Iraqi Journal of Science, 2021, Vol. 62, No. 9, pp: 3009-3019
DOI: 10.24996/ijs.2021.62.9.16

ISSN: 0067-2904

A Study of a Certain Family of Multivalent Functions Associated with Subordination

Reem O. Rasheed* and Kassim A. Jassim
Department of Mathematics, College of Science, University of Baghdad, Iraq

Received: 6/2/2020 Accepted: 12/5/2020

Abstract

The aim of this paper is to introduce a certain family of new classes of multivalent functions associated with subordination. The various results obtained here for each of these classes include coefficient estimates, radius of convexity, distortion and growth theorem.

Keywords: Harmonic functions, Fractional calculus operator, Univalent function.

دراسة لعائلة من الدوال متعددة التكافؤ المرتبطة بالتبعية التفاضلية

ريم عمران رشيد* , قاسم عبدالحميد جاسم
قسم الرياضيات, كلية العلوم, جامعة بغداد, بغداد, العراق

الخلاصة

الغرض الرئيسي من هذا البحث هو تقديم عائلة معينة من فئات جديدة من الدوال متعددة التكافؤ المرتبطة بالتبعية التفاضلية. تتضمن النتائج المختلفة التي تم الحصول عليها هنا لكل فئة من هذه الفئات تقديرات المعامل ، نصف قطر التحدب ، نظرية التشويه والنمو.

1. Introduction

Let A_ρ denotes a class of functions of the form:

$$f(z) = z^{\rho} - \sum_{\eta=\rho+1}^{\infty} a_\eta z^\eta \ , \qquad a_\eta \geq 0 \tag{1.1}$$

where $\rho \in \mathbb{N} = \{1,2,\ldots\}$, which are analytic in ρ -valent in the open unit disk $U=\{z\in \mathbb{C}; |z| < 1\}$.
A function f which belongs to the class A_ρ is said to be ρ -valent starlike of order α in U if and only if

$$\mathscr{R}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha\,, (0 \leq \alpha < \rho\,; z \in U). \tag{1.2}$$

Also a function f belonging to the class A_ρ is said to be ρ -valent convex of order α in U if and only if

$$\mathcal{R}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha\,, (0 \leq \alpha < \rho\,; z \in U). \tag{1.3}$$

For f and g are analytic in , we say that is subordinate to , written as
$f \prec g$ *in* U *or* $f(z) \prec g(z) \quad (z \in U)$,

*Email: al_reem_ms@yahoo.com

If there exist a Schwarz function w, that is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ in U such that $f(z) = g\big(w(z)\big), z \in U$. If g is univalent in U, then (see [1,2]):

$$f(z) \prec g(z) \quad (z \in U) \Longrightarrow (f(0) = g(0) \text{ and } f(U) \subset g(U).$$

We denote A_ρ^* by the class of all functions A_ρ in which are p-valent starlike of order α *in* U.
Further, $A_{\rho,\mu}^*$ denotes the subclass of A_ρ^* comprising the functions $f^\mu(z)$ of the form:

$$f^\mu(z) = \frac{\rho!}{(\rho-\mu)!} z^{\rho-\mu} - \sum_{\eta=\rho+1}^{\infty} \frac{\eta!}{(\eta-\mu)!} a_\eta z^{\eta-\mu}, where \;\; \mu = 0,1,2, \dots \dots \qquad (1.4)$$

$$f^{\mu+1}(z) = \frac{\rho!}{(\rho-\mu-1)!} z^{\rho-\mu-1} - \sum_{\eta=\rho+1}^{\infty} \frac{\eta!}{(\eta-\mu-1)!} a_\eta z^{\eta-\mu-1}, where \;\; \mu = 0,1,2, \dots \quad (1.5)$$

For $0 < Re\{\alpha\}$, $-1 \le B < A \le 1$, we say that a function $f(z) \in A_\rho$ is in the subclass $S(A, B, \alpha, \rho)$ in the event that fulfills the accompanying condition:

$$1 + \frac{1}{\alpha}\left\{\frac{zf^{\mu+1}(z)}{f^\mu(z)} - \rho + \mu\right\} \prec \frac{1 + Az}{1 + Bz} \qquad (1.6)$$

Furthermore ,a function $f(z) \in A_\rho$ is in the class $KS(A, B, \alpha, \rho)$ if $zf^{\mu+1}(z) \in S(A, B, \alpha, \rho)$.
By using the principle of subordination , Hanaa , Mohamed and Adela [3] investigated the subordination properties for multivalent functions associated with a generalized fractional differintegral operator . In addition, Wang , Aghalary, and Ibrahim[4] . Other authors introduced some properties of certain multivalent analytic functions involving Cho-Kwon-Srivastava operator [5, 6].
We also note that, as of late, researchers published many intriguing outcomes involving various linear and nonlinear operators associated with subordinations and their dual problrms (for details, see [7, 8]).
The theory of subordination has gained extraordinary consideration, especially in numerous subclasses of univalent and multivalent functions (see, for example, [9- 13]).

2. Coefficient estimates

The following theorem discussed the basic properties for the classes $S(A, B, \alpha, \rho)$ and $KS(A, B, \alpha, \rho)$.

Theorem (2.1): *A function $f(z)$, given by (1.1), is in $S(A, B, \alpha, \rho)$ if and only if*

$$\sum_{\eta=\rho+1}^{\infty} \left(\frac{\eta!}{(\eta-\mu)!} - \frac{\eta!}{(\eta-\mu-1)!}\right) |\alpha(A-B) - (1-B)[\alpha(\rho-\mu) - 1]| a_\eta$$
$$\le |\alpha(A-B) + B(\rho-\mu)(1+\alpha) - \alpha + 1| \qquad (2.1)$$

