Azeez and Abd Aubad . *Iraqi Journal of Science, 2021, Vol. 62, No. 8, pp: 2640-2652*
DOI: 10.24996/ijs.2021.62.8.16

ISSN: 0067-2904

Investigation of Commuting Graphs for Elements of Order 3 in Certain Leech Lattice Groups

Duha Abbas Azeez*, Ali Abd Aubad
Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq

Received: 15/2/2020 Accepted: 15/3/2020

Abstract
Assume that G is a finite group and X is a subset of G. The commuting graph is denoted by C(G,X) and has a set of vertices X with two distinct vertices x, y ∈ X, being connected together on the condition of xy = yx. In this paper, we investigate the structure of C(G,X) when G is a particular type of Leech lattice groups, namely Higman–Sims group HS and Janko group J_2, along with X as a G-conjugacy class of elements of order 3. We will pay particular attention to analyze the discs' structure and determinate the diameters, girths, and clique number for these graphs.

Keywords: Sporadic groups; commuting graph; diameter, cliques.
Mathematics Subject Classification 2010: 20D08,05C25 ,05C69.

التحقق من البيانات التبادلية لعناصر من الرتبة الثالثة في زمرLeech Lattice معينه

ضحى عباس عزيز *علي عبد عبيد
قسم الرياضيات , كلية العلوم , جامعة بغداد, بغداد ,العراق

الخلاصة
افترض ان G هو زمره منتهيه وX مجموعة جزئية من G. البيان التبادلي ويرمز له(C(G,X لدية المجموعة X تمثل الرؤوس وكل رأسين مختلفين في x,y∈X تكون مرتبطة بضلع تحت شرط xy=yx .في هذا البحث نجري تحقيق حول بناء البيان التبادلي C(G,X) عندما يكون G نوع خاص من زمر Leech lattice مثلا Higman–Sims group HS و Janko J2 group ويكون X صف تكافؤ من الرتبة الثالثة .سوف نبذل جهد خاص لتحليل البناء تحديد الاقطار , الخصور وعدد العصب لهذه البيانات.

1. Introduction and Preliminaries

It is believed that studying the action of a group on a graph is one of the best comprehensible ways of analyzing the structure of the group. Suppose that G is a group and X is a subset of G; the commuting graph denoted by C(G,X) has the set of vertices X with two vertices x, y ∈ X, which are connected if x ≠y, where xy = yx. The commuting graphs were first illustrated by Fowler and Brauer in a seminal paper [1]. They were eminent for giving evidence of a prescribed isomorphism of an involution centralizer, where there is a limited number of non-abelian groups capable of containing it. These graphs were extremely vital for the works of the Margulis-Platanov conjecture [2], as the graphs mentioned in [1] have X = G\{1} where 1 is the identity element of G). When X is a conjugacy class of involution (conjugacy class X of G means that for any two elements x, y ∈ X there are an element

*Email: it2017it@yahoo.com

$g \in G$ such that $x^g=y$, whereas involution means that all elements of X have order 2), then the commuting graph is known as the commuting involution graph. Rowley, Hart (nèe Perkins), Bates, and Bundy put their efforts into investigating the commuting involution graphs and supplying the diameters and disc sizes [3,4,5 6]. Suppose that X is a conjugacy class of elements of order 3. Nawawi and Rowley [7] analyzed the C(G,X) when G is either a symmetric group S_n or a sporadic group McL. The aim of this paper is to investigate the commuting graphs when G is a particular type of Leech lattice groups, such as the Higman–Sims group HS and Janko group J_2, along with X as a G-conjugacy class for elements of order 3. The research involves scrutinizing the discs' structures and calculating the diameters, girths, and clique number for these graphs. From now, we shall assume that G is one of the aforementioned groups. Also, we let t be an element of order 3 in G and $X = t^G$. It is clear that G, acting by conjugation, induces graph automorphisms of C(G,X) and is transitive on its vertices. For $x \in X$ and $i \in \mathbb{Z}^+$, $\Delta_i(x)$ denotes the set of vertices of C(G,X) which has a distance i from x. Using the usual distance function for graphs, this distance function will be denoted by d(;). We use $G_x(= C_G(x))$ to denote the stabilizer in G of x. Obviously, $\Delta_i(x)$ will be a union of certain Gx - orbits. Therefore, we are looking for finding the Gx-orbits of X. In the computational group theory of Magma [8] and GAP [9], packages are considered most commonly utilized. In most steps of the algorithm, Gap will be dominant in the implementation, while the permutation character of the centralizer of t in G ($C_G(t)$) may be verified using Magma and, hence, the number of $C_G(t)$-orbits (Permutation Rank on X) under the action of X on $C_G(t)$ is calculated. Finally, we will use the online Atlas of Group Representations [10] to get a class name of the groups and we refer to it as The Online Atlas.

