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On y-closed Dual Rickart Modules

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Abstract
In this paper, we develop the work of Ghawi on close dual Rickart modules and discuss y-closed dual Rickart modules with some properties. Then, we prove that, if M_1 and M_2 are y-closed simple R-modues and if M_1 is M_2-y-closed is a dual Rickart module, then either Hom (M_1, M_2) =0 or$M_1 \cong M_2$. Also, we study the direct sum of y-closed dual Rickart modules.

Keywords: Endomorphism ring, y-closed submodule, Image of endomorphism, y-closed simple, y-closed dual Rickart modules.

حول مقاسات الريكارتية الرديفة من النمط y– مغلق

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الخلاصة
في هذا البحث قمنا بتوسيع دراسة ثائر يونس غاوي في المقاسات الريكارتية الرديفة المغلقة حيث ناقشنا المقاسات الريكارتية الرديفة من النمط y– مغلق مع توسيع بعض الخواص حول هذا المفهوم. برهنا اذا كانت M_1 و M_2 هي مقاسات بسيطه من النمط y– مغلق و كان المقاس الريكارتي الرديف M_1 is M_2 من النمط y– مغلق فان اما Hom(M_1, M_2)=0 او $M_1 \cong M_2$. وايضا درسنا الجمع المباشر للمقاسات الريكارتية الرديفة من النمط y– مغلق .

1. INTRODUCTION

A module M is called a dual Rickart module if for every$\varphi \in End(M)$, then $Im\varphi = eM$ for some $e^2 = e \in S$. Equivalently, a module M is a dual Rickart module if and only if for every $\varphi \in End(M)$, then $Im\varphi$ is a direct summand of M [1]. A module M is called a closed dual Rickart module, if for any $f \in End(M)$, Imf is a closed submodule in M [1]. Recall that a submodule A of an R-module M is called a y-closed submodule of M if $\frac{M}{A}$ is nonsingular [2]. It is known that every y-closed submodule is closed.

In this paper, we give some results on the y-closed dual Rickart modules.

In section 2, we give the definition of the y-closed dual Rickart modules with some examples and basic properties. Moreover, we prove that for two R-modules M and N, and let B be a submodule of N if M is N-y-closed dual Rickart module, then M is B-y-closed dual Rickart module, see proposition (2.4) .

In section 3, we study the direct sum of y-closed dual Rickart modules. Furthermore, we prove that, let M and N be two R-modules, such that $M = A \oplus B$ if M is N-y-closed dual Rickart module, then A is L-y-closed dual Rickart module, for every submodule L of N, see proposition (3.1) .

Throughout this article, R is a ring with identity and M is a unital left R-module. For a left module$M$, $S = End_R(M)$ will denote the endomorphism ring of$M$.

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§2: Y-CLOSED DUAL RICKART MODULES

In this section, we introduce the concept of the y-closed dual Rickart modules and we illustrate it by some examples. Also, we give some basic properties. We start by the definition.

Definition 2.1: Let M and N be two R-modules. We say that M is N-y-closed dual Rickart module if for every homomorphism $0 \neq f: M \to N,\ Imf$ is a y-closed submodule of$N$.

For a module M. If M is M-y-closed dual Rickart module. Then we say that M is a y-closed dual Rickart module.

Examples 2.2

1- The module $Z_2 \oplus Z_2$ as Z_2-module is Z_2-y-closed dual Rickart module. To show that, let $0 \neq f: Z_2 \oplus Z_2 \to Z_2$ be any R-homomorphism. Then $Imf = Z_2$ is a y-closed submodule of Z_2.

2- Consider the module $Z\ as\ Z$-module and let $f: Z \to Z$ be a map defined by$f(n) = 4n,\ \forall n \in Z$. It is clear that f is an R-homomorphism and $Imf = 4Z$. But $\frac{Z}{4Z} \simeq Z_4$ and Z_4 as Z-module is singular, therefore $Imf = 4Z$ is not a y-closed submodule of Z. Thus Z is not y-closed dual Rickart module.