Proof : *Suppose that $f(z) \in S(A, B, \alpha, \rho)$.*
Therefore from (1.6) we have

$$p(z) = 1 + \frac{1}{\alpha}\left\{\frac{zf^{\mu+1}(z)}{f^\mu(z)} - \rho + \mu\right\} \prec \frac{1 + Az}{1 + Bz} \quad (2.2)$$

$$p(z) = \frac{1 + AK(z)}{1 + BK(z)}$$

where K(z) is Schwarz function

$$p(z) = \big(1 + BK(z)\big) = 1 + AK(z)$$
$$K(z)(Bp(z) - A) = 1 - p(z)$$
$$K(z) = \frac{p(z) - 1}{A - Bp(z)}$$
$$|K(z)| < 1$$
$$\left|\frac{\frac{1}{\alpha}\left\{\frac{zf^{\mu+1}(z)}{f^\mu(z)} - \rho + \mu\right\}}{A - B\left\{1 + \frac{1}{\alpha}\left\{\frac{zf^{\mu+1}(z)}{f^\mu(z)} - \rho + \mu\right\}\right\}}\right| < 1$$

$$\left|\frac{zf^{\mu+1}(z)-\alpha f^{\mu}(z)(\rho-\mu)}{[A-B(1-\rho+\mu)]\alpha f^{\mu}(z)-Bzf^{\mu+1}(z)}\right|<1 \tag{2.3}$$

$$zf^{\mu+1}(z)-\alpha f^{\mu}(z)(\rho-\mu)$$

$$=(1-\alpha)\frac{\rho!}{(\rho-\mu-1)!}z^{\rho-\mu}+\sum_{\eta=\rho+1}^{\infty}\left(\frac{\eta!}{(\eta-\mu)!}-\frac{\eta!}{(\eta-\mu-1)!}\right)(\alpha(\rho-\mu)-1)a_{\eta}z^{\eta-\mu} \tag{2.4}$$

Now consider

$$[A-B(1-\rho+\mu)]\alpha f^{\mu}(z)-Bzf^{\mu+1}(z)$$

$$=\{[A-B(1-\rho+\mu)]\alpha-B(\rho-\mu)\}\frac{\rho!}{(\rho-\mu-1)!}z^{\rho-m}$$

$$+\sum_{\eta=\rho+1}^{\infty}\left(\frac{\eta!}{(\eta-\mu-1)!}-\frac{\eta!}{(\eta-\mu)!}\right)(B[1+B\alpha(1-\rho+\mu)]-A\alpha)a_{\eta}z^{\eta-\mu} \tag{2.5}$$

From (2.3) we have

$$\left|\frac{(1-\alpha)\frac{\rho!}{(\rho-\mu)!}z^{\rho-\mu}+\sum_{\eta=\rho+1}^{\infty}\left(\frac{\eta!}{(\eta-\mu)!}-\frac{\eta!}{(\eta-\mu-1)!}\right)(\alpha(\rho-\mu)-1)a_{\eta}z^{\eta-\mu}}{\begin{array}{c}\{[A-B(1-\rho+\mu)]\alpha-B(\rho-\mu)\}\frac{\rho!}{(\rho-\mu-1)!}z^{\rho-m}\\ +\sum_{\eta=\rho+1}^{\infty}\left(\frac{\eta!}{(\eta-\mu-1)!}-\frac{\eta!}{(\eta-\mu)!}\right)(B[1+\alpha(1-\rho+\mu)]-A\alpha)a_{\eta}z^{\eta-\mu}\end{array}}\right|<1$$

Since $\{z\}<|z|$ *, and after considering the values of z on real axis and letting* $z\longrightarrow 1$, *we get*

$$(1-\alpha)\frac{\rho!}{(\rho-\mu)!}z^{\rho-\mu}+\sum_{\eta=\rho+1}^{\infty}\left(\frac{\eta!}{(\eta-\mu)!}-\frac{\eta!}{(\eta-\mu-1)!}\right)|\alpha(\rho-\mu)-1|a_{\eta}$$

$$\leq\{[A-B(1-\rho+\mu)]\alpha-B(\rho-\mu)\}\frac{\rho!}{(\rho-\mu-1)!}$$

$$+\sum_{\eta=\rho+1}^{\infty}\left(\frac{\eta!}{(\eta-\mu-1)!}-\frac{\eta!}{(\eta-\mu)!}\right)|B[1+\alpha(1-\rho+\mu)]-A\alpha|a_{\eta}$$

That is

$$\sum_{\eta=\rho+1}^{\infty}\left(\frac{\eta!}{(\eta-\mu)!}-\frac{\eta!}{(\eta-\mu-1)!}\right)|\alpha(A-B)-(1-B)[\alpha(\rho-\mu)-1]|a_{\eta}$$

$$\leq|\alpha(A-B)+B(\rho-\mu))(1+\alpha)-\alpha+1|$$

Corollary (2.2): *Let* $f(z)\in S(A,B,\alpha,\rho)$, *then*

$$a_{\eta}\leq\frac{|\alpha(A-B)+B(\rho-\mu)(1+\alpha)-\alpha+1|}{\left(\frac{\eta!}{(\eta-\mu)!}-\frac{\eta!}{(\eta-\mu-1)!}\right)|\alpha(A-B)-(1-B)[\alpha(\rho-\mu)-1]|},\eta\geq\rho+1$$

and the same holds for

$$f(z)=z^{\rho}-\frac{|\alpha(A-B)+B(\rho-\mu)(1+\alpha)-\alpha+1|}{\left(\frac{\eta!}{(\eta-\mu)!}-\frac{\eta!}{(\eta-\mu-1)!}\right)|\alpha(A-B)-(1-B)[\alpha(\rho-\mu)-1]|}z^{\eta} \tag{2.6}$$

Theorem (2.3): $f(z)=z^{\rho}-\sum_{\eta=\rho+1}^{\infty}a_{\eta}z^{\eta}$, $a_{\eta}\geq 0$ *is in* $K\,S(A,B,\alpha,\rho)$ *if and only if*

$$\sum_{\eta=\rho+1}^{\infty}\left(\frac{\eta!\,(\eta-\mu)}{(\eta-\mu-q)}-\frac{\eta!\,(\eta-\mu)}{(\eta-\mu-q-1)!}\right)|\alpha(A-B)+(\rho-\mu)-B(\rho+\mu)|a_{\eta}$$

$$\leq|\alpha||(A-B)-B(\rho+\mu)+\rho-\mu|\frac{\rho!\,(\rho-\mu)}{(\rho-\mu-q)(\rho-\mu-q-1)!}a_{\eta} \tag{2.7}$$

proof: *Suppose that* $f(z)\in K\,S(A,B,\alpha,\rho)$ *if* $zf^{\mu+1}(z)\in KS(A,B,\alpha,\rho)$
Let $g(z)=zf^{\mu+1}(z)$
Therefore from (2.2)

$$p(z)=1+\frac{1}{\alpha}\left\{\frac{zg^{q+1}(z)}{g^{q}(z)}-\rho+\mu\right\}\prec\frac{1+Az}{1+Bz} \tag{2.8}$$

where

$$g^q(z) = \frac{\rho!\,(\rho-\mu)}{(\rho-\mu-q)!} z^{\rho-\mu-q} - \sum_{\eta=\rho+1}^{\infty} \frac{\eta!\,(\eta-\mu)}{(\eta-\mu-q)!} a_\eta z^{\eta-\mu-q} \tag{2.9}$$