For the aforesaid groups, the sizes of conjugacy classes for elements of order 3 and the permutation ranks on each class, as well as the structure of the centralizer of t in G, which can be seen in [11], are given in the next table.

Table 1-Class size and Permutation Rank of X

Group	Class	Size of Class	Permutation Rank	$C_G(t)$
HS	3A	12300	399	$C_3 \times S_5$
J_2	3A	560	10	$C_3 . A_6$
	3B	16800	531	$C_3 \times A_4$

2. Computational Method

Let $x \in \Delta_i(t)$ and $z \in C_G(t)$, then one can see immediately that $x^z \in \Delta_i(t)$. Thus, for a finite group G, each disc $\Delta_i(t)$ of the commuting graph C(G,X) is a union of specific $C_G(t)$-orbits.

The size of $C_G(t)$ -orbits under the action of conjugation on t^G can be calculated by using the character table of the group, as we can see in the following result:

Proposition 2.1. [12]. Let G be a group acting transitively on a finite set Ω, with a permutation character χ. Suppose that $\alpha \in \Omega$ and G_α has exactly k orbits on Ω. Then $<\chi,\chi> = k$.

The quantity k in Proposition 2.1 is called the permutation rank of G_α on Ω .Therefore, the permutation rank of $C_G(t)$ on X is the number of $C_G(t)$-orbits under the conjugation action on X.

Now, let C be a G-conjugacy class. It is obvious that the set $X_C = \{x \in X : tx \in C\}$ under the conjugation action of $C_G(t)$ breaks up into sub orbits. Thus, to find all the sub orbits of X, we have to identify the $C_G(t)$-orbits of X_C, for all those C, such that $X_C \neq \emptyset$. The next definition gives us the size of the set X_C and, therefore, the size of sub orbits of X_C:

Definition 2.2. [13]. Let C_i , C_j and C_k be the conjugacy classes of a finite group G. Then, for a fixed element $g \in C_k$, the following set is defined:

$a_{ijk} = |\{(g_i, g_j) \in C_i \times C_j \mid g_i g_j = g\}|$.

Then, for all possible i, j, and k, the value a_{ijk} is called a class structure constant for G.

The next lemma will be used to compute the class structure constants for G.

Lemma 2.3. [13]. Let G be a finite group with n conjugacy classes $C_1, C_2, \ldots, C_n$. Then for all i, j, and k, we have g_i as follows:

$$a_{ijk} = \frac{|G|}{|C_G(gi)||C_G(gj)|} \Sigma_{\chi \in Irr(G)} \frac{\chi(gi)\chi(gj)\overline{\chi(gk)}}{\chi(I)}$$

where g_i , g_j, and g_j are respectively in C_i, C_j and C_k, and $\chi \in Irr(G)$ is the irreducible character table in G.

For the proof of the above lemma, see [13], page 128, Lemma 2.12.. Now, since $|X_C| = |\{(c, x) \in C \times X \mid cx = t\}| = | a_{ijk} |$, then by employing Lemma 2.3, we get

$$|X_C| = \frac{|G|}{|C_G(g)||C_G(t)|} \quad \Sigma_{\chi \varepsilon Irr(G)} \frac{\chi(g)\chi(t)^2}{\chi(I)}$$

Therefore, from the complex character table of G, which is available in the GAP character table library, and using the GAP function of "Class Multiplication Coefficient", we immediately obtain the size of X_C.

