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Approximation of Modified Baskakov Operators Based on Parameter s

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Abstract

In this article, we define and study a family of modified Baskakov type operators based on a parameter $s > -\frac{1}{2}$. This family is a generalization of the classical Baskakov sequence. First, we prove that it converges to the function being approximated. Then, we find a Voronovsky-type formula and obtain that the order of approximation of this family is $O\left(n^{-(2s+1)}\right)$. This order is better than the order of the classical Baskakov sequence $O(n^{-1})$ whenever $s > 0$. Finally, we apply our sequence to approximate two test functions and analyze the numerical results obtained.

Keywords: Baskakov operators, Voronovsky-type asymptotic formula, Order of approximation

تقريب بمؤثرات Baskakov المحدثة المعتمدة على معامل s

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الخلاصه

في هذا البحث، نقدم وندرس عائلة مؤثرات من النمط Baskakov المحدثة المعتمدة على المعامل $s > -1/2$. هذه العائلة تعميم لمتتابعة مؤثرات Baskakov الاعتيادية. أولا، نبرهن تقاربها للدالة المقربة. ثم، نجد نمط صيغة Voronovsky ونجد رتبة تقريب هذه العائلة هو $O\left(n^{-(2s+1)}\right)$. هذه الرتبة أفضل من رتبة تقريب متتابعة Baskakov الاعتيادية $O(n^{-1})$ عندما $s > 0$. أخيرا، نطبق متتابعتنا لتقريب دالتين اختباريتين ونحلل النتائج الحاصلة

1. Introduction

The well-known classical Baskakov sequence is defined as [1]

$$M_n(f;x) = \sum_{\kappa=0}^{\infty} p_{n,\kappa}(x) f\left(\frac{\kappa}{n}\right), \tag{1.1}$$

where $p_{n,\kappa}(x) = \frac{(n+\kappa-1)!}{\kappa!(n-1)!} x^{\kappa}(1+x)^{-n-\kappa}, x \in [0,\infty)$.

Many modifications to the above sequence were applied by several researchers, all reaching the same order of approximation $O(n^{-1})$ [2, 3, 4]. Indeed, there are some techniques, such as the linear combination and Micchelli combination, that were defined and studied for many sequences of positive and linear operators to modify the approximation order by these sequences. But these techniques

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increase the arithmetic operations in the computer programs, which decreases the advantage of these techniques [5- 8].

Pallini [9] presented a modification of the sequence of classical Bernstein polynomials with a different order of approximation. His sequence depends on a parameter $s > -1/2$ and is defined as follows

$$B_{n,s}(f;x) = \sum_{\kappa=0}^{n} \frac{n!}{\kappa!\,(n-\kappa)!} x^{\kappa}(1-x)^{n-\kappa} f\left(x + \frac{1}{n^s}\left(\frac{\kappa}{n} - x\right)\right), \tag{1.2}$$

where $x \in [0,1]$ and $f \in C[0,1]$. The parameter s is called the convenient approximation coefficient.

The sequence (1.2) has a better approximation order of $O\left(n^{-(2s+1)}\right)$ when $s > 0$. [9]

In this study, we define a family of modified Baskakov type operators, such as the family (1.2), as follows

$$L_{n,s}(f;x) = \sum_{\kappa=0}^{\infty} p_{n,\kappa}(x) f\left(x + \frac{1}{n^s}\left(\frac{\kappa}{n} - x\right)\right), \tag{1.3}$$

where $f \in C_\alpha[0,\infty) = \{f \in C[0,\infty): f(t) = O((1+t)^\alpha), \text{for some } \alpha > 0\}$ and $x \in [0,\infty)$.

The space $C_\alpha[0,\infty)$ is normed by the norm $\|f\|_{C_\alpha} = \sup_{t\in[0,\infty)}|f(t)|(1+t)^{-\alpha}$.

Note that $L_{n,0}(.;x) = M_n(.,x)$.

Here, we prove some theorems for the family $L_{n,s}(.;x)$ in simultaneous approximation, i.e. the convergence theorem and Voronovsky-type asymptotic formula. It turns out that the order of approximation by this family is modified to the order $O\left(n^{-(2s+1)}\right)$ when $s > 0$. Also, we support this study by applying the sequence $L_{n,s}(.;x)$ on two test functions to approximate them. The numerical results show that the sequence $L_{n,s}(f;x), s > 0$ is more accurate and faster than the classical Baskakov sequence.

