^{in}(C)$ has one component. It follows from the Proposition 2.5 that the Hurwitz spaces $H_r^{in}(C)$ are connected.
Lemma 4.3: For the group L(3,2), the Hurwitz spaces, $H_r^{in}(C)$, are disconnected if $r = 3$ and $n = 7$.
Proof : By lemma 3.6, the group L(3,2), when it acts on the right coset of the maximal subgroup S_4 , has a ramification type of length $r \leq 7$. If $r \leq 2$, then by proposition 2.10, G is cyclic group, and then it will be eliminated. Hence, the length of the ramification types is between $3 \leq r \leq 7$. As shown in Table 3, the types C=(3A,4A,7A) and C=(3A,4A,7B) have two braid orbits on Nielsen classes N (L(3,2) $, C$), which implies that the Hurwitz spaces $H_r^{in}(C)$ has two components. By Proposition 2.5, we obtain that the Hurwitz spaces $H_r^{in}(C)$ are disconnected.
Lemma 4.4: For the group L(3,2), the Hurwitz spaces $H_r^{in}(C)$ are connected if $r \geq 4$ and $n = 7$.
Proof : It follows from the fact that the Nielsen classes $N\,(C)$ are the disjoint union of braid orbits, but for L(3,2) we have only one braid orbit for all types C, as given in Table 3. From Proposition 2.5, we obtain that the Hurwitz spaces $H_r^{in}(C)$ are connected.
Lemma 4.5: For the group L(3,2), the Hurwitz spaces $H_r^{in}(C)$ are disconnected if $r \geq 3$ and $n = 8$.
Proof : By Lemma 3.7, $r \leq 4$. This means that the length of ramification types is between $3 \leq r \leq 4$. Since some type C ramifications in Table 4 have two braid orbits on Nielsen classes N (L(3,2) $, C$), the Hurwitz spaces $H_r^{in}(C)$ have two components. By proposition 2.5, we obtain that the Hurwitz spaces $H_r^{in}(C)$ are disconnected.
Lemma 4.6: For the group A_7, the Hurwitz spaces $H_r^{in}(C)$ are disconnected if $r \geq 3$ and $n = 7$.
*Proof :*By Lemma 3.8, $r \leq 7$. Since some type C ramifications in Table 5 have at least two braid orbits on Nielsen classes $N\,(A_7, C)$, the Hurwitz spaces $H_r^{in}(C)$ have at least two components. By Proposition 2.5, we obtain that the Hurwitz spaces $H_r^{in}(C)$ are disconnected.
Note that the Hurwitz spaces for all of ramification types C of group A_7 are disconnected when A_7 acts on the right coset of the maximal subgroup A_6.
Lemma 4.7: For the group A_7, the Hurwitz spaces $H_r^{in}(C)$ are disconnected if $r \geq 3$ and $n = 15$.
*Proof :*By Lemma 3.9, $r \leq 4$. Similar to above, when A_7 acts on the right cosets of the maximal subgroup $L_2(7)$, it has at least two braid orbits for some types C on Nielsen classes $N\,(A_7, C)$, as given in Table 6. Therefore, the Hurwitz spaces $H_r^{in}(C)$ has at least two components. By proposition 2.5, we obtain that the Hurwitz spaces $H_r^{in}(C)$ are disconnected.
Lemma 4.8: For the group A_7, the Hurwitz spaces $H_r^{in}(C)$ are disconnected if $r = 3$ and connected if r=4 where $n = 21$.
*Proof :*By lemma 3.9, $r \leq 4$. Thus the length of the ramification types C of, A_7 when acts on the right cosets of maximal subgroup S_5 between $3 \leq r \leq 4$. Now if r=3, then A_7 has at least two braid orbits for some types C on Nielsen classes $N\,(A_7, C)$. However, if r=4, then A_7 has one braid orbits for all types C on Nielsen classes $N\,(A_7, C)$, as a presented in Table7. From Proposition 2.5, we obtain that the Hurwitz spaces $H_r^{in}(C)$ are disconnected if r=3 and connected if r=4.
Lemma 4.9: For the group A_7, the Hurwitz spaces $H_r^{in}(C)$ are disconnected if $r = 3$ and $n = 35$.
*Proof :*By Lemma 3.10, $r \leq 3$. It follows that the group A_7 has four ramification types of length 3, as shown in Table 8. Note that all types have at least two braid orbits on Nielsen classes $N\,(A_7, C)$. Therefore, the Hurwitz spaces $H_r^{in}(C)$ have at least two components. Hence, by Proposition 2.5, we obtain that the Hurwitz spaces $H_r^{in}(C)$ are disconnected for all type C.
Proof of Theorem 2: By Lemmas 4.1 to 4.9, we could obtain the results in Tables-(1 to 8).