3. **Diameters, Girths, and Clique Number**

To determinate the diameters, girths, and clique number for the C(G,X), we will generated the graph by using the gap package YAGS [14] **(**specifically, Graph By Relation) in the next algorithm, which can be realized by using the definition of the commuting graph. The algorithm is given as follows:

Algorithm 1
Input: The group G, $t \in G$ (the elements of order 3);
i: $X = t^G$ the G-conjugacy class of t. **ii**: Relation = {the set of elements satisfies the condition : $x \neq y$ and $x*y = y*x$} **iii**: C (G,X) is the graph generated by X and relation. **v**: Calculate Diameter(C(G,X)), Girth(C (G,X)) and Clique number (C (G,X)) .
Output: The diameter, girth and the clique number of C (G;X).

The results in table 2 are obtained by applying Algorithm 1 on the groups specified in the table.

Table 2-Diameter, Girth, and clique number of commuting graphs

Graph	Diameter	Girth	clique number	group
C(HS,3A)	4	3	8	$C_3 \times C_3$
C(J_2,3A)	disconnected	no girth (forest)	2	C_3
C(J_2,3B)	8	3	6	$C_3 \times C_3$

The last column of Table-2 represents the maximum elementary abelian group generated by the maximum clique.

4. **Analyzing the Discs Structures**

This section is dedicated to analyze the structures of the Δi(t) of the commuting graph C(G, X).

4.1. **C_G(t)-Orbits of Δ_i(t)**

The next algorithm is employed to break $\Delta_i(t)$ into $C_G(t)$-sub orbits of X_C , for all those C (G-Conjugacy class), such that $X_C \neq \varphi$, where their sizes are also provided.

Algorithm 2.
Input: the group G, $t \in G$, (the elements of order 3), C (G-Conjugacy Class)
i: $X \to t^G$ the G-conjugacy class of t **ii**: $C_G(t) \to$ centralizer in G of t. **iii**: $O \to$ the orbits in $C_G(t)$ of X. **iv**: $\|X_C\| =\to$ **"Class Multiplication Coefficient"** of C in X. **v**: **For** $i \to 1$ to size (O) **Do** **vi**: **If** $t * O[i][1]$ Conjugate to $C \to Y_C \cup O[i]$ **vii**: **Repeat** the steps vi, vii until $\|X_C\| = \|Y_C\|$. **viii**: $X_C \to X_C = Y_C$. ix: For $j \to 1$ to size O **Do** **x**: **If** t in $O[j] \to t = O_j$ (there is only one) **xi**: $Y_0 \to Y_0 \cup j$ **xii**: **For** i in O[j] Do **xiii**: **For** $h \to 1$ to size (O) **Do** xiv: **If** $d(O[h][1], i) = 1 \to \{Y_1 \cup \{h\}\}\backslash\{Y_0\}$ **xv**: **For** x in $Y_1 \to \Delta_1(t) \cup O[x]$ **xvi**: **for** j in Y_1 **Do** **xvii**: **Repeat** the steps x1, x2 and **xviii**: **If** $d(O[h][1], i) = 1 \to \{Y_2 \cup \{h\}\} \backslash\{Y_1 \cup Y_0\}$ **xix**: **For** x in $Y_2 \to \Delta_2(t) \cup O[x]$ **xx**: **Repeat** the above steps and replace the Y_{i+1} with Y_i and $\Delta_{i+1}(t)$ with $\Delta_i(t)$
Output: The positions of the sets X_C in the $\Delta_i(t)$ with their sizes.

For each graph in Table-2**,** we provide information about the discs structure. We should note that, in the next tables, the value $n * m$ means the number and the size of $C_G(t)$-orbits in certain $\Delta_i(t)$, respectively. The exceptional case is $C(J_2, 3A)$ as the graph is disconnected. The cases are:

1- $C(J_2, 3A)$: Form Table- 1, the permutation rank of 3A is 10 and $C_G(t) \cong C_3 . A_6$. In Table- 2, one can see that the graph is disconnected and, by Algorithm 1, there are 280 connected 2-components of $C(J_2, 3A)$.