2. Auxiliary Results

This section gives some Lemmas that are needed in the proof of the main theorems in this study.

Lemma 2.1 [2] For $x \in [0,\infty)$, $n \in N := \{1,2,\ldots\}$ and $e_m = t^m$, $m \in N^0 := N \cup \{0\}$, we have

(i) $L_n(e_0;x) = 1$, $L_n(e_1;x) = x$ and $L_n(e_2;x) = \left(1 + \frac{1}{n}\right)x^2 + \frac{x}{n}$.

(ii) $L_n(e_m;x) = \frac{(n+m-1)!}{(n-1)!\,n^m}x^m + \frac{m(m-1)}{2}\frac{(n+m-2)!}{(n-1)!n^{m-1}}x^{m-1} + O(n^{-2})$.

Lemma 2.2 [2] For $m \in N^0$, the definition of the moment of order m for the sequence $L_n(f;x)$ is given as

$$U_{n,m}(x) = \sum_{\kappa=0}^{\infty} p_{n,\kappa}(x)\left(\frac{\kappa}{n} - x\right)^m.$$

Then $U_{n,0}(x) = 1$, $U_{n,1}(x) = 0$, $U_{n,2}(x) = \frac{(1+x)x}{n}$, $nU_{n,m+1}(x) = x(1+x)\left\{mU_{n,m-1}(x) + U'_{n,m}(x)\right\}$ holds for $m \in N$.

Consequently

(i) $U_{n,m}(x)$ are polynomials and degree $U_{n,m}(x) \le m$.

(ii) $U_{n,m}(x) = O\left(\frac{1}{n^{[(m+1)/2]}}\right)$, where $[(m+1)/2]$ means the integer part of $(m+1)/2$.

Lemma 2.3 For $x \in [0,\infty)$, $e_m = t^m$ and $m \in N^0$, we get

(i) $L_{n,s}(e_0;x) = 1$, $L_{n,s}(e_1;x) = x$, $L_{n,s}(e_2;x) = x^2 + \frac{x(1+x)}{n^{2s+1}}$,

(ii) $L_{n,s}(e_m;x) = \frac{1}{n^{sm}}\sum_{j=0}^{m}\binom{m}{j}\frac{(n^s-1)^{m-j}}{n^j}\left\{(n)_j x^m + \frac{(j-1)j}{2}(n)_{j-1}x^{m-1} + O(n^{-2})\right\}$,

where $(n)_j = \frac{(n+j-1)!}{(n-1)!}$.

Proof: The property (i) follows immediately by the direct evaluation and Lemma 2.1, hence, the steps are omitted.

To prove (2), using Lemma 2.1

$$L_{n,s}(e_m;x)=\frac{1}{n^{sm}}\sum_{\kappa=0}^{\infty}p_{n,\kappa}(x)\left(\frac{\kappa}{n}+x(n^s-1)\right)^m$$
$$=\frac{1}{n^{sm}}\sum_{j=0}^{m}\binom{m}{j}\left(x(n^s-1)\right)^{m-j}L_n(e_j;x)$$
$$=\frac{1}{n^{sm}}\sum_{j=0}^{m}\binom{m}{j}x^{m-j}(n^s-1)^{m-j}\left(\frac{(n+m-1)!}{n^j(n-1)!}x^j+\frac{(n+m-2)!}{n^{j-1}(n-1)!}\frac{m(m-1)}{2}x^{j-1}+O(n^{-2})\right).$$

Hence, the consequence (ii) is held.

Note that, from Korovkin's theorem [3] and the value of $L_{n,s}(e_2;x)$ in Lemma 2.3 (i), we have that the sequence $L_{n,s}(f;x)\to f(x)$ as $n\to\infty$ whenever $s>-1/2$.