This is equivalent to (since $|K(z)| < 1$ *)*

$$\left| \frac{\frac{1}{\alpha}\left\{\frac{zg^{q+1}(z)}{g^q(z)} - \rho + \mu\right\}}{A - B\left\{1 + \frac{1}{\alpha}\left\{\frac{zg^{q+1}(z)}{g^q(z)} - \rho + \mu\right\}\right\}} \right| < 1$$

$$\left| \frac{zg^{q+1}(z) - \alpha g^q(z)(\rho-\mu)}{[\alpha(A-B) - B\alpha(\rho+\mu)]g^q(z) + zg^{q+1}(z)} \right| < 1 \tag{2.10}$$

$$\begin{aligned} &zg^{q+1}(z) - \alpha g^q(z)(\rho-\mu) \\ &= (\rho-\mu-q-\alpha(\rho-\mu)) \frac{\rho!\,(\rho-\mu)}{(\rho-\mu-q)(\rho-\mu-q-1)!} z^{\rho-\mu-q} \\ &+ \sum_{\eta=\rho+1}^{\infty} \left(\frac{\eta!\,(\eta-\mu)}{(\eta-\mu-q)!} - \frac{\eta!\,(\eta-\mu)}{(\eta-\mu-q-1)!}\right)(\alpha(\rho-\mu)-1) a_\eta z^{\eta-\mu-q} \end{aligned} \tag{2.11}$$

Now consider

$$\begin{aligned} &[\alpha(A-B) - B\alpha(\rho+\mu)]g^q(z) + zg^{q+1}(z) \\ &= [\alpha(A-B) - B\alpha(\rho-\mu) + \rho-\mu-q] \frac{\rho!\,(\rho-\mu)}{(\rho-\mu-q-1)!} a_\eta z^{\eta-\mu-q} \\ &+ \textstyle\sum_{\eta=\rho+1}^{\infty} \left(\frac{\eta!(\eta-\mu)}{(\eta-\mu-q)!} - \frac{\eta!(\eta-\mu)}{(\eta-\mu-q-1)!}\right)[\alpha(A-B) - B\alpha(\rho+\mu)+1] a_\eta z^{\eta-\mu-q} \end{aligned} \tag{2.12}$$

From (2.10), we obtain

$$\left| \frac{\begin{array}{c}(\rho-\mu-q-\alpha(\rho-\mu)) \frac{\rho!\,(\rho-\mu)}{(\rho-\mu-q)(\rho-\mu-q-1)!} z^{\rho-m-q} \\ + \sum_{\eta=\rho+1}^{\infty} \left(\frac{\eta!\,(\eta-\mu)}{(\eta-\mu-q)!} - \frac{\eta!\,(\eta-\mu)}{(\eta-\mu-q-1)!}\right)(\alpha(\rho-\mu)-1) a_\eta z^{\eta-\mu-q}\end{array}}{\begin{array}{c}[\alpha(A-B) - B\alpha(\rho+\mu) + \rho-\mu-q] \frac{\rho!\,(\rho-\mu)}{(\rho-\mu-q-1)!} a_\eta z^{\eta-\mu-q} \\ + \sum_{\eta=\rho+1}^{\infty} \left(\frac{\eta!\,(\eta-\mu)}{(\eta-\mu-q)!} - \frac{\eta!\,(\eta-\mu)}{(\eta-\mu-q-1)!}\right)[\alpha(A-B) - B\alpha(\rho+\mu)+1] a_\eta z^{\eta-\mu-q}\end{array}} \right| < 1$$

Since $Re(z) < |z|$ *and by letting* $z \longrightarrow 1$ *on real axis , we get*

$$\begin{aligned} &(\rho-\mu-q-\alpha(\rho-\mu)) \frac{\rho!\,(\rho-\mu)}{(\rho-\mu-q)(\rho-\mu-q-1)!} \\ &+ \sum_{\eta=\rho+1}^{\infty} \left(\frac{\eta!\,(\eta-\mu)}{(\eta-\mu-q)!} - \frac{\eta!\,(\eta-\mu)}{(\eta-\mu-q-1)!}\right)(\alpha(\rho-\mu)-1) a_\eta \\ \leq\ &[\alpha(A-B) - B\alpha(\rho+\mu) + \rho-\mu-q] \frac{\rho!\,(\rho-\mu)}{(\rho-\mu-q-1)!} a_\eta \\ &+ \sum_{\eta=\rho+1}^{\infty} \left(\frac{\eta!\,(\eta-\mu)}{(\eta-\mu-q)!} - \frac{\eta!\,(\eta-\mu)}{(\eta-\mu-q-1)!}\right)[\alpha(A-B) - B\alpha(\rho+\mu)+1] a_\eta \end{aligned}$$

That is,

$$\begin{aligned} &\sum_{\eta=\rho+1}^{\infty} \left(\frac{\eta!\,(\eta-\mu)}{(\eta-\mu-q)!} - \frac{\eta!\,(\eta-\mu)}{(\eta-\mu-q-1)!}\right)|\alpha(A-B) + (\rho-\mu) - B(\rho+\mu)| a_\eta \\ &\leq |\alpha||(A-B) - B(\rho+\mu) + \rho-\mu| \frac{\rho!\,(\rho-\mu)}{(\rho-\mu-q)(\rho-\mu-q-1)!} a_\eta \end{aligned}$$

Corollary (2.4): *Let* $f(z) \in KS(A,B,\alpha,\rho)$ *then*

$$a_\eta \le \frac{|\alpha(A-B)+B(\rho+\mu)+\rho-\mu| \frac{\rho!\,(\rho-\mu)}{(\rho-\mu-q)(\rho-\mu-q-1)!}}{\left(\frac{\eta!}{(\eta-\mu)!}-\frac{\eta!}{(\eta-\mu-1)!}\right)|\alpha(A-B)+(\rho-\mu)-B(\rho+\mu)|}, \eta \ge \rho+1$$

and the same holds for

$$f(z) = z^\rho - \frac{|\alpha(A-B)+B(\rho+\mu)+\rho-\mu| \frac{\rho!(\rho-\mu)}{(\rho-\mu-q)(\rho-\mu-q-1)!}}{\left(\frac{\eta!}{(\eta-\mu)!}-\frac{\eta!}{(\eta-\mu-1)!}\right)|\alpha(A-B)+(\rho-\mu)-B(\rho+\mu)|} z^\eta \qquad (2.13)$$

3. Growth and Distortion theorem

The following results are giving the distortion and growth properties for our class.

