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The Ranking Function for Solving the Fuzzy Hotelling T^2 Test

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Abstract

A multivariate control chart is measured by many variables that are correlated in production, using the quality characteristics in any product. In this paper, statistical procedures were employed to find the multivariate quality control chart by utilizing fuzzy Hotelling T^2 test**.** The procedure utilizes the triangular membership function to treat the real data, which were collected from Baghdad Soft Drinks Company in Iraq. The quality of production was evaluated by using a new method of the ranking function.

Keywords: Multivariate Quality Control, Fuzzy Set Theory, Hotelling T^2 , Ranking function.

الدالة الرتبية المقترحة لحل اختبار Hotelling T^2 الضبابي

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الخلاصة:

تم قياس مخطط المراقبة متعدد المتغيرات من خلال العديد من المتغيرات المترابطة في الإنتاج باستخدام خصائص الجودة لأي منتج. في هذا البحث، تم استخدام الأساليب الإحصائية للإيجاد مخطط مراقبة الجودة لمتعدد المتغيرات من خلال استخدام Hotelling T^2 الضبابي. وتم تطبيق أسلوب Hotelling T^2 الضبابي باستخدام الدالة العضوية الضبابية الثلاثية للبيانات الحقيقية التي تم أخذها من شركة بغداد للمشروبات الغازية في العراق و تم تقييم جودة الإنتاج باستخدام طريقة جديدة للدالة الرتبية

1- Introduction

Control Chart is one of the statistical procedures, which is used to monitor the production process and determine whether it is subjected to specifications of quality control or not, since the units produced differ in quality. If these imbalances and deviations are minor, the production is acceptable, but if these differences and deviations exceed certain limits, the production is not acceptable**.**
The statistician Shewhart used the statistical methods for the first time in the field of quality control in 1924. He used a single variable or separate and independent variables with neglecting the relationship between them. In fact, the product cannot be controlled through a single variable and the neglect of the rest of the variables are actions that are made in most of the industrial products. Thus, Hotelling (1947) used the multivariable method of controlling product quality by taking the relationship between variables [1].

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Control chart is one of the most common statistical methods used in terms of monitoring the changes that occur during the stages of production process. It acts by determining whether the process is statistically accurate or not, through observations recorded from the samples drawn.
The data for CO_2 and Brix were taken from Baghdad Soft Drinks Company in Iraq. It is a laboratory for testing produced drinking water bottles and detect the specifications and efficiency of production accurately and quickly.
Zadeh was the first discover the fuzzy set theory in 1965 [2]. Bradshaw (1983) used the fuzzy sets for the first time as a basis for the explanation of the measurement of conformity for each product unit with its specifications [3]. Raz and Wang (1990) attempted to extend the use of control charts to allow for linguistic variables [4]. Cheng *et al*. (1995) proposed an economic statistical np-control chart design [5]. Franceschine and Romano (1999) proposed a method for the online control of qualitative characteristics of a product/service using control charts for linguistic variables [6]. Latva-Kayra (2001) proposed EWMA (the exponentially weighted moving average) and CUSUM (the cumulative sum) with fuzzy control limits and used their fuzzy combination [7]. Gulbayand and Kahraman (2006) proposed a direct fuzzy approach to fuzzy control charts without any defuzziffication, in addition to fuzzy abnormal pattern rules based on the probabilities of fuzzy events [8]. Wang and Kuo (2007), multiresolution relied on robust fuzzy clustering approach[9]. Alizadeh, Khamseh, and Ghomi (2010) developed multivariate variable control charts in fuzzy modes [10]. Taylan and Darrab (2012) described the use of artificial intelligence (AI) methods, such as fuzzy logic and neural networks, in quality control and improvement [11]. M. Hossein and et al. (2014) provided a literature review of the control chart under a fuzzy environment with proposing several classifications and analyses [12]. Fernández and et al. (2015) reported that the use of fuzzy control charts becomes inevitable when the considered statistical data are vague or affected by uncertainty [13]. Madadi and Mahmoudzadeh (2017) presented control and development of the fuzzy statistical process for the attribute quality control charts by using Monte Carlo simulation method [14]. The goal of this research is carrying out the fuzzy Hotelling T^2 technique utilizing the triangular membership function, then using a new ranking function to find the Hotelling T^2 .
This paper is organized as follows. Section 2 presents the Hotelling T^2 technique. Section 3 describes the fuzzy set theory. In section 4, we introduce the new method of ranking function. In section 5, an introduction to the application of real data is presented. In section 6, numerical results are shown. In section 7, conclusions are given.