For $m\in N^0$, the moment of order m for the sequence (1.3) is denoted and defined as:

$$T_{n,m}(x)=L_{n,s}((t-x)^m;x)=\frac{1}{n^{ms}}U_{n,m}(x).$$

Lemma 2.4 For the function $T_{n,m}(x)$, we have

(i) $T_{n,0}(x)=1$, $T_{n,1}(x)=0$ and $T_{n,2}(x)=\frac{x(x+1)}{n^{2s+1}}$.

(ii) We have $n^{s+1}T_{n,m+1}(x)=(1+x)x\left\{\frac{m}{n^s}T_{n,m-1}(x)+T'_{n,m}(x)\right\}, m\in N$.

(iii) The function $T_{n,m}(x)$ is a polynomial of degree $\le m$.

(iv) $T_{n,m}(x)=O\left(n^{-\left[\frac{(2s+1)m+1}{2}\right]}\right)\ \forall\, x\in[0,\infty)$.

Proof Using Lemma 2.2 and the direct computation, the consequences (i), (ii) and (iii) can hold immediately.

For $x\in[0,\infty)$ and applying Lemma 2.2, we get

$$T_{n,m}(x)=\frac{1}{n^{sm}}U_{n,m}(x)=\frac{1}{n^{sm}}O\left(n^{-\left[\frac{m+1}{2}\right]}\right).$$
$$=\begin{cases}\frac{1}{n^{sm}}A_1n^{-\left(\frac{m+1}{2}\right)}, & m \text{ odd}\\ \frac{1}{n^{sm}}A_2n^{-\left(\frac{m}{2}\right)}, & m \text{ even}\end{cases},$$

where A_1 and A_2 are positive constants.

Hence, $T_{n,m}(x)=O\left(n^{-\left[\frac{(2s+1)m+1}{2}\right]}\right)$.

3. Theoretical Results

Here, we give some theorems in simultaneous approximation for the sequence $L_{n,s}(f;x)$. First, one shows that the r-th derivative of $L_{n,s}(f;x)$ is an approximate process of the $f^{(r)}$, $r\in N$.

Theorem 3.1 Suppose that $f\in C_\alpha[0,\infty)$ and $f^{(r)}$ exists at $x\in(0,\infty)$, the following limit holds

$$\lim_{n\to\infty}L_{n,s}^{(r)}(f;x)=f^{(r)}(x). \quad (3.1)$$

Also, the limit (3.1) holds uniformly on $[a,b]$ if $f^{(r)}$ is continuous on the interval $(a-\mu,b+\mu)\subset(0,\infty),\mu>0$.

Proof We can expand the function $f(t)$ by Taylor series, as follows

$$f(t)=\sum_{h=0}^{r}f^{(h)}(x)\frac{(t-x)^h}{h!}+(t-x)^r\varepsilon(t,x),$$

where $\varepsilon(t,x)\to 0$ as $t\to x$. Then,

$$L_{n,s}^{(r)}(f;x)=\sum_{h=0}^{r}\frac{f^{(h)}(x)}{h!}L_{n,s}^{(r)}\big((t-x)^h;x\big)+L_{n,s}^{(r)}((t-x)^r\varepsilon(t,x);x):=E_1+E_2.$$

Then,

$$E_1=\sum_{h=0}^{r}\frac{f^{(h)}(x)}{h!}L_{n,s}^{(r)}\big((t-x)^h;x\big)=\sum_{h=0}^{r}\frac{f^{(h)}(x)}{h!}\sum_{j=0}^{h}\binom{h}{j}(-x)^{h-j}L_{n,s}^{(r)}\big(t^j;x\big)$$

From Lemma 2.3, when $j < r$, we get $L_{n,s}^{(r)}(t^j; x) = 0$.
Hence,

$$E_1 = \frac{f^{(r)}(x)}{r!} L_{n,s}^{(r)}(t^r; x)$$
$$= \left[\frac{1}{n^{sr}} \sum_{j=0}^{r} \frac{1}{n^j} \binom{m}{j} (n^s - 1)^{r-j} \left\{\frac{(n+j-1)!}{(n-1)!} x^r + \frac{j(j-1)}{2} \frac{(n+j-2)!}{(n-1)!} x^{r-1} + O(n^{-2})\right\}\right],$$

we get $E_1 = o(1), j > 0$ and, when $j = 0$, we have

$$E_1 = \frac{f^{(r)}(x)}{r!} \left[\frac{(n^s-1)^r}{n^{sr}} \left\{r! + O\left(\frac{1}{n^2}\right)\right\}\right] \to f^{(r)}(x) \text{ as } n \to \infty.$$

$$E_2 = L_{n,s}^{(r)}(\varepsilon(t,x)(t-x)^r; x) = \frac{1}{n^{sr}} \sum_{k=0}^{\infty} p_{n,k}^{(r)}(x) \varepsilon\left(x + \frac{(t-x)}{n^s}, x\right)(t-x)^r = 0$$