3- Consider the modules Z_p and $Z_{P\infty}$ as Z-modules. The module Z_p is not $Z_{P\infty}$-y-closed dual Rickart module. To show that, let $i: (\frac{1}{p} + Z) \to Z_{P\infty}$ be the inclusion map. Since $Z_{P\infty}$ is singular, then $\frac{Z_{P\infty}}{(\frac{1}{p}+Z)}$ is singular, by [2]. Therefore $(\frac{1}{p} + Z)$ is not a y-closed submodule of $Z_{P\infty}$. But $Z_p \simeq (\frac{1}{p} + Z)\ as\ Z$-module, therefore, is not $Z_{P\infty}$-y-closed dual Rickart module.

Remark 2.3: A dual Rickart module needs not to be a y-closed dual Rickart module. For example, the module $Z_6\ as\ Z$-module is a dual Rickart module, where $Z_6\ as\ Z$-module is semisimple. Claim that $Z_6\ as\ Z$-module is not y-closed dual Rickart module. To show that, let $f: Z_6 \to Z_6$ be a map defined by $f(x) = 2x,\ \forall x \in Z_6$. It is clear that f is a homomorphism and $Imf = \{\bar{0}, \bar{2}, \bar{4}\}$. But $\frac{Z_6}{Imf} \simeq Z_2$ and Z_2 as Z-module is singular, therefore Imf is not a y-closed submodule of Z_6 . Thus Z_6 is not y-closed dual Rickart module.

Proposition 2.4: Let M and N be two R-modules and let B be a submodule of N. If M is N-y-closed dual Rickart module, then M is B-y-closed dual Rickart module.

Proof. Let $f: M \to B$ be an R-homomorphism and let $i: B \to N$ be the inclusion map. Consider the map $i \circ f: M \to N$. Since M is N-y-closed dual Rickart module, then $Imf = Im\, i \circ f$ is a y-closed submodule of N and hence $\frac{N}{Imf}$ is nonsingular. But $\frac{B}{Imf}$ is a submodule of $\frac{N}{Imf}$, therefore $\frac{B}{Imf}$ is nonsingular and hence Imf is a y-closed submodule of B. Thus M is B-y-closed dual Rickart module.

Definition 2.5: Let M be an R-module, than M is called a y-closed simple if M and 0 are the only y-closed submodules of M.

Proposition 2.6: Let M be an R-module and let N be a y-closed simple R-module. If M is N-y-closed dual Rickart module, then either

(1) Hom(M,N)=0 or

(2) Every nonzero R-homomorphism from M to N is an epimorphism.

Proof. Assume that Hom(M,N) ≠0 and let $f: M \to N$ be a non-zero R-homomorphism. Since M is N-y-closed dual Rickart, then Imf is y-closed submodule of N. But N is y-closed simple, therefore $Imf = N$ and f is an epimorphism.

Recall that an R-module M is called a Co-Quasi-Dedekind R-module if every nonzero endomorphism of M is an epimorphism, see[3, p2].

Proposition 2.7: Let $M_1\ and\ M_2$ be R-modules such that M_2 is y-closed simple and $M_1$is M_2-y-closed dual Rickart module. If Hom$(M_1, M_2) \neq 0$, then M_2 is Co-Quasi-Dedekind R-module.

Proof. Assume that there is an R-homomorphism $0 \neq f: M_1 \to M_2$. Then by proposition (2.6), $Imf = M_2$. Now let $0 \neq g: M_2 \to M_2$ be an R-homomorphism . Consider the map $g \circ f: M_1 \to M_2$. Since $M_1\ is\ M_2$-y-closed dual Rickart module, then $Img \circ f$ is a y-closed submodule of M_2. But f is an epimorphism, therefore $Img \circ f = Img$ is a y-closed submodule of M_2. Since M_2 is y-closed simple, then $Img = M_2$. Thus M_2 is Co-Quasi-Dedekind R-module.