2- Hotelling T^2

Hotelling 1947 suggested this technique [15]. It has the advantage of having only a higher limit which represents a value of T^2. If one or more of these observations exceeds the upper limit, the production process is out of control. In this case, it is necessary to reveal the reasons for these deviations and take the necessary measures to correct the production process
The Hotelling T^2 relation is expressed as follows:
$T^2 = (X - \bar{X})^T \sum^{-1} (X - \bar{X})$
Let X=$(x_1, x_2, x_3, \dots, x_L)$ represents a row vector of random variables that represent the quality of the product and x is distributed as multivariate normal distribution with $X \sim N(M, \sum)$ where M is the vector and $\sum$ is the v8ariance- covariance matrix.

The variance-covariance matrix can be compensated for by the matrix S, which is an unbiased estimate of the matrix $\sum$. This matrix of S is

$$\sum = \begin{pmatrix} {S_1}^2 & S_{12} & \cdots & S_{1L} \\ S_{21} & {S_2}^2 & \cdots & S_{2L} \\ \vdots & \vdots & \ddots & \vdots \\ S_{L1} & S_{L2} & \cdots & {S_L}^2 \end{pmatrix} \quad \dots\dots\dots\dots\dots \quad (1)$$

such that

$$S_{ij} = \frac{\sum_{L=1}^{n}(x_{iL} - \bar{x}_i)(x_{jL} - \bar{x}_j)}{n-1} \quad \text{for } i \neq j \quad \dots\dots\dots\dots\dots\dots \quad (2)$$

$\bar{X}_i = \frac{\sum x_i}{n}$

$${S_i}^2 = \frac{\sum_{L=1}^{n}(x_{iL} - \bar{x}_i)^2}{n-1} \quad \dots\dots\dots\dots\dots\dots\dots \quad (3)$$

The null and alternative hypotheses are:
H_0: The production process is out of control.
H_1: The production process is in control.
Then we apply T^2 test as follows:
$T^2 = (X - X)^T \sum^{-1} (X - \bar{X})$ ………………. (4)
The calculated values of T^2 test are compared with the value of the control term T^2 test, extracted by the following formula:
$T^2 = \frac{L(n-1)}{n-L}$ F (L, n-L,1-α) ………………… (5)
L: represents the number of variables.
n: represents the sample size.
F: the tabular value of F at the 1-α level of significance and L, n-L degree of freedom.
If the calculated values of T^2 are greater than the value of the control term, we reject the null hypothesis that the production process is in the control limit and accept the alternative hypothesis that the production process is out of control if the calculated value of T^2 is smaller than the value of the control term.

3- Fuzzy set [15]

Let X be the universal set. A fuzzy set in X is a set of ordered pairs, A={(x,$\mu_A(x)); x \epsilon X$} , where $\mu_A : X \rightarrow [0,1]$ is called the membership set.

α-Cats of the Fuzzy Set [15]

The crisp set that contains all the elements of X that have non-zero membership grades in a fuzzy set A is called the support of the set A, denoted by Supp(A). i.e., Supp(A)={x ϵX : A(x) ≥ 0}.
The triangular membership function is used as follows:

$$\mu_A(x) = \begin{cases} \frac{\lambda(x-a)}{(b-a} & a \leq x \leq b \\ \lambda & x = b \\ \frac{\lambda(c-x)}{(c-b)} & b \leq x \leq c \end{cases} \quad \text{………..} \ (6)$$

- A new ranking function [8,16]