Theorem (3.1): *If $f(z) \in S(A,B,\alpha,\rho)$ then*

$$|z|^\rho - |z|^{\rho+1} \frac{|\alpha(A-B)+B(\rho-\mu)(1+\alpha)-\alpha+1|}{\left(\frac{(\rho+1)!}{(\rho-\mu+1)!}-\frac{(\rho+1)!}{(\rho-\mu)!}\right)|\alpha(A-B)-(1-B)[\alpha(\rho-\mu)-1]|} \le |f(z)|$$

$$\le |z|^\rho + |z|^{\rho+1} \frac{|\alpha(A-B)+B(\rho-\mu)(1+\alpha)-\alpha+1|}{\left(\frac{(\rho+1)!}{(\rho-\mu+1)!}-\frac{(\rho+1)!}{(\rho-\mu)!}\right)|\alpha(A-B)-(1-B)[\alpha(\rho-\mu)-1]|} \qquad (3.1)$$

The same holds for

$$f(z) = z^\rho - z^{\rho+1} \frac{|\alpha(A-B)+B(\rho-\mu)(1+\alpha)-\alpha+1|}{\left(\frac{(\rho+1)!}{(\rho-\mu+1)!}-\frac{(\rho+1)!}{(\rho-\mu)!}\right)|\alpha(A-B)-(1-B)[\alpha(\rho-\mu)-1]|} \qquad (3.2)$$

Proof : $f(z) \in S(A,B,\alpha,\rho)$

Therefore, from theorem (2.1) we have

$$\sum_{\eta=\rho+1}^{\infty} a_\eta \le \frac{|\alpha(A-B)+B(\rho-\mu)(1+\alpha)-\alpha+1|}{\left(\frac{\eta!}{(\eta-\mu)!}-\frac{\eta!}{(\eta-\mu-1)!}\right)|\alpha(A-B)-(1-B)[\alpha(\rho-\mu)-1]|}$$

$$f(z) = z^\rho - \sum_{\eta=\rho+1}^{\infty} a_\eta z^\eta$$

$$|f(z)| \ge |z|^\rho - \sum_{\eta=\rho+1}^{\infty} |a_\eta||z|^\eta \ge |z|^\rho - |z|^{\rho+1} \sum_{\eta=\rho+1}^{\infty} |a_\eta|$$

$$\ge |z|^\rho - |z|^{\rho+1} \frac{|\alpha(A-B)+B(\rho-\mu)(1+\alpha)-\alpha+1|}{\left(\frac{(\rho+1)!}{(\rho-\mu+1)!}-\frac{(\rho+1)!}{(\rho-\mu)!}\right)|\alpha(A-B)-(1-B)[\alpha(\rho-\mu)-1]|}$$

Similarly ,

$$|f(z)| \le |z|^\rho + \sum_{\eta=\rho+1}^{\infty} |a_\eta||z|^\eta \le |z|^\rho + |z|^{\rho+1} \sum_{\eta=\rho+1}^{\infty} |a_\eta|$$

$$\le |z|^\rho + |z|^{\rho+1} \frac{|\alpha(A-B)+B(p-m)(1+\alpha)-\alpha+1|}{\left(\frac{(\rho+1)!}{(\rho-\mu+1)!}-\frac{(\rho+1)!}{(\rho-\mu)!}\right)|\alpha(A-B)-(1-B)[\alpha(\rho-\mu)-1]|}$$

Therefore ,

$$|z|^\rho - |z|^{\rho+1} \frac{|\alpha(A-B)+B(\rho-\mu)(1+\alpha)-\alpha+1|}{\left(\frac{(\rho+1)!}{(\rho-\mu+1)!}-\frac{(\rho+1)!}{(\rho-\mu)!}\right)|\alpha(A-B)-(1-B)[\alpha(\rho-\mu)-1]|} \le |f(z)|$$

$$\le |z|^\rho + |z|^{\rho+1} \frac{|\alpha(A-B)+B(\rho-\mu)(1+\alpha)-\alpha+1|}{\left(\frac{(\rho+1)!}{(\rho-\mu+1)!}-\frac{(\rho+1)!}{(\rho-\mu)!}\right)|\alpha(A-B)-(1-B)[\alpha(\rho-\mu)-1]|}$$

Thus the verification is finished.

Theorem (3.2): *If $f(z) \in KS(A,B,\alpha,\rho)$ then*

$$|z|^\rho - |z|^{\rho+1} \frac{|\alpha(A-B)+B(\rho+\mu)+(\rho-\mu)| \frac{\rho!\,(\rho-\mu)}{(\rho-\mu-q)(\rho-\mu-q-1)!}}{\left(\frac{(\rho+1)!\,(\rho-\mu+1)}{(\rho-\mu-q+1)!}-\frac{(\rho+1)!\,(\rho-\mu+1)}{(\rho-\mu-q)!}\right)|\alpha(A-B)-(\rho-\mu)-B(\rho+\mu)|}$$

$$\le |f(z)| \le$$

$$|z|^{\rho} + |z|^{\rho+1}\frac{|\alpha(A-B)+B(\rho+\mu)+(\rho-\mu)\,|\dfrac{\rho!\,(\rho-\mu)}{(\rho-\mu-q)(\rho-\mu-q-1)!}}{\left(\dfrac{(\rho+1)!\,(\rho-\mu+1)}{(\rho-\mu-q+1)!}-\dfrac{(\rho+1)!\,(\rho-\mu+1)}{(\rho-\mu-q)!}\right)|\alpha(A-B)-(\rho-\mu)-B(\rho+\mu)|} \quad (3.3)$$

With the same holding for

$$f(z) = z^{\rho} - z^{\rho+1}\frac{|\alpha(A-B)+B(\rho+\mu)+(\rho-\mu)\,|\frac{\rho!(\rho-\mu)}{(\rho-\mu-q)(\rho-\mu-q-1)!}}{\left(\frac{(\rho+1)!(\rho-\mu+1)}{(\rho-\mu-q+1)!}-\frac{(\rho+1)!(\rho-\mu+1)}{(\rho-\mu-q)!}\right)|\alpha(A-B)-(\rho-\mu)-B(\rho+\mu)|} \quad (3.4)$$