Proposition 2.8: Let M_1 and M_2 be y-closed simple R-modues. If $M_1\ is\ M_2$ y-closed dual Rickart module, then either Hom(M_1, M_2)=0 or $M_1 \cong M_2$.

Proof. Assume that Hom $(M_1, M_2) \neq 0$ and let $0 \neq f: M_1 \to M_2$ be an R-homomorphism. Since M_1 is M_2 y-closed dual Rickart module, then by Proposition (2.7), f is an epimorphism. Now consider the following short exact sequence

$$0 \longrightarrow kerf \xrightarrow{i} M_1 \xrightarrow{f} M_2 \longrightarrow 0$$

where i is the inclusion map. Since M_2 is nonsingular, then $M_2 \cong \frac{M_1}{kerf}$ is nonsingular. Hence $kerf$ is y-closed submodule of M_1. But M_1 *is* y-closed simple and $M_1 \neq kerf$, therefore $kerf = 0$. Thus $M_1 \cong M_2$.

§3 DIRECT SUM OF Y-CLOSED DUAL RICKART MODULES

In this section, we study the direct sum of the y-closed dual Rickart modules. we begin with the following theorem .

Theorem 3.1: Let M *and* N be two R-modules such that $M = A \oplus B$. If M is N-y-closed dual Rickart module, then A *is* L-y-closed dual Rickart module, for every submodule L of N.

Proof. Let M be N-y-closed dual Rickart module and $f: A \to L$ be an R-homomorphism. Let $p: M \to A$ be the projection map and $i: L \to N$ be the inclusion map. Consider the map $(i \circ f \circ p): M \to N$. Since M is N-y-closed dual Rickart, then $Im(i \circ f \circ p)$ is a y-closed submodule of N. But

$$\begin{aligned} Im(i \circ f \circ p) &= \{ i \circ f \circ p(x),\ x \in M\} \\ &= \{ i(f(p(a+b))),\quad a \in A, b \in B\} \\ &= \{ f(a)\ ,\ a \in A\} = Imf \end{aligned}$$

Therefore $Im(i \circ f \circ p) = Imf$ is a y-closed submodule of N. Hence Imf is a y-closed of L. Thus A *is* L-y-closed dual Rickart module.

Proposition 3.2: Let $M = \oplus_{i\in I} M_i$ and $N = \oplus_{i\in I} N_i$ be two R-modules, such that $f(M_i) \subseteq N_i, \forall i \in I$. Then M is N-y-closed dual Rickart module if and only if each M_i is N_i -y-closed dual Rickart module.

Proof. ⇒ Clear by Propositions (2.4) and (3.1)

For the converse, let f: $M \to N$ be an R-homomorphism. We want to show that Imf is a y-closed submodule of N. Since $f(M_i) \subseteq N_i, \forall i \in I$, then we can consider $f|_{M_i}: M_i \to N_i\ \forall i \in I$. First, claim that $Im(f|_{M_i}) = Imf \cap N_i, \forall i \in I$. To show that, let $f(x_i) \in Im(f|_{M_i})$, $x_i \in Mi$, then $f(x_i) \in (Imf \cap N_i)$. Now let $f(x) \in (Imf \cap N_i)$. Then $x = \sum_{j\in I} x_j$, where $x_j \in Mi$, for each $j \in I$ and $x_j \neq 0$ for at most a finite number of $j \in I$. Now $f(x) = f(\sum_{j\in I}(x_j)) = \sum_{j\in I} f(x_j) \in \oplus_{j\in I} N_j$. But $f(x) \in N_i$. Therefore $f(x_j) = 0,\ \forall j \neq i$ and $f(x) = f(x_j)$. Hence $f(x) \in Im(f|_{M_i})$. Thus $Im(f|_{M_i}) = Imf \cap N_i, \forall i \in I$. Claim that $Imf = \oplus_{i\in I} (Imf \cap N_i)$. To show that, let $f(x) \in Imf, x \in M$. Then $x = \sum_{i\in I} x_i$, where $x_i \in M_i$, $\forall i \in I$ and $x_i \neq 0$, for at most a finite number of $i \in I$. Hence $f(x) = f(\sum_{i\in I} x_i) = \sum_{i\in I} f(x_i)$, for all $i \in I$. By our assumption $f(x_i) \in Imf \cap N_i,\ \forall i \in I$. Hence $f(x) \in \oplus_{i\in I} (Imf \cap N_i)$. Thus $Imf = \oplus_{i\in I} (Imf \cap N_i) = \oplus_{i\in I} Im(f|_{M_i})$. Since M_i is N_i -y-closed dual Rickart module, for each $i \in I$, then $Im(f|_{M_i})$ is a y-closed submodule of N_i and hence $\oplus_{i\in I} Im(f|_{M_i})$ is a y-closed submodule of N, by [4, proposition (2.1.20), p29]. So, Imf is a y-closed submodule of N. Thus M is N-y-closed dual Rickart module.