2- $C(J_2, 3B)$: Form Table- 1, the permutation rank of 3B is 531. Form Table- 2, the graph is connected with diameter 8. The centralizer of $t \in 3B$ is isomorphic to $C_3 \times A_4$. The disc structure of the $C(J_2, 3B)$ is described in Table-3.

Table 3-Discs structure of $C(J_2, 3B)$

Class Name	$\Delta_1(t)$	$\Delta_2(t)$	$\Delta_3(t)$	$\Delta_4(t)$	$\Delta_5(t)$	$\Delta_6(t)$	$\Delta_7(t)$	$\Delta_8(t)$
1A	1							
2A				3*2	12			
2B		12*2		36*2				
3A	4*2	12*2						
3B	4*2,1	12*2	36*2 12*2	36*4 12*4,3*2	36*2, 12	36*4	12*6	
4A			36*2	36*3		12*4	36	12*2
5A					12*2	36	12*2,3	12
5B					12*2	36	12*2,3	12
5C				36*2	36	36*4		36
5D				36*2	36	36*4		36

6A			36*4 12*2	12*4		36*12	36*4	
6B		12*2	36*4		36*4	36*20		
7A				36*3	36*36	36*24	36*8	
8A				36*4	36*28	36*32		
10A					36*6	36*12		
10B					36*6	36*12		
10C				36*10	36*12	36*26	36*2	
10D				36*10	36*12	36*26	36*2	
12A				36*6	36*16	36*4	36*2, 12*12	
15A				36*4	36*10, 12*2	36*9, 12*6	36*4, 12*4	
15B				36*4	36*10, 12*2	36*9, 12*6	36*4, 12*4	

3- C(HS, 3A): Form Table-1, the permutation rank of 3A is 399. Form Table-2, the graph is connected with diameter 4. The centralizer of t ∈ 3A is isomorphic to C_3 x S_5. The discs structure of the C(HS, 3B) is described in Table-4.

Table 4-Discs structure of C(HS,3A)

Class Name	$\Delta_1(t)$	$\Delta_2(t)$	$\Delta_3(t)$	$\Delta_4(t)$
1A	1			
2A		120,15	90	
2B				18*2
3A	20*3,1	120*4,15	90	360*4,90*4,18*2
4B			360	360*3,90*5
4C			360*8,180	360*6,90*2
5A			18*2	
5B		60,120	360,180	
5C			360*18	360*27
6B		120*3	360*10	360*4,90*3
7A			360*11	360*70
8A			360*4	360*16,90*8
8B			360*2	360*12
8C			360*2	360*12
10A			360*6	360*18,90*6
10B				360*2
11A			360*3	360*25
11B			360*3	360*25
12A				360*16,90*8
15A		120*3		360*8
20A			360*2,90*2	360*2,90*2
20B			360*2,90*2	360*2,90*2

4.2. The subgroup < t, x >

For any x, y in the arbitrary $C_G(t)$-orbit, the subgroup generated by t and x, i.e. <t,x>, is conjugated to the subgroup generated by t and y. This is obvious because if we conjugate one of them by t, we get the other. For each $C_G(t)$-sub orbit of X_C, the next algorithm provides the algebraic structure of the subgroup generated by t and random elements of $C_G(t)$-sub orbits.

We give the algebraic structure of each subgroup generated by < t, x > for a random element x in a $C_G(t)$-suborbits of X_C, with their numbers and sizes, for each connected graph mentioned in Tables- 5 and 6. For more information about these subgroups, we refer to several previous works [11, 13, 15].

Algorithm 3 .
Input: The Sub orbits O_C in the Set X_C ;
i: **For** i → 1 to Size O_C **Do** **ii: If** Subgroup (t, O_C [i][1]]))) equal Subgroup (t, O_C [1][1]]))) Then **iii**: Subgroup$_{< t, O_C >}$→ (t, O_C [i][1]]))); **iv**: Set_C → Set_C ∪ O_C [i] **v**: **If** Size Set_C equal Size X_C **Stop** **vi**: **Else** X_C → X_C \ Set_C **vii**: repeat the steps i to iv until the step v is true
Output: The group algebraic structure of the subgroup generated by t and the suborbits of the sets X_C in the $\Delta_i(t)$.