Proof : $f(z) \in KS(A,B,\alpha,\rho)$

Therefore, from theorem (2.3) we have

$$\sum_{\eta=\rho+1}^{\infty} a_{\eta} \le \frac{|\alpha(A-B)+B(\rho+\mu)+(\rho-\mu)|\dfrac{\rho!\,(\rho-\mu)}{(\rho-\mu-q)(\rho-\mu-q-1)!}}{\left(\dfrac{\eta!}{(\eta-m)!}-\dfrac{\eta!}{(\eta-m-1)!}\right)|\alpha(A-B)+(\rho-\mu)-B(\rho+\mu)|}$$

$$f(z) = z^{\rho} - \sum_{\eta=\rho+1}^{\infty} a_{\eta} z^{\eta}$$

$$|f(z)| \ge |z|^{\rho} - \sum_{\eta=\rho+1}^{\infty} |a_{\eta}||z|^{\eta} \ge |z|^{\rho} - |z|^{\rho+1}\sum_{\eta=\rho+1}^{\infty} |a_{\eta}|$$

Therefore ,

$$\ge |z|^{\rho} - |z|^{\rho+1}\frac{|\alpha(A-B)+B(\rho+\mu)+(\rho-\mu)\,|\dfrac{\rho!\,(\rho-\mu)}{(\rho-\mu-q)(\rho-\mu-q-1)!}}{\left(\dfrac{(\rho+1)!\,(\rho-\mu+1)}{(\rho-\mu-q+1)!}-\dfrac{(\rho+1)!\,(\rho-\mu+1)}{(\rho-\mu-q)!}\right)|\alpha(A-B)-(\rho-\mu)-B(\rho+\mu)|}$$

Similarly ,

$$|f(z)| \le |z|^{\rho} + \sum_{\eta=\rho+1}^{\infty} |a_{\eta}||z|^{\eta} \le |z|^{\rho} + |z|^{\rho+1}\sum_{\eta=\rho+1}^{\infty} |a_{\eta}|$$

$$\le |z|^{\rho} + |z|^{\rho+1}\frac{|\alpha(A-B)+B(\rho+\mu)+(\rho-\mu)\,|\dfrac{\rho!\,(\rho-\mu)}{(\rho-\mu-q)(\rho-\mu-q-1)!}}{\left(\dfrac{(\rho+1)!\,(\rho-\mu+1)}{(\rho-\mu-q+1)!}-\dfrac{(\rho+1)!\,(\rho-\mu+1)}{(\rho-\mu-q)!}\right)|\alpha(A-B)-(\rho-\mu)-B(\rho+\mu)|}$$

Therefore ,

$$|z|^{\rho} - |z|^{\rho+1}\frac{|\alpha(A-B)+B(\rho+\mu)+(\rho-\mu)\,|\dfrac{\rho!\,(\rho-\mu)}{(\rho-\mu-q)(\rho-\mu-q-1)!}}{\left(\dfrac{(\rho+1)!\,(\rho-\mu+1)}{(\rho-\mu-q+1)!}-\dfrac{(\rho+1)!\,(\rho-\mu+1)}{(\rho-\mu-q)!}\right)|\alpha(A-B)-(\rho-\mu)-B(\rho+\mu)|}$$

$$\le |f(z)| \le$$

$$|z|^{\rho} + |z|^{\rho+1}\frac{|\alpha(A-B)+B(\rho-\mu)(1+\alpha)-\alpha+1|}{\left(\dfrac{(\rho+1)!}{(\rho-\mu+1)!}-\dfrac{(\rho+1)!}{(\rho-\mu)!}\right)|\alpha(A-B)-(1-B)[\alpha(\rho-\mu)-1]|}$$

Thus the verification is finished.

Theorem (3.3): *If* $f(z) \in S(A,B,\alpha,\rho)$ *then*

$$\rho|z|^{\rho-1} - (\rho+1)|z|^{\rho}\varphi(\rho+1) \le |f'(z)| \le \rho|z|^{\rho-1} + (\rho+1)|z|^{\rho}\varphi(\rho+1) \quad (3.5)$$

Proof: $f(z) \in S(A,B,\alpha,\rho)$ *, therefore from theorem (2.1) we have*

$$\sum_{\eta=\rho+1}^{\infty} a_{\eta} \le \frac{|\alpha(A-B)+B(\rho-\mu)(1+\alpha)-\alpha+1|}{\left(\dfrac{\eta!}{(\eta-\mu)!}-\dfrac{\eta!}{(\eta-\mu-1)!}\right)|\alpha(A-B)-(1-B)[\alpha(\rho-\mu)-1]|}$$

$$f'(z) = \rho z^{\rho-1} - \sum_{\eta=\rho+1}^{\infty} \eta a_{\eta} z^{\eta-1}$$

$$|f'(z)| \geq \rho|z|^{\rho-1} - \sum_{\eta=\rho+1}^{\infty} \eta|a_\eta||z|^{\eta-1}$$

$$\geq \rho|z|^{\rho-1} - (\rho+1)|z|^{\rho} \sum_{\eta=\rho+1}^{\infty} |a_\eta|$$

$$\geq |z|^{\rho-1} - (\rho+1)|z|^{\rho} \frac{|\alpha(A-B) + B(\rho-\mu)(1+\alpha) - \alpha + 1|}{\left(\frac{(\rho+1)!}{(\rho-\mu+1)!} - \frac{(\rho+1)!}{(\rho-\mu)!}\right)|\alpha(A-B) - (1-B)[\alpha(\rho-\mu)-1]|}$$

Similarly ,

$$|f'(z)| \leq \rho|z|^{\rho-1} + \sum_{\eta=\rho+1}^{\infty} \eta|a_\eta||z|^{\eta-1}$$

$$\leq \rho|z|^{\rho-1} + (\rho+1)|z|^{\rho} \sum_{\eta=\rho+1}^{\infty} |a_\eta|$$

$$\leq |z|^{\rho-1} + (\rho+1)|z|^{\rho} \frac{|\alpha(A-B) + B(\rho-\mu)(1+\alpha) - \alpha + 1|}{\left(\frac{(\rho+1)!}{(\rho-\mu+1)!} - \frac{(\rho+1)!}{(\rho-\mu)!}\right)|\alpha(A-B) - (1-B)[\alpha(\rho-\mu)-1]|}$$

Therefore ,

$$\rho|z|^{\rho-1} - (\rho+1)|z|^{\rho}\varphi(\rho+1) \leq |f'(z)| \leq \rho|z|^{\rho-1} + (\rho+1)|z|^{\rho}\varphi(\rho+1)$$

where ,

$$\varphi(\rho+1) = \frac{|\alpha(A-B) + B(\rho-\mu)(1+\alpha) - \alpha + 1|}{\left(\frac{(\rho+1)!}{(\rho-\mu+1)!} - \frac{(\rho+1)!}{(\rho-\mu)!}\right)|\alpha(A-B) - (1-B)[\alpha(\rho-\mu)-1]|}$$

Thus the verification is finished.