Proposition 3.3: Let M, N be two R-modules with the property that the sum of any two y-closed submodule of N is a y-closed submodule of N. The following statements are equivalent

(a) M is a y-closed dual Rickart module,

(b) $\sum_{f\in I} f(M)$ is y-closed submodule of M, where I is a finitely generated left ideal of $End_R(M)$.

Proof. $(\boldsymbol{a}) \Rightarrow (\boldsymbol{b})$. Let $I = (f_1, \dots, f_n)$ be a finitely generated left ideal of $End_R(M)$. Since M is a y-closed dual Rickart module, then $Im(f_j)$ is a y-closed submodule of N, $\forall\ 1 \leq j \leq n.$ But $Im(f_j) = Im(f_1 + \cdots + f_n)$. Hence $\sum_{j=1} f_j(M)$ is a y-closed submodule of N.

$(\boldsymbol{b}) \Rightarrow (\boldsymbol{a})$. Clear.

Recall that an R-module M is called a faithful module if $ann(M) = 0$, where $ann(M) = \{r \in R \mid rx = 0, \forall x \in M\}$, see [5, p206].

Before we give our next result, let us recall that an R-module M is called dualizable if $Hom(M, R) \neq 0$, see [6, p10] .

Proposition 3.4: Let M be a y-closed simple, faithful R-modue. If M is y-closed dual Rickart module. Then M is divisible.

Proof. Suppose that M is y-closed simple, faithful and y-closed dual Rickart module. Let R must be commutative. $f(m) = rm$, $\forall m \in M$. R must be considered not zero divisor. It is clear that f is an R-homomorphism. Since $M$is a y-closed dual Rickart module, then $Imf = rM$ is a y-closed submodule of M. Since M is a faithful module, then $rM \neq 0$. But M is y-closed simple, therefore $rM = M$. Thus M is divisible.

Recall that an R-module M is called $1/2$ cancellation module if it is faithful and for any ideal A of R such that $AM = M$ implies $A = R$, see [7] .

Proposition 3.5: Let M be a faithful, finitely generated and y-closed simple R-module, where R is not a field. Then M is not y-closed dual Rickart module.

Proof. Assume that M is a y-closed dual Rickart module and let $0 \neq r \in R$ such that $R \neq (r)$. Define $f: M \to M$ by $f(m) = rm$, $\forall m \in M$. It is clear that f is an epimorphism, then $Imf = rM$ is a y-closed sbmodule of M. Since M is a faithful module, then $rM \neq 0$. But M is an y-closed simple module, therefore $rM = M$. Since M is finitely generated and faithful, then M is $1/2$ cancellation, by [7]. So, $R = (r)$, which is a contradiction. Thus M is not y-closed dual Rickart module.