Table 1-Connected Components $\mathcal{H}_{r,1}^{in}(C)$ of 7:3

Group	Ramification type	N. of orbits	Length of orbits	Ramification type	N. of orbits	Length of orbits
7:3	(3A,3B,7A)	1	1	(3A,3B,7B)	1	1

Table 2- Connected Components $\mathcal{H}_{r,1}^{in}(C)$ of AGL(1,7)

Group	Ramification type	N. of orbits	Length of orbits	Ramification type	N. of orbits	Length of orbits
AGL(1,7)	(3B,6A,6A)	1	1	(3A,6B,6B),	1	1
	(2A,2A,3A,3B)	1	8			

Table 3-Connected Components $\mathcal{H}_{r,1}^{in}(C)$ of L(3,2), n=7.

Group	Ramification type	N. of orbits	Length of orbits	Ramification type	N. of orbits	Length of orbits
L(3,2)	(3A,3A,7A)	1	1	(3A,3A,7B)	1	1
	(3A,4A,7A)	2	1	(3A,4A,7B)	2	1
	(4A,4A,7A)	1	1	(4A,4A,7B)	1	1
	(2A,7A,7A)	1	1	(2A,7B,7B)	1	1
	(2A,3A,3A,3A)	1	120	(2A,3A,3A,4A)	1	84
	(2A,3A,4A,4A)	1	66	(2A,4A,4A,4A)	1	36
	(2A,2A,3A,7B)	1	21	(2A,2A,3A,7A)	1	21
	(2A,2A,4A,7B)	1	14	(2A,2A,4A,7A)	1	14
	(2A,2A,2A,3A,3A)	1	864	(2A,2A,2A,3A,4A)	1	648
	(2A,2A,2A,4A,4A)	1	456	(2A,2A,2A,2A,7A)	1	147
	(2A,2A,2A,2A,7B)	1	147	(2A,2A,2A,2A,2A,3A)	1	6480
	(2A,2A,2A,2A,2A,4A)	1	4800	(2A,2A,2A.2A,2A,2A,2A)	1	48960

Table 4- Connected Components $\mathcal{H}_{r,1}^{in}(C)$ of L(3,2), n=8.

Group	Ramification type	N. of orbits	Length of orbits	Ramification type	N. of orbits	Length of orbits
L(3,2)	(2A,4A,7A)	1	1	(2A,4A,7B)	1	1
	(2A,7A,7A)	1	1	(2A,7B,7B)	1	1
	(3A,4A,4A)	2	1	(3A,4A,7A)	2	1
	(3A,7A,7A)	1	1	(3A,4A,7B)	2	1
	(3A,7A,7B)	1	1	(3A,7B,7B)	1	1
	(2A,2A,3A,3A)	1	30	(2A,3A,3A,3A)	1	120
	(3A,3A,3A,3A)	2	144,90			