Theorem (3.4): *If $f(z) \in KS(A,B,\alpha,\rho)$ then*

$$\rho|z|^{\rho-1} - (\rho+1)|z|^{\rho}\omega(\rho+1) \leq |f'(z)| \leq \rho|z|^{\rho-1} + (\rho+1)|z|^{\rho}\omega(\rho+1) \qquad (3.6)$$

Proof: *$f(z) \in KS(A,B,\alpha,\rho)$, therefore from theorem (2.1) we have*

$$\sum_{\eta=\rho+1}^{\infty} a_\eta \leq \frac{|\alpha(A-B) + B(\rho+\mu) + \rho - \mu| \frac{\rho!\,(\rho-\mu)}{(\rho-\mu-q)(\rho-\mu-q-1)!}}{\left(\frac{\eta!}{(\eta-\mu)!} - \frac{\eta!}{(\eta-\mu-1)!}\right)|\alpha(A-B) + \alpha(\rho-\mu) - \alpha B(\rho+\mu)|}$$

$$f'(z) = \rho|z|^{\rho-1} - \sum_{\eta=\rho+1}^{\infty} \eta a_\eta z^{\eta-1}$$

$$\left|f'(z)\right| \geq \rho|z|^{\rho-1} - \sum_{\eta=\rho+1}^{\infty} \eta|a_\eta||z|^{\eta-1}$$

$$\geq \rho|z|^{\rho-1} - (\rho+1)|z|^{\rho} \sum_{\eta=\rho+1}^{\infty} |a_\eta|$$

$$\geq \rho|z|^{\rho-1} - (\rho+1)|z|^{\rho} \frac{|\alpha(A-B) + B(\rho+\mu) + \rho - \mu| \frac{\rho!\,(\rho-\mu)}{(\rho-\mu-q)(\rho-\mu-q-1)!}}{\left(\frac{\eta!}{(\eta-\mu)!} - \frac{\eta!}{(\eta-\mu-1)!}\right)|\alpha(A-B) + \alpha(\rho-\mu) - \alpha B(\rho+\mu)|}$$

Similarly ,

$$\left|f'(z)\right| \leq \rho|z|^{\rho-1} + \sum_{\eta=\rho+1}^{\infty} \eta|a_\eta||z|^{\eta-1}$$

$$\leq \rho|z|^{\rho-1} + (\rho+1)|z|^{\rho} \sum_{\eta=\rho+1}^{\infty} |a_\eta|$$

$$\leq \rho|z|^{\rho-1} + (\rho+1)|z|^{\rho} \frac{|\alpha(A-B) + B(\rho+\mu) + \rho - \mu| \frac{\rho!\,(\rho-\mu)}{(\rho-\mu-q)(\rho-\mu-q-1)!}}{\left(\frac{\eta!}{(\eta-\mu)!} - \frac{\eta!}{(\eta-\mu-1)!}\right)|\alpha(A-B) + \alpha(\rho-\mu) - \alpha B(\rho+\mu)|}$$

Therefore ,

$$\rho|z|^{\rho-1} - (\rho+1)|z|^{\rho}\omega(\rho+1) \leq \left|f'(z)\right| \leq \rho|z|^{\rho-1} + (\rho+1)|z|^{\rho}\omega(\rho+1)$$

where ,

$$\varphi(\rho+1) = \frac{|\alpha(A-B) + B(\rho+\mu) + \rho - \mu| \frac{\rho!\,(\rho-\mu)}{(\rho-\mu-q)(\rho-\mu-q-1)!}}{\left(\frac{\eta!}{(\eta-\mu)!} - \frac{\eta!}{(\eta-\mu-1)!}\right)|\alpha(A-B) + \alpha(\rho-\mu) - \alpha B(\rho+\mu)|}$$

Thus the verification is finished.

4 closure theorems

In the following theorems, we obtain close to convex and the radii of starlikenessand convexity of the functions $f(z) \in S(A,B,\alpha,\rho)$.

Theorem(4.1): *If* $f(z) \in S(A,B,\alpha,\rho)$ *, then* f *is close to convex function of order* δ *in* $|z| < r_1$
r_1

$$= inf_{\eta}\left\{\frac{[\rho|z|^{\rho-1} - (2-\delta)]\left[\left(\frac{\eta!}{(\eta-\mu)!} - \frac{\eta!}{(\eta-\mu-1)!}\right)|\alpha(A-B) - (1-B)[\alpha(\rho-\mu) - 1]|\right]}{\eta|z|^{\rho-1}(|\alpha(A-B) + B(\rho-\mu)(1+\alpha) - \alpha + 1|)}\right\}^{\frac{1}{\eta-\rho}} \quad (4.1)$$

Proof: *It is enough to prove that* $\left|f'(z) - 1\right| < 1 - \delta$

That is

$$\left|f'(z) - 1\right| \leq \rho|z|^{\rho-1} - \sum_{\eta=\rho+1}^{\infty} \eta\left|a_{\eta}\right||z|^{\eta-1} - 1 < 1 - \delta$$

$$\left|f'(z) - 1\right| \leq \rho|z|^{\rho-1} - \sum_{\eta=\rho+1}^{\infty} \eta\left|a_{\eta}\right||z|^{\eta-1} < 2 - \delta \quad (4.2)$$

From theorem (2.1)

$$\sum_{\eta=\rho+1}^{\infty} a_{\eta} \leq \frac{|\alpha(A-B) + B(\rho-\mu)(1+\alpha) - \alpha + 1|}{\left(\frac{\eta!}{(\eta-\mu)!} - \frac{\eta!}{(\eta-\mu-1)!}\right)|\alpha(A-B) - (1-B)[\alpha(\rho-\mu) - 1]|} \quad ,\eta \geq \rho + 1$$

That is

$$\sum_{\eta=\rho+1}^{\infty} \frac{\left(\frac{\eta!}{(\eta-\mu)!} - \frac{\eta!}{(\eta-\mu-1)!}\right)|\alpha(A-B) - (1-B)[\alpha(\rho-\mu) - 1]|}{|\alpha(A-B) + B(\rho-\mu)(1+\alpha) - \alpha + 1|} a_{\eta} \leq 1 , \quad (4.3)$$