Proposition 3.6: Let M be an R-module such that R is M-y-closed dual Rickart module. Then every cyclic submodule of M is a y-closed submodule.

Proof. Suppose that M is an R-module such that R *is* M-y-closed dual Rickart module and let $0 \neq m \in M$. Define $f: R \to Rm$ by $f(r) = rm$, $r \in R$. Let $i: Rm \to M$ be the inclusion map. Consider the map $i \circ f: R \to M$. It is clear that $Im(i \circ f) = Rm$. Since R is M-y-closed dual Rickart, then $Im\, i \circ f$ is a y-closed submodule of M. Thus Rm is a y-closed submodule of M.

Recall that an R-module M is called y-extending if for any submodule A of M there exists a direct summand K of M such that $A \cap K$ is essential in A and $A \cap K$ is essential in K, see [8] .

Proposition 3.7: Let M be a y-extending R-module. If $\oplus_I R$ is M-y-closed dual Rickart module, for every index set I, then M is a semisimple module.

Proof. Let N be a submodule of M and let $\{n_\propto; \propto \in \Lambda\}$ be a set of generators of N. For each $\propto \in \Lambda$, define $f_\propto: R \to Rn_\propto$ by $f_\propto(r) = rn_\propto$,$\forall r \in R$. Now define $f: \oplus_I R \to N$ by $((r_\propto)_{\propto \in \Lambda}) = \sum r_\propto n_\propto$. Its is clear that f is an epimorphism. Let $i: N \to M$ be the inclusion map. Consider $i \circ f: \oplus_I R \to M$. Since $\oplus_I R$ *is* M-y-closed dual Rickart module, then $Im\, i \circ f = N$ is a y-closed submodule of M. But M is a y-extending module, therefore N is a direct summand of M. Thus M is semisimple.

Proposition 3.8: Let M be y-extending and a self-generator R-module. If $\oplus_I M$ is a y-closed dual Rickart module for every index set I, then M is semisimple.

Proof. Let N be a submodule of M. Since M is a self-generator, then there exists a family $\{f_\propto\}_{\propto \in \Lambda}$ where $f_\propto: M \to N$ is an R-homomorphism such that $\sum_{\propto \in \Lambda} Imf_\propto = N$. Define $f: \oplus M_{\propto \in \Lambda} \to N$ by $f((m_\propto)_{\propto \in \Lambda} = \sum_{\propto \in \Lambda} f_\propto(m_\propto)$. It is clear that f is an epimorphism. Let $i: N \to M$ be the inclusion map. Consider the map $i \circ f: \oplus_{\propto \in \Lambda} M \to M$. Since $\oplus_{\propto \in \Lambda} M \oplus M$ is a y-closed dual Rickart module, then $Im\, i \circ f = Imf = N$ is a y-closed submodule of M. But M is a y-extending module, therefore N is a direct summand of M. Thus M is semisimple.

Now, we give the following characterization.

Theorem 3.9: Let M_1 and M_2 be two R-modules. Then the following statements are equivalent.

(1) M_1 is M_2-y-closed dual Rickart module;

(2) For every submodule N of M_2 , every direct summand K of M_1 is N-y-closed dual Rickart module;

(3) For every direct summand K *of* M_1, every y-closed submodule L of M_2 , and for every $f \in Hom_R(M_1, L)$, the Image of the restricted map $f|_K$ is a y-closed submodule of K.