Since $\varepsilon\left(x + \frac{t-x}{n^s}, x\right) = \varepsilon(x,x) = 0$ as $t \to x$. Hence, (3.1) is held.

The uniformity property of the limit (3.1) can be followed because δ in the proof above is depending only on ε, i.e. δ is independent of x.

The next theorem is a Voronovsky-type asymptotic formula for $L_{n,s}^{(r)}(f; x)$.

Theorem 3.2 Let $f \in C_\alpha[0,\infty)$ for some $\alpha > 0$ and $x \in (0,\infty)$. If $f^{(r+2)}(x)$ exists, then

$$\lim_{n\to 0} n^{2s+1} \left(L_{n,s}^{(r)}(f; x) - f^{(r)}(x)\right)$$
$$= \frac{1}{2}(r-1) r f^{(r)}(x) + \frac{1}{2} r(1+2x) f^{(r+1)}(x) + \frac{1}{2} x(1+x) f^{(r+2)}(x). \quad (3.2)$$

Also, the limit (3.2) holds uniformly on $[a,b]$ if $f^{(r)}$ is continuous on the interval $(a-\mu, b+\mu) \subset (0,\infty), \mu > 0$.

Proof: By Taylor's expansion of $f(t)$, we get

$$L_{n,s}^{(r)}(f; x) = \sum_{h=0}^{r+2} L_{n,s}^{(r)}((t-x)^h; x) + L_{n,s}^{(r)}((t-x)^{r+2} \varepsilon(t,x); x) := I_1 + I_2,$$

where $\varepsilon(t,x) \to 0$ *as* $t \to x$.
By Lemma 2.3, we get

$$I_1 = \sum_{h=r}^{r+2} \frac{f^{(i)}(x)}{h!} \sum_{j=r}^{h} \binom{h}{j} (-x)^{h-j} L_{n,s}^{(r)}(t^j; x).$$
$$= \frac{f^{(r)}(x)}{r!} L_{n,s}^{(r)}(t^r; x) + \frac{f^{(r+1)}(x)}{(r+1)!}\Big((r+1)(-x) L_{n,s}^{(r)}(t^r; x) + L_{n,s}^{(r)}(t^{r+1}; x)\Big).$$
$$+ \frac{f^{(r+1)}(x)}{(r+2)!}\left(\frac{(r+2)(r+1)}{2} x^2 L_{n,s}^{(r)}(t^r; x) + (r+2)(-x) L_{n,s}^{(r)}(t^{r+1}; x) + L_{n,s}^{(r)}(t^{r+2}; x)\right)$$
$$= \frac{f^{(r)}(x)}{n^{sr} r!}\left(\sum_{j=0}^{r} \binom{r}{j} \frac{(n^s-1)^{r-j}}{n^j} \frac{(n+j-1)!}{(n-1)!}\right) r!$$
$$+ \frac{f^{(r+1)}(x)}{(r+1)!}\left(\frac{(r+1)(-x)}{n^{sr}} \sum_{j=0}^{r} \binom{r}{j} \frac{(n^s-1)^{r-j}}{n^j} \frac{\big(n+r-(j+1)\big)!}{(n-1)!} r!\right.$$
$$+ \frac{x}{n^{s(r+1)}} \sum_{j=0}^{r+1} \binom{r+1}{j} \frac{(n^s-1)^{r+1-j}}{n^j} \frac{(n+j-1)!}{(n-1)!} (r+1)!$$
$$\left.+ \frac{1}{n^{s(r+1)}} \sum_{j=0}^{r+1} \binom{r+1}{j} \frac{(n^s-1)^{r+1-j}}{n^j} \frac{(n+j-1)!}{(n-1)!} \frac{j(j-1)}{2} r!\right)$$