Table 5-Subgroup Structure of < t, x > in C(HS, 3A)

$\Delta_i(t)$	Class of t *x	Number of Orbits	Size of Orbits	Subgroup< t, x>
$\Delta_1(t)$	1A	1	1	C_3
	3A	3	60	C_3 x C_3
	3A	1	1	C_3
$\Delta_2(t)$	2A	2	135	A_4
	3A	3	360	C_3 x A_4
	3A	2	135	A_4
	5B	1	60	A_5
	5B	1	120	GL(2,4)
	6B	3	360	C_3 x A_4
	15A	3	360	GL(2,4)
$\Delta_3(t)$	2A	1	90	A_4
	3A	1	90	A_4
	4B	1	360	PSL(3,2)
	4C	5	1620	A_6
	4C	4	1440	PSL(3,2)
	5A	2	36	A_5
	5B	1	360	A_7
	5B	1	180	A_6
	5C	4	1440	(C_2 x C_2 x C_2 x C_2) : A_6
	5C	4	1440	A_7

	5C	2	720	A_5
	5C	4	1440	A_6
	5C	4	1440	$(C_2 \times C_2 \times C_2 \times C_2) : A_5$
	6B	4	1440	$(C_2 \times C_2 \times C_2) : (C_7 : C_3)$
	6B	5	1800	A_7
	6B	1	360	$A_4 \times A_4$
	7A	4	1440	$(C_2 \times C_2 \times C_2) : (C_7 : C_3)$
	7A	2	720	A_7
	7A	4	1440	PSL(3,4)
	7A	1	360	PSL(3,2)
	8B	2	720	$(C_2 \times C_2 \times C_2)$. PSL(3,2)
	8A	4	1440	$(C_2 \times C_2 \times C_2 \times C_2) : A_6$
	8C	2	720	$(C_2 \times C_2 \times C_2)$. PSL(3,2)
	10A	6	2160	HS
	11A	2	720	M_{22}
	11A	1	360	HS
	11B	1	360	HS
	11B	2	720	M_{22}
	20A	2	720	HS
	20A	2	180	$(((C_2 \times C_2 \times C_2) : (C_2 \times C_2)) : C_2) : A_5$
	20B	2	720	HS
	20B	2	180	$(((C_2 \times C_2 \times C_2) : (C_2 \times C_2)) : C_2) : A_5$
$\Delta_4(t)$	2B	2	36	A_4
	3A	6	1080	$(C_4 \times C_4) : C_3$
	3A	2	720	$C_7 : C_3$
	3A	2	36	A_4
	4B	2	720	A_6
	4B	4	360	$(C_4 \times C_4) : C_3$
	4B	1	360	PSL(3,2)
	4B	1	90	SL(2,3)
	4C	4	1440	PSL(3,2)
	4C	2	180	SL(2,3)
	4C	2	720	$(C_4 \times C_4) : C_3$
	5C	8	2880	$(C_2 \times C_2 \times C_2 \times C_2) : A_5$
	5C	2	720	A_6
	5C	12	4320	PSL(3,4)
	5C	4	1440	PSL(2,11)
	5C	1	360	A_5
	6B	4	1440	PSL(2,11)