Observe that (4.3) is true if

$$\frac{\eta\,|z|^{\eta-1-\rho+\rho}}{\rho|z|^{\rho-1} - (2-\delta)} < \frac{\left(\frac{\eta!}{(\eta-\mu)!} - \frac{\eta!}{(\eta-\mu-1)!}\right)|\alpha(A-B) - (1-B)[\alpha(\rho-\mu) - 1]|}{|\alpha(A-B) + B(\rho-\mu)(1+\alpha) - \alpha + 1|}$$

That is

$$|z|^{\eta-\rho} < \frac{[\rho|z|^{\rho-1} - (2-\delta)]\left[\left(\frac{\eta!}{(\eta-\mu)!} - \frac{\eta!}{(\eta-\mu-1)!}\right)|\alpha(A-B) - (1-B)[\alpha(\rho-\mu) - 1]|\right]}{\eta|z|^{\rho-1}(|\alpha(A-B) + B(\rho-\mu)(1+\alpha) - \alpha + 1|)}$$

Therefore

$$|z| \leq \left\{\frac{[\rho|z|^{\rho-1} - (2-\delta)]\left[\left(\frac{\eta!}{(\eta-\mu)!} - \frac{\eta!}{(\eta-\mu-1)!}\right)|\alpha(A-B) - (1-B)[\alpha(\rho-\mu) - 1]|\right]}{\eta|z|^{\rho-1}(|\alpha(A-B) + B(\rho-\mu)(1+\alpha) - \alpha + 1|)}\right\}^{\frac{1}{\eta-\rho}}$$

Thus the verification is finished.

Theorem(4.2): *If* $f(z) \in S(A,B,\alpha,\rho)$ *, then* f *is starlike function of order* δ *in* $|z| < r_2$

$$r_2 = inf_\eta |z|$$

$$\leq \left\{ \frac{(\rho+\delta-2)}{(\eta+\delta-2)} \left(\frac{\left(\frac{\eta!}{(\eta-\mu)!} - \frac{\eta!}{(\eta-\mu-1)!}\right) |\alpha(A-B) - (1-B)[\alpha(\rho-\mu)-1]|}{|\alpha(A-B) + B(\rho-\mu)(1+\alpha) - \alpha + 1|} \right) \right\}^{\frac{1}{\eta-\rho}} \quad (4.4)$$

Proof: *We have to demonstrate that*

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1-\delta$$

That is

$$\frac{(\rho-1)\,|z|^\rho - \sum_{\eta=\rho+1}^{\infty}(\eta-1)|a_\eta||z|^\eta}{|z|^\rho - \sum_{\eta=\rho+1}^{\infty}|a_\eta||z|^\eta} \leq 1-\delta \quad (4.5)$$

Hence (4.5) holds if

$$(\rho-1)\,|z|^\rho - \sum_{\eta=\rho+1}^{\infty}(\eta-1)|a_\eta||z|^\eta \leq (1-\delta)\left[|z|^\rho - \sum_{\eta=\rho+1}^{\infty}|a_\eta||z|^\eta\right]$$

Or equivalently

$$\sum_{\eta=\rho+1}^{\infty} \frac{(\eta+\delta-2)}{(\rho+\delta-2)} |a_\eta||z|^{\eta-\rho} \leq 1 \quad (4.6)$$

From theorem (2.1) we have

$$\sum_{\eta=\rho+1}^{\infty} \frac{\left(\frac{\eta!}{(\eta-\mu)!} - \frac{\eta!}{(\eta-\mu-1)!}\right) |\alpha(A-B) - (1-B)[\alpha(\rho-\mu)-1]|}{|\alpha(A-B) + B(\rho-\mu)(1+\alpha) - \alpha + 1|} a_\eta \leq 1\,, \quad (4.7)$$

Hence by using (4.6) and (4.7) we get

$$\frac{(\eta+\delta-2)}{(\rho+\delta-2)}\,|z|^{\eta-\rho} \leq \frac{\left(\frac{\eta!}{(\eta-\mu)!} - \frac{\eta!}{(\eta-\mu-1)!}\right) |\alpha(A-B) - (1-B)[\alpha(\rho-\mu)-1]|}{|\alpha(A-B) + B(\rho-\mu)(1+\alpha) - \alpha + 1|}$$

$$|z|^{\eta-\rho} \leq \frac{(\eta+\delta-2)}{(\rho+\delta-2)} \left(\frac{\left(\frac{\eta!}{(\eta-\mu)!} - \frac{\eta!}{(\eta-\mu-1)!}\right) |\alpha(A-B) - (1-B)[\alpha(\rho-\mu)-1]|}{|\alpha(A-B) + B(\rho-\mu)(1+\alpha) - \alpha + 1|} \right)$$

$$|z| \leq \left\{ \frac{(\rho+\delta-2)}{(\eta+\delta-2)} \left(\frac{\left(\frac{\eta!}{(\eta-\mu)!} - \frac{\eta!}{(\eta-\mu-1)!}\right) |\alpha(A-B) - (1-B)[\alpha(\rho-\mu)-1]|}{|\alpha(A-B) + B(\rho-\mu)(1+\alpha) - \alpha + 1|} \right) \right\}^{\frac{1}{\eta-\rho}}$$

Theorem (4.3): *If* $f(z) \in S(A,B,\alpha,p)$ *, then* f *is convex function of order* δ *in* $|z| < r_3$

$$r_3 = inf_\eta |z|$$

$$\leq \left\{ \frac{(\rho+\delta-2)}{(\eta+\delta-2)} \left(\frac{\left(\frac{\eta!}{(\eta-\mu)!} - \frac{\eta!}{(\eta-\mu-1)!}\right) |\alpha(A-B) - (1-B)[\alpha(\rho-\mu)-1]|}{|\alpha(A-B) + B(\rho-\mu)(1+\alpha) - \alpha + 1|} \right) \right\}^{\frac{1}{\eta-\rho}} \quad (4.8)$$

Proof: *We have to demonstrate that*

$$\left| \frac{zg'(z)}{g(z)} - 1 \right| \leq 1-\delta$$

Where $g(z) = zf^{\lambda+1}(z)$
That is

$$\frac{\frac{\rho!}{(\rho-\lambda-1)!}(\rho-\lambda-1)|z|^{\rho-\lambda}-\sum_{\eta=\rho+1}^{\infty}\frac{\eta!}{(\eta-\lambda-1)!}(\eta-\lambda+1)|a_\eta||z|^{\eta-\lambda}}{\frac{\rho!}{(\rho-\lambda-1)!}|z|^{\rho-\lambda}-\sum_{\eta=\rho+1}^{\infty}\frac{\eta!}{(\eta-\lambda-1)!}|a_\eta||z|^{\eta-\lambda}} \le 1-\delta \qquad (4.9)$$