$$
\begin{aligned}
&+\frac{f^{(r+2)}(x)}{(r+2)!}\Bigg(\frac{(r+1)(r+2)x^2}{2n^{rs}}\sum_{j=0}^{r}\binom{r}{j}\frac{(n^s-1)^{r-j}}{n^j}\frac{(n+j+1)!}{(n-1)!}r! \\
&\quad-\frac{(r+2)x^2}{n^{(r+1)(s+1)}}\sum_{j=0}^{r+1}\binom{r+1}{j}\frac{(n^s-1)^{r+1-j}}{n^j}\frac{(n+j-1)!}{(n-1)!}(r+1)! \\
&\quad-\frac{1}{n^{s(r+1)}}\sum_{j=0}^{r+1}\binom{r+1}{j}\frac{(n^s-1)^{r+1-j}}{n^j}\frac{(n+j-1)!}{(n-1)!}\frac{(r+1-j)j(j-1)}{(n-1)!\,2}r! \\
&\quad+\frac{x^2}{2n^{s(r+2)}}\sum_{j=0}^{r+2}\binom{r+2}{j}\frac{(n^s-1)^{r+2-j}}{n^j}\frac{(n+j+1)!}{(n-1)!} \\
&\quad+(r+2)!\frac{(r+1)!}{n^{s(r+2)}}\sum_{j=2}^{r+2}\binom{r+2}{j}\frac{(n^s-1)^{r+2-j}}{n^j}\frac{(n+j-1)!}{(n-1)!}\frac{j(j-1}{2}\Bigg).
\end{aligned}
$$

Using the same technique of Theorem 3.1, we get $L_{n,s}^{(r)}((t-x)^{r+2}\varepsilon(t,x);x)\to 0$ as $n\to\infty$. Then, (3.2) holds.

The uniformity property of the limit (3.2) can be followed from the fact that δ in all steps in the proof above is depending only on ε, i.e. δ is independent of x.

4. Numerical Results

In this part, we give numerical applications of the three sequences $L_{n,s}(f;x)$, $s=0,1,1.3,2$ to approximate the two test functions of $f(x)=2\cos(2x)$ and $g(x)=|(x-3)^2-3|$ for three values of $n=5,10,20$. We describe the results by the graphics of each test function and its three approximations for each value of n.

Example 4.1
The test function $f(x)=2\cos(2x)$ and $x\in[0,20]$.