	6B	3	270	SL(2,3)
	7A	24	8640	M_{22}
	7A	4	1440	$(C_2 \times C_2 \times C_2)$. PSL(3,2)
	7A	20	7200	PSL(3,4)
	7A	8	2880	A_7
	7A	8	2880	$(C_4 \times C_4 \times C_4) : (C_7 : C_3)$
	7A	4	1440	PSL(3,2)
	7A	2	720	$C_7 : C_3$
	8B	6	2160	PSU(3,5)
	8B	4	1440	M_{11}
	8B	2	720	$(C_2 \times C_2 \times C_2)$. PSL(3,2)
	8A	16	5760	M_{22}
	8A	8	720	$(C_4 . (C_4 \times C_4)) : C_3$
	8C	6	2160	PSU(3,5)
	8C	4	1440	M_{11}
	8C	2	720	$(C_2 \times C_2 \times C_2)$. PSL(3,2)
	10A	12	4320	PSU(3,5)
	10A	6	2160	HS
	10A	2	180	$((C_2 \times ((C_4 \times C_2) : C_2)) : C_2) : A_5$
	10A	2	180	$(((C_2 \times C_2 \times C_2) : (C_2 \times C_2)) : C_2) : A_5$
	10A	2	180	SL(2,5)
	10B	2	720	HS
	11A	16	5760	M_{22}
	11A	8	2880	HS
	11A	1	360	PSL(2,11)
	11B	8	2880	HS
	11B	16	5760	M_{22}
	11B	1	360	PSL(2,11)
	12A	8	2880	HS
	12A	8	2880	$(C_4 \times C_4 \times C_4) : (C_7 : C_3)$
	12A	8	720	$(C_4 . (C_4 \times C_4)) : C_3$
	15A	6	2160	HS
	15A	2	720	A_8
	20A	2	720	HS
	20A	2	180	$((C_2 \times ((C_4 \times C_2) : C_2)) : C_2) : A_5$
	20B	2	720	HS
	20B	2	180	$((C_2 \times ((C_4 \times C_2) : C_2)) : C_2) : A_5$

Table-6 illustrates the subgroup structure of < t, x > in the graph $C(J_2, 3B)$.

$\Delta_i(t)$	Class of t *x	Number of Orbits	Size of Orbits	Subgroup < t, x>
$\Delta_1(t)$	1A	1	1	C_3
	3A	2	8	C_3 x C_3
	3B	2	8	C_3 x C_3
	3B	1	1	C_3
$\Delta_2(t)$	2B	2	24	A_4
	3A	2	24	C_3 x A_4
	3B	2	24	A_4
	6B	2	24	C_3 x A_4
$\Delta_3(t)$	3B	2	24	C_3 x A_4
	3B	2	72	(C_4 x C_4) : C_3
	4A	2	72	(C_4 x C_4) : C_3
	6A	4	144	A_4 x A_4
	6A	2	24	C_3 x A_4
	6B	4	144	A_4 x A_4
$\Delta_4(t)$	2A	2	6	A_4
	2B	2	72	A_4
	3B	4	78	A_4
	3B	2	72	(C_4 x C_4) : C_3
	3B	4	48	C_3 x A_4
	4A	2	72	(C_4 x C_4) : C_3
	4A	1	36	PSL(3,2)
	5C	2	72	HJ
	5D	2	72	HJ
	6A	4	48	C_3 x A_4
	7A	1	36	PSL(3,2)
	7A	2	72	HJ
	8A	4	144	PSU(3,3)
	10C	10	360	HJ
	10D	10	360	HJ
	12A	4	144	PSU(3,3)
	12A	2	72	HJ
	15A	4	144	HJ
	15B	4	144	HJ
$\Delta_5(t)$	2A	1	12	A_4
	3B	2	72	(C_5 x C_5) : C_3
	3B	1	12	A_4

	5A	2	24	GL(2,4)
	5B	2	24	GL(2,4)
	5C	1	36	$(C_5 \times C_5) : C_3$
	5D	1	36	$(C_5 \times C_5) : C_3$
	6B	4	144	HJ
	7A	28	1008	HJ
	7A	8	288	PSU(3,3)
	8A	20	720	HJ
	8A	8	288	PSU(3,3)
	10A	2	72	$A_5 \times A_4$
	10A	4	144	HJ
	10B	4	144	HJ
	10B	2	72	$A_5 \times A_4$
	10C	12	432	HJ
	10D	12	432	HJ
	12A	16	576	HJ
	15A	8	288	HJ
	15A	2	72	$A_5 \times A_4$
	15A	2	24	GL(2,4)
	15B	2	72	$A_5 \times A_4$
	15B	8	288	HJ
	15B	2	24	GL(2,4)
$\Delta_6(t)$	3B	2	72	$(C_5 \times C_5) : C_3$
	3B	2	72	$C_7 : C_3$
	4A	4	48	$C_3 . A_6$
	5A	1	36	$(C_5 \times C_5) : C_3$
	5B	1	36	$(C_5 \times C_5) : C_3$
	5C	4	144	HJ
	5D	4	144	HJ
	6A	8	288	PSU(3,3)
	6A	4	144	HJ
	6B	20	720	HJ
	7A	22	792	HJ
	7A	2	72	$C_7 : C_3$
	8A	28	1008	HJ
	8A	4	144	PSU(3,3)
	10A	12	432	HJ
	10B	12	432	HJ
	10C	26	936	HJ
	10D	26	936	HJ
	12A	4	144	PSU(3,3)
	15A	9	324	HJ
	15A	4	48	GL(2,4)