Table 5- Connected Components $\mathcal{H}_{r,1}^{in}(C)$ of A_7,n=7

Group	Ramification type	N. of orbits	Length of orbits	Ramification type	N. of orbits	Length of orbits
A_7	[5A,5A,7A]	12	1	[5A,5A,7B]	12	1
	[4A,5A,7A]	18	1	[4A,5A,7B]	18	1
	[4A,4A,7A]	22	1	[4A,4A,7B]	22	1
	[3B,5A,7A]	8	1	[3B,5A,7B]	8	1
	[3B,4A,7A]	6	1	[3B,4A,7B]	6	1
	[5A,6A,7A]	7	1	[5A,6A,7B]	7	1
	[4A,6A,7A]	8	1	[4A,6A,7B]	8	1
	[3B,6A,7A]	3	1	[3B,6A,7B]	3	1
	[6A,6A,7A]	2	1	[6A,6A,7B]	2	1
	[3A,7A,7A]	2	1	[3A,7A,7B]	3	1

	[3A,7B,7B]	2	1	[2A,7A,7A]	2	1
	[2A,7A,7B]	1	1	[2A,7B,7B]	2	1
	[3A,5A,5A,5A]	2	450,180	[3A,4A,5A,5A]	2	600,600
	[3A,4A,4A,5A]	2	600,600	[3A,4A,4A,4A]	2	1260,1260
	[3A,3B,5A,5A]	2	180,330	[3A,3B,4A,5A]	2	410,410
	[3A,3B,4A,4A]	2	532,590	[3A,3B,3B,5A]	2	240,130
	[3A,3B,3B,4A]	2	248,248	[3A,3B,3B,3B]	3	126,42,42
	[3A,5A,5A,6A]	1	550	[3A,4A,5A,6A]	1	700
	[3A,4A,4A,6A]	1	894	[3A,3B,5A,6A]	1	330
	[3A,3B,4A,6A]	1	408	[3A,3B,3B,6A]	1	188
	[3A,5A,6A,6A]	1	230	[3A,5A,4A,6A]	1	304
	[3A,3B,6A,6A]	1	130	[3A,6A,6A,6A]	1	102
	[3A,3A,5A,7A]	2	35,70	[3A,3A,5A,7B]	2	35,70
	[3A,3A,4A,7A]	2	84,84	[3A,3A,4A,7B]	2	84,84
	[3A,3A,3B,7A]	2	49,28	[3A,3A,3B,7B]	2	49,28
	[3A,3A,6A,7A]	1	70	[3A,3A,6A,7B]	1	70
	[2A,4A,5A,5A]	2	660,640	[2A,5A,5A,5A]	3	360,480,360
	[2A,2A,6A,7B]	3	56,56,28	[2A,4A,4A,5A]	3	960,960,960
	[2A,4A,4A,4A]	3	1260,1260,1248	[2A,3B,4A,4A]	1	1548
	[2A,3B,5A,5A]	1	900	[2A,3B,4A,5A]	1	1290
	[2A,3B,3B,5A]	1	540	[2A,3B,3B,4A]	1	576
	[2A,3B,3B,3B]	1	104	[2A,5A,5A,6A]	3	250,250,330
	[2A,4A,5A,6A]	3	340,365,365	[2A,4A,4A,6A]	3	468,468,424
	[2A,3B,5A,6A]	1	510	[2A,3B,4A,6A]	1	600
	[2A,3B,3B,6A]	1	254	[2A,5A,6A,6A]	3	95,95,160
	[2A,4A,6A,6A]	3	140,150,150	[2A,3B,6A,6A]	1	192
	[2A,6A,6A,6A]	3	30,30,80	[2A,3A,5A,7A]	1	175
	[2A,3A,5A,7B]	1	175	[2A,3A,4A,7A]	1	266
	[2A,3A,4A,7B]	1	266	[2A,3A,3B,7A]	1	119
	[2A,3A,3B,7B]	1	119	[2A,3A,6A,7A]	1	105
	[2A,3A,6A,7B]	1	105	[2A,3A,3A,5A,5A]	1	11350
	[2A,2A,5A,7A]	3	105,105,70	[2A,2A,5A,7B]	3	105,105,70
	[2A,2A,4A,7A]	3	112,126,126	[2A,2A,4A,7B]	3	112,126,126
	[2A,2A,3A,7A]	1	126	[2A,2A,3B,7A]	1	126
	[2A,2A,6A,7A]	3	56,56,28	[3A,3A,3A,5A,5A]	2	4050,1800
	[3A,3A,3A,3A,7B]	2	588,294	[3A,3A,3A,4A,5A]	2	5400,5400
	[3A,3A,3A,4A,4A]	2	8064,8268	[3A,3A,3A,3B,5A]	2	3150,1650
	[3A,3A,3A,3B,4A]	2	3804,3804	[3A,3A,3A,4A,4A]	2	2160,1170
	[3A,3A,3A,5A,6A]	1	4950	[3A,3A,3A,4A,6A]	1	6552
	[3A,3A,3A,3B,6A]	1	3120	[3A,3A,3A,6A,6A]	1	2262
	[3A,3A,3A,3A,7A]	2	588,194	[2A,3A,3A,6A,6A]	1	3552
	[2A,3A,3A,4A,5A]	1	18100	[2A,3A,3A,4A,4A]	1	26676
	[2A,3A,3A,3B,5A]	1	8250	[2A,3A,3A,3B,4A]	1	12060
	[2A,3A,3A,3B,3B]	1	5380	[2A,3A,3A,5A,6A]	1	7350