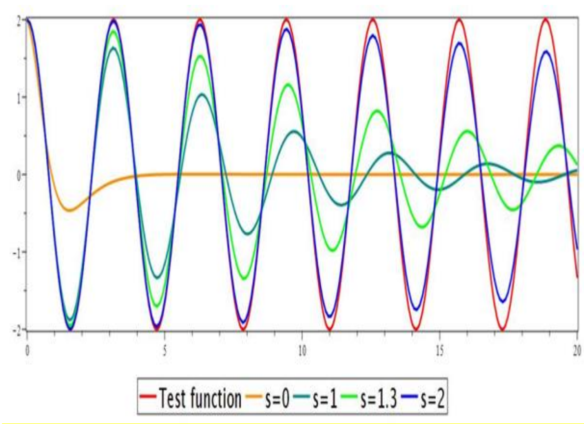

Figure 4.1- $(n=5)$

Example 4.2
The test function $g(x)=|(x-3)^2-3|$ and $x\in[0,6]$

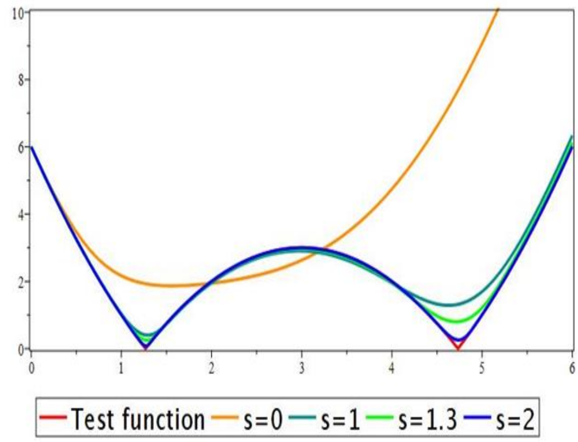

Figure 4.4- $(n=5)$

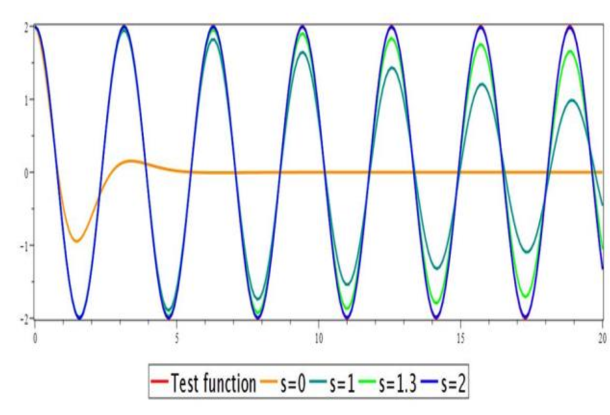

Figure 4.2- $(n = 10)$

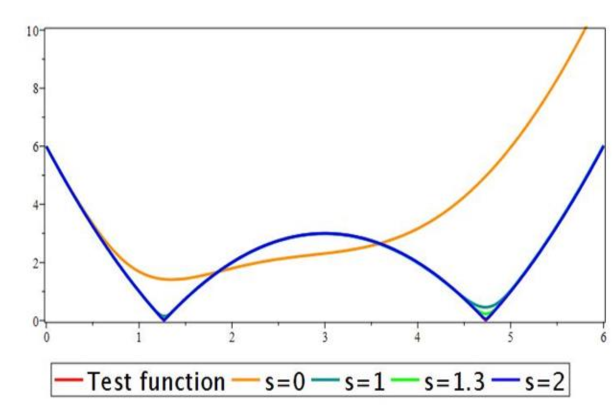

Figure 4.5 $-(n = 10)$

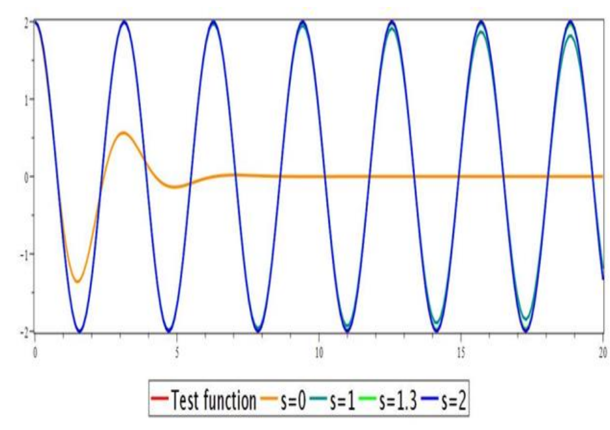

Figure 4.3- $(n = 20)$

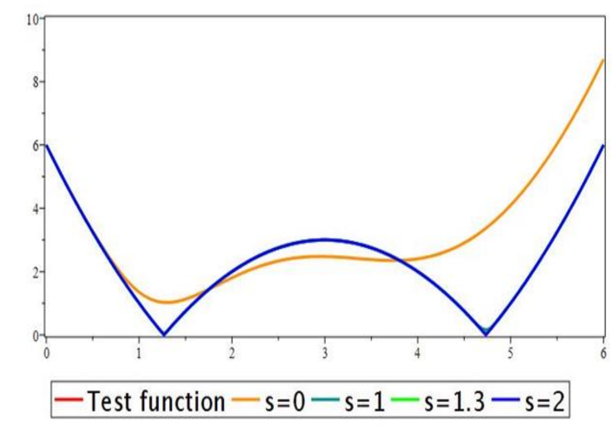

Figure 4.6- $(n = 20)$

5. Conclusions

The numerical results showed that the sequence $L_{n,s}(f;x)$ becomes faster and more accurate when s increases. Hence, this sequence is more efficient than the classical Baskakov sequence $L_n(f;x)$ because it has $O\left(n^{-(2s+1)}\right)$ when $s > 0$. We recommend using this sequence in the related applications instead of the classical one.

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