	15A	2	24	$C_3 . A_6$
	15B	9	324	HJ
	15B	4	48	GL(2,4)
	15B	2	24	$C_3 . A_6$
$\Delta_7(t)$	3B	6	72	$(C_3 \times C_3) : C_3$
	4A	1	36	PSL(3,2)
	5A	2	24	$C_3 . A_6$
	5A	1	3	A_5
	5B	2	24	$C_3 . A_6$
	5B	1	3	A_5
	6A	4	144	HJ
	7A	8	288	PSU(3,3)
	10C	2	72	HJ
	10D	2	72	HJ
	12A	12	144	$C_3 . A_6$
	12A	2	72	HJ
	15A	4	48	$C_3 . A_6$
	15A	4	144	HJ
	15B	4	144	HJ
	15B	4	48	$C_3 . A_6$
$\Delta_8(t)$	4A	2	24	$C_3 . A_6$
	5A	1	12	$C_3 . A_6$
	5B	1	12	$C_3 . A_6$
	5C	1	36	A_5
	5D	1	36	A_5

4.3. Collapsed Adjacency Matrix

$\Delta_i(t)$ is a union of certain $C_G(t)$-orbits. For any $\Delta_i(t)$ of C(G, X), let Δ^1_i, Δ^2_i... Δ^r_i be the set of the $C_G(t)$-orbits in $\Delta_i(t)$ of size r. If n is the number of the $C_G(t)$-orbits of C(G, X), then the collapsed adjacency matrix for C(G, X) is n×n with entry (s, h) th. The number of vertices in the row is indexed by Δ^h_i, which is connected to a single vertex in the column, indexed by Δ^s_j. For this purpose, we create the following algorithm**.**

Algorithm 4
Input: The suborbits Δ^h_i and Δ^s_j;
i: **For** x → Random in Δ^h_i Do **ii**: **For** y → in Δ^s_j Do **iii**: **If** x commute with y Then **iv**: entry $entry_{s,h}$ → $entry_{s,h} \cup \{y\}$ **v**: (s, h) th → size of $entry_{s,h}$.
Output: entry (s, h)th the number of vertices in row, indexed by Δ^h_i, is connected to a single vertex in column, indexed by Δ^s_j.

The collapsed adjacency matrix is an n × n matrix, such that n is the permutation rank given in the Table-1. The matrix is too big to fit in the paper. For this reason, we only give an example for applying Algorithm 4 to find the entry of the collapsed adjacency matrix. Now, we calculate some entries of the 399 × 399 collapsed adjacency matrix of C(HS, 3B). We provide the entries for such matrix since the technique we use is similar for any matrix size. Therefore, we just give 10 × 10 parts of 399 × 399 the collapsed adjacency matrix of C(HS, 3B), as showing below:

Table 7-Part of Collapsed Adjacency Matrix C(HS, 3A)

$C_G(t)$-orbits	$t = \Delta_0$	Δ^1_1	Δ^2_1	Δ^1_2	Δ^2_2	Δ^1_3	Δ^2_3	Δ^1_4	Δ^2_4
$t = \Delta_0$	1	1	20	0	0	0	0	0	0
Δ^1_1	1	1	20	0	0	0	0	0	0
Δ^2_1	1	1	2	0	6	0	0	0	0
Δ^1_2	0	0	0	1	0	0	0	0	0
Δ^2_2	0	0	6	0	1	0	0	0	0
Δ^1_3	0	0	0	0	0	1	1	0	0
Δ^2_3	0	0	0	0	0	1	1	0	0
Δ^1_4	0	0	0	0	0	0	0	1	0
Δ^2_4	0	0	0	0	0	0	0	0	1

5. **Main Results**

The graph $C(J_2, 3A)$ is disconnected with 280 connected 2- components, as seen above. For a connected commuting involution graph C(G, X), given in Table-1, the graph structure is described in the following theorem.