The ranking function is ordering the elements which maps
D:F(R) → R for each fuzzy number into the real line, where a natural order exist. We defined orders on by:
D(Ã) > D($\tilde{B}$) ↔ Ã > $\tilde{B}$
D(Ã) < D($\tilde{B}$) ↔ Ã < $\tilde{B}$
D(Ã)= D($\tilde{B}$) ↔ Ã ≈ $\tilde{B}$
Where $\widetilde{A}, \widetilde{B}$ are in F(R) .
Assume that the triangular fuzzy numbers are represented by Ã= (a, b, c), where b is the whole sample, a is the lift width, and c is the right width.
Now, we present the arbitrary fuzzy numbers Ã(α) by an ordered pair of functions [$\tilde{A}^L(\alpha), \tilde{A}^U(\alpha)$] , where $\tilde{A}^L(\alpha)$ is a bounded left continuous nondecreasing function over [0,1] and $\tilde{A}^U(\alpha)$ is a bounded left continuous nonincreasing function over [0,1]; $\tilde{A}^L(\alpha) \leq \tilde{A}^U(\alpha)$,
where $\tilde{A}^L(\alpha) = inf\{x|\tilde{A}(x) \geq \alpha\}$ and $\tilde{A}^U(\alpha) = sup\{x|\tilde{A}(x) \geq \alpha\}$.
Now, we utilize the triangular membership function to find the new ranking function, which is as follows:

$\alpha = \frac{\lambda(x-a)}{(b-a)}$ α=$\frac{\lambda(c-x)}{(c-b)}$

$x = \frac{\alpha}{\lambda}(b-a) + a$ $x = c - \frac{\alpha}{\lambda}((c-b)$

$\tilde{A}^L(\alpha) = a + \frac{\alpha}{\lambda}(b-a)$ $\tilde{A}^u(\alpha) = c - \frac{\alpha}{\lambda}(c-b)$

$$R(\tilde{A}) = \frac{\int_w^\lambda [\tilde{A}^L(\alpha) + \tilde{A}^u(\alpha)]\, d\alpha}{\int_w^\lambda \alpha\, d\alpha} = \frac{\int_w^\lambda \left[a + \frac{\alpha}{\lambda}(b-a) + c - \frac{\alpha}{\lambda}(c-b)\right] d\alpha}{\int_w^\lambda \alpha\, d\alpha}$$

$$= \frac{\int_w^\lambda \left[(a+c) + \frac{\alpha}{\lambda}(2b-a-c)\right] d\alpha}{\int_w^\lambda \alpha\, d\alpha} = \frac{\left[(a+b)\alpha + \frac{\alpha^2}{2\lambda}(2b-a-c)\right]\Big|_w^\lambda}{\frac{\alpha^2}{2}\Big|_w^\lambda}$$

$$=\frac{(a+c)(\lambda-w)+\frac{(\lambda^2-w^2)}{2\lambda}(2b-a-c)}{\frac{\lambda^2-w^2}{2}}=\frac{(a+c)(\lambda-w)+\frac{(\lambda-w)(\lambda+w)}{2\lambda}(2b-a-c)}{\frac{(\lambda-w)(\lambda+w)}{2}}$$

$$=\frac{2\lambda(a+b)+(\lambda+w)(2b-a-c)}{2\lambda}*\frac{2}{(\lambda+w)}=\frac{2\lambda(a+b)+(\lambda+w)(2b-a-c)}{\lambda(\lambda+w)}$$

$$=\frac{\lambda(2b+a+c)+w(2b-a-c)}{\lambda(\lambda+w)} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

5- Description of the Company

Baghdad Soft Drinks Company is one of the most important companies operating in the province of Baghdad /Zaafaraniya. The establishment of this company dates back to the 1960s, as one of the establishments of the Ministry of Industry and Minerals which issued the founding license no. 474 in 1961/12/21 from the Directorate General of Industrial Development. However, it became a mixed company in 1989, in accordance with the Companies Law no. 36 in 1983. The procedures of establishing the company were completed by issuing the certificate of incorporation under the decision of the Registrar of Companies in the Ministry of Commerce no. m.s/3315 in 23/3/1989.