	[2A,3A,3A,4A,6A]	1	10108	[2A,3A,3A,3B,6A]	1	1698
	[2A,3A,3A,3A,7A]	1	1519	[2A,3A,3A,3A,7B]	1	1519
	[2A,2A,3A,5A,5A]	1	19850	[2A,2A,3A,4A,5A]	1	29900
	[2A,2A,3A,4A,4A]	1	42016	[2A,2A,3A,3B,5A]	1	13650
	[2A,2A,3A,3B,4A]	1	18996	[2A,2A,3A,3B,3B]	1	8290
	[2A,2A,3A,5A,6A]	1	11650	[2A,2A,3A,4A,6A]	1	15508
	[2A,2A,3A,3B,6A]	1	7056	[2A,2A,3A,6A,6A]	1	5350
	[2A,2A,3A,3A,7A]	1	2548	[2A,2A,3A,3A,7B]	1	2548
	[2A,2A,2A,4A,5A]	3	16200, 16200, 15600	[2A,2A,2A,4A,4A]	3	21816, 21816, 20736
	[2A,2A,2A,3B,5A]	1	21600	[2A,2A,2A,3B,4A]	1	27648
	[2A,2A,2A,6A,6A]	3	2364,2364, 3120	[2A,2A,2A,5A,6A]	3	6600,5700 ,5700
	[2A,2A,2A,3B,6A]	1	10314	[2A,2A,2A,3B,3B]	1	11088
	[2A,2A,2A,2A,7A]	3	1568,2205, 2205	[2A,2A,2A,2A,7B]	3	1568,2205 ,2205
	[2A,2A,2A,3A,7A]	1	4116	[2A,2A,2A,3A,7B]	1	4116
	[2A,2A,2A,5A,5A]	3	10800,1080 0,12000	[2A,2A,2A,2A,3A,5 A]		511,500
	[3A,3A,3A,3A,3A,5A]	2	37500,1750 0	[3A,3A,3A,3A,3A,4 A]	2	49280,492 80
	[3A,3A,3A,3A,3A,3B]	2	14880,2880 0	[3A,3A,3A,3A,3A,6 A]	1	44720
	[2A,3A,3A,3A,3A,6A]	1	68756	[2A,3A,3A,3A,3A,5 A]	1	105000
	[2A,3A,3A,3A,3A,4A]	1	165696	[2A,3A,3A,3A,3A,3 B]	1	76128
	[2A,2A,3A,3A,3A,5A]	1	184250	[2A,2A,3A,3A,3A,4 A]	1	276016
	[2A,2A,3A,3A,3A,3B]	1	125418	[2A,2A,3A,3A,3A,6 A]	1	107454
	[2A,2A,2A,3A,3A,5A]	1	311250	[2A,2A,2A,3A,3A,5 A]	1	446112
	[2A,2A,2A,3A,3A,3B]	1	202968	[2A,2A,2A,3A,3A,6 A]	1	167916
	[2A,2A,2A,2A,3A,4A]	1	710112	[2A,2A,2A,2A,3A,3 B]	1	318,438
	[2A,2A,2A,2A,2A,5A]	1	511500	[2A,2A,2A,2A,2A,4 A]	1	1,093,120
	[2A,2A,2A,2A,2A,3B]	1	479,520	[2A,2A,2A,2A,2A,6 A]	1	392,040
	[2A,2A,2A,2A,3A,6A]	1	258,864	[2A,2A,2A,2A,2A,2 A,3A]	1	12,157,56 0
	[2A,2A,3A,3A,3A,3A ,3A]	1	171960	[2A,2A,2A,3A,3A,3 A,3A]	1	2,904,744
	[2A,2A,2A,2A,2A,3A ,3A]	1	7,690,320	[2A,2A,2A,2A,3A,3 A,3A]	1	4765610
	[2A,2A,2A,2A,2A,2A ,2A]	1	18,828,480	[2A,3A,3A,3A,3A,3 A,3A]	1	988960
	[3A,3A,3A,3A,3A,3A ,3A]	2	353520, 169530			