Corollary 5.1. For G is one of the groups of Table-2, we have the following results:

• Diam $C(J_2, 3B)$ = 8 and $|\Delta_1|$ =,18 $|\Delta_2|$ =96 , $|\Delta_3|$ = 480 , $|\Delta_4|$ = 2052, $|\Delta_5|$ =5304 , $|\Delta_6|$ =7392 , $|\Delta_7|$ =1338 ,$|\Delta_8|$ =120.

• Diam C(HS, 3A) = 6 and $|\Delta_1|$ =62, $|\Delta_2|$ = 1530, $|\Delta_3|$ =27216, $|\Delta_4|$ =94392.

Proof. Each of $\Delta_i(t)$ of the commuting graph C(G, X) is a union of specific $C_G(t)$-orbits. Thus, using the previous tables, we obtain the proof.

6. References

1. Brauer, R. and Fowler, K. A. **1955**. *On Groups of Even Order*. Ann. of Math., **62**(3):.565-583.

2. Rapinchuk. S, Segev. Y and Seitz. M. **2002**. Finite quotients of the multiplicative group of a finite dimensional division algebra are solvable. *J. Amer. Math. Soc*., **15**(4): 929-978.

3. 3.Bates, C., Bundy, D., Hart, S., and Rowley, P. **2007**. Commuting involution graphs for sporadic simple groups. *J. Algebra*, **316**(2): 849-868.

4. Bates, C., Bundy, D., Hart, S., and Rowley, P. **2003.**Commuting involution graphs for finite Coxeter groups. *J. Group Theory*, **6**(4):461-476.

5. Bates, C., Bundy, D., Hart, S., and Rowley, P. **2003.**Commuting involution graphs for symmetric groups. *J. Algebra*, **266**(1): 133-153.

6. Bates, C., Bundy, D., Hart, S., and Rowley, P. **2004.** Commuting involution graphs in special linear groups. *Comm. Algebra*, **32**(11): 4179-4196.

7. 7.Nawawi, A. and Rowley, P. **2012**. *On commuting graphs for elements of order 3 in symmetric groups*. PhD thesis, University of Manchester.

8. Brinkman, J., Bosma, W., Cannon, J., and Playoust, C. **1997**.The Magma algebra system. I. The user language. J. *Symbolic Comput*. **24**(3-4): 235-265.

9. The GAP Group. **2008**. GAP Groups, Algorithms, and Programming,. Version 4.4.12, http://www.gap-system.org.

10. Tripp, J., Suleiman, I., Rogers, S., Parker, R., Norton, S., Nickerson, S., Linton, S., Bray, J., Wilson, A., and Walsh, P. A world wide web atlas of group representations., http://brauer. maths. qmul. ac.uk/Atlas/v3/.

11. Conway, H., Curtis, R.T., Norton, S. P., and Parker, R. A. **1985**. *ATLAS of Finite Groups: Maximal Subgroups and Ordinary Characters for Simple Groups*. Oxford. Clarendon press.

12. Isaacs, I. M. **1976**. *Character theory of finite Simple group*. New York. Academic Press.

13. Gorenstein, D. **1980**.*Finite Groups*. 2nd ed, Chelsea, Chelsea Publishing Company.

14. Cedillo, C., MacKinney-Romero, R., Pizaa, M.A., Robles, I.A. and Villarroel-Flores.R.. **2019.**Yet Another Graph System,YAGS, Version 0.0.5. http://xamanek.izt.uam.mx/yags/.

15. Wilson, R .**2009**.*The Finite Simple Groups*. 1nd ed. London. Graduate Texts in Mathematics. Springer.