The company has 5 factories, each of which contains production lines, which are:

a) Degla factory: consists of four production lines.
b) Euphrates factory: consists of four production lines.
c) Shatt al-Arab factory: consists of two production lines.
d) Rafidain factory: consists of one production line.
e) Aquafina factory: consists of two production lines.

In addition, there are three factories that manufacture carbon dioxide gas.

The company is licensed to produce soft drinks from Pepsi Co. International, and the latter takes samples from the market and examine them to assess the quality of production. The company is about to distribute its products in central and southern Iraq.

6- Numerical Results

After our visiting to Baghdad Soft Drinks Company, we are able to get many data:

The rate CO_2 and Brix located in Miranda Apple for a full day as the concentration are check every half hour.

CO_2: Carbon dioxide gas levels, with a range of 2.6-3.0 unit.

Brix: The amount of solids dissolved in the sample, with a range of 12.2-12.6 unit.

Table 1-Data of CO_2 and Brix levels obtained from Baghdad Soft Drinks Company.

number of data	1	2	3	4	5	6	7	8	9	10	11	12	13
CO_2	2.75	2.74	2.74	2.76	2.74	2.78	2.73	2.74	2.74	2.76	2.74	2.78	2.74
Brix	12.39	12.4	12.36	12.41	12.4	12.4	12.37	12.4	12.36	12.41	12.4	12.4	12.4
number of data	14	15	16	17	18	19	20	21	22	23	24	25	
CO_2	2.77	2.74	2.8	2.79	2.76	2.75	2.77	2.74	2.8	2.79	2.76	2.75	
Brix	12.37	12.4	12.36	12.37	12.35	12.38	12.37	12.4	12.36	12.37	12.35	12.38	

Then, we fuzzified the data to convert them to vague numbers by using the membership function in equation (6):

$$\left.\begin{array}{l}(a,b,c)=(2.6, all\ samples\ ,3.0)\\(a,b,c)=(12.2, allsamples, 12.6)\end{array}\right\} \quad \ldots\ldots\ldots\ldots (8)$$

The first fuzzy numbers of CO_2 levels are 2.6, 2.75, 3, and so on until reaching to sample (25).

The first fuzzy numbers of Brix levels are 12.2, 12.39, 12.6, and so on until reaching to sample (25).

Therefore, the new ranking function in equation (7) were applied to transform the fuzzy numbers to crisp numbers, but the new ranking function depends upon w, λ∈[0,1].

Now, we compute a new ranking function by utilizing the values of w and λ.
The values of wand λ are w=0.1and λ=0.9, respectively.
Then, we use equation (7) to find the ranking function.

Table 2-Fuzzy ranking functions of CO_2 and Brix values samples, when w=0.1and λ=0.9

number of data	1	2	3	4	5	6	7	8	9
CO_2	11.089	11.067	11.067	11.111	11.067	11.156	11.044	11.067	11.067
Brix	49.578	49.6	49.51	49.62	49.6	49.6	49.53	49.6	49.51
number of data	10	11	12	13	14	15	16	17	18
CO_2	11.111	11.067	11.156	11.067	11.133	11.067	11.2	11.178	11.111
Brix	49.62	49.6	49.6	49.6	49.53	49.6	49.51	49.53	49.489
number of data	19	20	21	22	23	24	25		
CO_2	11.089	11.133	11.067	11.2	11.178	11.111	11.089		
Brix	49.56	49.53	49.6	49.51	49.53	49.489	49.56		

Now, the Hotelling T^2 test in equation (4) is applied to all the samples presented in Table-2.
First, we find S_{12} by equation (2) and ${S_1}^2$, ${S_2}^2$ by equation (3). The obtained values are S_{12}=-0.000679835, ${S_1}^2$=0.002209053, ${S_2}^2$=0.001904527.
Second, we compute $\sum^{-1}$ by equation (1):

$$\Sigma^{-1} = \begin{pmatrix} 0.002209053 & 0.000679835 \\ 0.000679835 & 0.001904527 \end{pmatrix}^{-1} = \begin{pmatrix} 508.5484309 & 181.5302523 \\ 181.5302523 & 589.8634349 \end{pmatrix}$$