Table 6- Connected Components $\mathcal{H}_{r,1}^{in}(C)$ of A_7, n=15

Group	Ramification type	N. of orbits	Length of orbits	Ramification type	N. of orbits	Length of orbits
A_7	[3A,3B,7A]	1	1	[3A,3B,7A]	1	1
	[3B,4A,6A]	6	1	[3B,4A,6A] 10	10	1
	[3B,4A,7A]	6	1	[3B,4A,7B]	6	1
	[4A,4A,4A]	24	1	[2A,6A,7A]	2	1
	[2A,5A,7A]	2	1	[2A,7A,7A]	2	1
	[2A,6A,7B]	2	1	[2A,5A,7B]	2	1
	[2A,7A,7B]	1	1	[2A,7B,7B]	2	1
	[2A,3B,3B,3B]	1	104	[2A,2A,3B,4A]	1	192
	[2A,2A,2A,7A]	2	21,21	[2A,2A,2A,7B]	2	21,21

Table 7- Connected Components $\mathcal{H}_{r,1}^{in}(C)$ of A_7 n=21

Grou p	Ramification type	N. of orbits	Length of orbits	Ramification type	N. of orbits	Length of orbits
A7	[3B,4A,4A]	8	1	[4A,4A,4A]	24	1
	[3B,3B,6A]	2	1	[3B,4A,6A]	6	1
	[4A,4A,6A]	12	1	[3B,6A,6A]	2	1
	[4A,6A,6A]	4	1	[6A,6A,6A]	2	1
	[3A,3B,7A]	1	1	[3A,4A,7A]	2	1
	[3A,3B,7B]	1	1	[3A,4A,7B]	2	1
	[3A,6A,7A]	1	1	[3A,6A,7B]	1	1
	[2A,5A,7A]	2	1	[2A,5A,7B]	2	1
	[2A,2A,2A,7A]	1	21	[2A,2A,2A,7B]	1	21

Table 8-Connected Components $\mathcal{H}_{r,1}^{in}(C)$ of A_7, when n=35

Grou p	Ramification type	N. of orbits	Length of orbits	Ramification type	N. of orbits	Length of orbits
A7	[2A,6A,7A]	2	1	[2A,6A,7B]	2	1
	[3A,4A,4A]	8	1	[3A,3A,6A]	2	1

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