Proof. $(\mathbf{1}) \Rightarrow (\mathbf{2})$ Let K be a direct summand of M_1, N be a submodule of M_2, and $f: K \to N$ be an R-homomrphism. Let $M = K \oplus K_1$ for some submodule $K_1$of M. Define $g: M_1 \to M_2$ by

$$g(x) = \left\{ \begin{array}{ll} f(x), & if\ x \in K \\ 0 & if\ x \in K_1 \end{array} \right\}$$

Clearly, g is an R-homomrphism. Since M_1 is M_2-y-closed dual Rickart module, then Img is a y-closed submodule of M_2 and hence $\frac{M_2}{Img}$ is nonsinular. But,
$Img = \{ g(a+b), \quad a \in K, \quad b \in K_1\} = \{ f(a), \quad a \in K\} = Imf$. So Imf is a y-closed submodule of M_2. Hence $\frac{M_2}{Imf}$ is nonsingular. But $\frac{N}{Imf}$ is a submodule of $\frac{M_2}{Imf}$, therefore $\frac{N}{Imf}$ is nonsingular. Thus Imf is a y-closed submodule of N.
$(\mathbf{2}) \Rightarrow (\mathbf{3})$. Let K be a direct summand of M_1 and L is y-closed submodule of M_2. Let $f: M_1 \to L$ be R-homomrphism. Since $f|_K: K \to L$ and K is L-y-closed dual Rickart module, then $Im(f|_K)$ is a y-closed submodule of L.
$(\mathbf{3}) \Rightarrow (\mathbf{1})$ Let $f\colon M_1 \to M_2$ be an R-homomrphism. Take $K = M_1$ and $L = M_2$. Since $f|_K: K \to L$ and L is a y-closed submodule of M_2, therefore Imf is a y-closed submodule of M_2. Thus M_1 is M_2-y-closed dual Rickart module.

Remark 3.10: let M and N be two R-modules and $f: M \to N$ be an R-homomorphism. Let $A_M = M \oplus 0$, $B_N = 0 \oplus N$, $\bar{f}: A_M \to B_N$ be a map defined by $\bar{f}(m, 0) = (0, f(m))$, for every $m \in M$ and $T_f = \{x + \bar{f}(x), x \in A_M\}$. Then,

1- $M \oplus N = A_M \oplus B_N$

2-$\bar{f}$ is an R-homomorphism

3- $ker\bar{f} = kerf \oplus 0$

4- T_f is a submodule of $M \oplus N$

5- $A_M + T_f = A_M \oplus Im\bar{f}$.

In this paper, by $A_M, B_M, \bar{f}, T_f$, we mean the same concepts in the previous above remark.

Now, we will give characterization for the notion that M is N-y-closed dual Rickart module.

Theorem 3.11: Let M and N be two R-modules. Then M is N-y-closed dual Rickart module if and only if, for every homomorphism $f: M \to N$, $A_M + T_f$ is a y-closed of $M \oplus N$.

Proof. Let $f: M \to N$ be an R-homomorphism. Since M is N-y-closed dual Rickart, then Imf is y-closed submodule of N and so $0 \oplus Imf$ is y-closed submodule of $0 \oplus N$. Therefore $Im\bar{f}$ is y-closed submodule of $0 \oplus N$. Hence $A_M \oplus Im\bar{f}$ is y-closed submodule of $A_M \oplus B_N$. So $A_M \oplus Im\bar{f}$ is y-closed submodule of $M \oplus N$. By the same argument of the proof of the theorem in [9, Theorem(2.2)], $A_M \oplus Im\bar{f} = A_M + T_f$. Thus $A_M + T_f$ is a y-closed sumodule of $M \oplus N$.

For the converse, Let $f: M \to N$ be R-homomrphism. Since $A_M + T_f$ is y-closed submodule of $M \oplus N$, and $A_M + T_f = A_M \oplus Im\bar{f}$, therefore $\frac{M \oplus N}{A_M + T_f} = \frac{A_M \oplus B_N}{A_M \oplus Im\bar{f}} \simeq \frac{B_N}{Im\bar{f}}$ is nonsingular. Therefore $Im\bar{f}$ is y-closed submodule of $0 \oplus N$. Hence $\frac{0 \oplus N}{Im\bar{f}} = \frac{0 \oplus N}{0 \oplus Imf} \simeq \frac{N}{Imf}$ is nonsingular. So Imf is y-closed submodule of M. Thus M is N-y-closed dual Rickart module.

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