After that, we compute the first Hotelling ${T_1}^2$ value by equation (4), and so on for the other samples.

$${T_1}^2 = \begin{pmatrix} -0.0187 \\ 0.0169 \end{pmatrix}^T \begin{pmatrix} 508.5484309 & 181.5302523 \\ 181.5302523 & 589.8634349 \end{pmatrix} \begin{pmatrix} -0.0187 \\ 0.0169 \end{pmatrix}$$

Table 3-Hotelling T^2 test when w=0.1, λ=0.9

number of data	1	2	3	4	5	6	7	8	9	10	11
T^2	0.231	1.172	3.051	2.305	1.172	2.756	3.105	1.172	3.051	2.305	1.172
number of data	12	13	14	15	16	17	18	19	20	21	22
T^2	2.756	1.707	1.606	3.051	3.869	3.729	0.016	0.812	1.606	3.051	3.869
number of data	23	24	25								
T^2	3.729	0.016	0.23								

Now, we compute the value of the control term T^2 test shown in equation (5), as follows:

$$T^2 = \frac{L(n-1)}{n-L} \text{F(L,n-L,1-α)} = \frac{2*24}{23} * F(2,23,0.95) = \frac{48}{23} * 3.42 = 7.137391$$

Then, we compare the calculated and tabulated values of all T^2 test results.
We found that all values of T^2 indicate that the production process is under control, which implies that this process is in accordance with the specifications of quality control.
Finally, we draw the charts of quality control for samples taking from another source, namely Miranda green apple factory, depending on the results of Hotelling T^2 test.

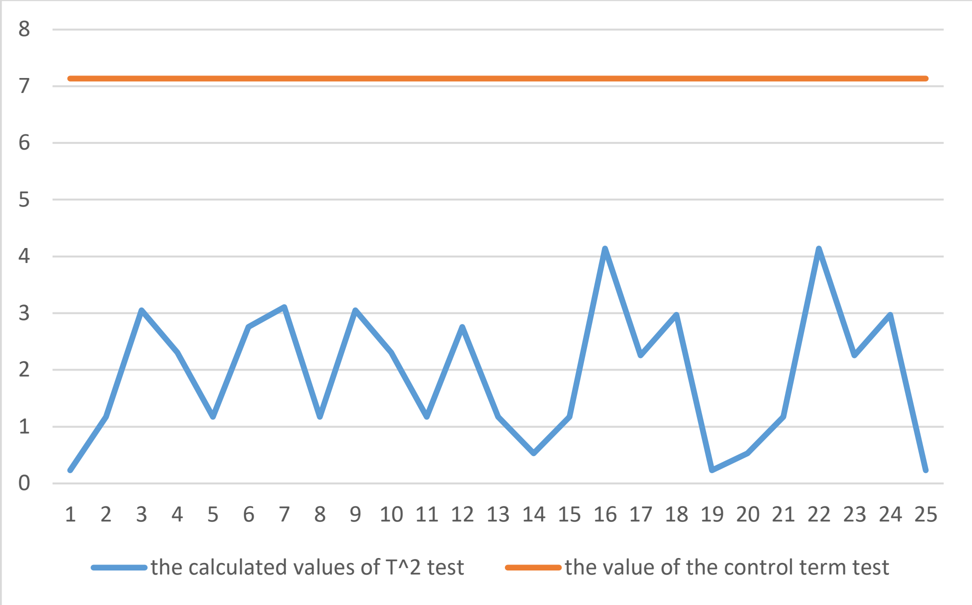

Figure 1-The chart of ranking function when w=0.1, λ=0.9

The results showed 69that the fuzzy control chart is more accurate and economically faster in controlling the quality of production, leading to the detection of defective units during the production process, which helps to detect errors quickly.

7- Conclusions

The triangular membership function was applied to find the fuzzy numbers for CO_2 and Brix to real data. Then, the proposed ranking function was used to transform the fuzzy numbers to crisp numbers. Hence, the Hotelling T^2 test was employed to find whether the production is under control or not. Finally, represents chart was drawn that includes all samples of production to illustrate the level of compliance of samples to control limits.

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