Hence (4.9) holds if

$$\frac{\rho!}{(\rho-\lambda-1)!}(\rho-\lambda-1)|z|^{\rho-\lambda}-\sum_{\eta=\rho+1}^{\infty}\frac{\eta!}{(\eta-\lambda-1)!}(\eta-\lambda+1)|a_\eta||z|^{\eta-\lambda} \le (1-\delta)\left[\frac{\rho!}{(\rho-\lambda-1)!}|z|^{\rho-\lambda}-\sum_{\eta=\rho+1}^{\infty}\frac{\eta!}{(\eta-\lambda-1)!}|a_\eta||z|^{\eta-\lambda}\right]$$

Or equivalently

$$\sum_{\eta=\rho+1}^{\infty}\frac{\frac{\eta!}{(\eta-\lambda-1)!}(\lambda-\eta-\delta)}{\frac{\rho!}{(\rho-\lambda-1)!}(\lambda-\rho-\delta+2)}|a_\eta||z|^{\eta-\rho} \le 1 \qquad (4.10)$$

From theorem (2.1) we have

$$\sum_{\eta=\rho+1}^{\infty}\frac{\left(\frac{\eta!}{(\eta-\mu)!}-\frac{\eta!}{(\eta-\mu-1)!}\right)|\alpha(A-B)-(1-B)[\alpha(\rho-\mu)-1]|}{|\alpha(A-B)+B(\rho-\mu)(1+\alpha)-\alpha+1|}a_\eta \le 1\,, \qquad (4.11)$$

Hence by using (4.10) and (4.11) we get

$$\frac{\frac{\eta!}{(\eta-\lambda-1)!}(\lambda-\eta-\delta\,)}{\frac{p!}{(p-m-1)!}(\lambda-\rho-\delta+2)}|z|^{\eta-\rho} \le \frac{\left(\frac{\eta!}{(\eta-\mu)!}-\frac{\eta!}{(\eta-\mu-1)!}\right)|\alpha(A-B)-(1-B)[\alpha(\rho-\mu)-1]|}{|\alpha(A-B)+B(\rho-\mu)(1+\alpha)-\alpha+1|}$$

$$|z|^{\eta-\rho} \le \frac{\frac{\rho!}{(\rho-\lambda-1)!}(\lambda-\rho-\delta+2)}{\frac{\eta!}{(\eta-\mu-1)!}(\lambda-\eta-\delta\,)}\left(\frac{\left(\frac{\eta!}{(\eta-\mu)!}-\frac{\eta!}{(\eta-\mu-1)!}\right)|\alpha(A-B)-(1-B)[\alpha(\rho-\mu)-1]|}{|\alpha(A-B)+B(\rho-\mu)(1+\alpha)-\alpha+1|}\right)$$

$$|z| \le \left\{\frac{\frac{\rho!}{(\rho-\lambda-1)!}(\lambda-\rho-\delta+2)}{\frac{\eta!}{(\eta-\mu-1)!}(\lambda-\eta-\delta\,)}\left(\frac{\left(\frac{\eta!}{(\eta-\mu)!}-\frac{\eta!}{(\eta-\mu-1)!}\right)|\alpha(A-B)-(1-B)[\alpha(\rho-\mu)-1]|}{|\alpha(A-B)+B(\rho-\mu)(1+\alpha)-\alpha+1|}\right)\right\}^{\frac{1}{\eta-\rho}}$$

Thus the verification is finished.

References

1. Bulboacˇa, T. **2005**. *Differential Subordinations and Superordinations*, Recent Results; House of Scientific Book Publ.: Cluj-Napoca, Romania,.
2. Miller, S.S.; Mocanu, P.T. **2000**. *Differential Subordination*: *Theory and Applications*, *Series on Monographs and Textbooks in Pure and Applied Mathematics*; Marcel Dekker Inc.: New York, NY, USA**;** Volume 225.
3. Hanaa, Mohamed and Adela. **2018**. Subordination Properties for Multivalent Functions Associated with a Generalized Fractional Differintegral Operator , *Axioms — Open Access Journal, Axioms*, **7**(27).
4. Wang, Z. G., Aghalary, R. and Ibrahim, R. W. **2009**. Some properties of certain multivalent analytic functions involving Cho-Kwon-Srivastava operator, *Mathematical and Computer Modelling*, **49**: 1969-1984.

5. B. C. Carlson and D. B. Shaffer. **1984**. Starlike and prestarlike hypergeometric functions, *SIAM J. Math Anal.* **15**: 737-745. https://doi.org/10.1137/0515057

6. He, P. and Zhang, D. **2014**. Some properties of multivalent functions associated with a certain operator, *Ukrainian Mathematical Journal,* **65**(11): 1749-1754

7. Hanaa M. Zayed. **2019**. Subordination and inclusion theorems for higher order derivatives of a generalized fractional differ integral operator, *Zayed Journal of the Egyptian Mathematical Society*, **27**: 17.

8. Vinod Kumar , Prachi Srivastava. **2019**. Some Applications of Multivalent Functions Defined by Extended Fractional Differintegral Operator ,*International Journal of Applied Engineering Research,* **14**(12): 2915-2924 .

9. Mostafa, A.O.; Aouf, M.K.; Zayed, H.M.; Bulboacˇa, T. **2017**. Multivalent functions associated with Srivastava-Saigo -Owa fractional differintegral operator. *RACSAM* .

10. Mostafa, A.O.; Aouf, M.K.; Zayed, H.M. **2018**. Subordinating results for p-valent functions associated with the Srivastava–Saigo–Owa fractional differintegral operator. *Afr. Mat.*

11. Wang, Z.; Shi, L **2017**. Some properties of certain extended fractional differintegral operator. *RACSAM***,** 1–11.

12. Adnan Aziz and Kassim Abdulhameed. **2019**. On A Certain Class of Meromorphic Multivalent Functions Defined by Fractional Calculus Operator. *Iraqi Journal of Science*, **60**(12): 2685-2696.

13. Sattar Kamil and Kassim Abdulhameed. **2019**. On a Class of Meromorphic Multivalent Functiions Convoluted with Higher Derivat ives of Fractional Calculus Operator . *Iraqi Journal of Science,* **60**(10): 2237-2245.