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On y-closed Rickart Modules

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Abstract

In a previous work, Ali and Ghawi studied closed Rickart modules. The main purpose of this paper is to define and study the properties of y-closed Rickart modules .We prove that, Let M and N be two R-modules such that N is singular. Then M is N-y-closed Rickart module if and only if $\mathrm{Hom}(M, N) = 0$. Also, we study the direct sum of y-closed Rickart modules.

Keywords: y-closed submodule, y-closed simple, y- closed Rickart modules.

حول المقاسات الريكارتية المغلقة من النمط−y

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الخلاصة

درس الباحثان علي و غاوي المقاسات الريكارتية المغلقة. الهدف الرئيسي من هذا البحث هو تعريف ودراسة خواص المقاسات الريكارتية المغلقة من النمط −y . برهنا على انه اذا كان M و N اي مقاسين بحيث ان N شاذه فان المقاس الريكارتي M يكون مغلق من النمط −y بالنسبة ل N اذا وفقط اذا $\mathrm{Hom}(M, N) = 0$.وايضا درسنا الجمع المباشر للمقاسات الريكارتية المغلقة من النمط −y.

1.INTRODUCTION

A module M is called closed Rickart if for any $f \in \mathrm{End}(M)$, $\mathrm{ann}_M(f) = \mathrm{Ker}f$ is closed submodule of M [1]. Recall that a submodule A of an R-module M is called a y-closed submodule of M if $\frac{M}{A}$ is nonsingular [2]. It is known that every y-closed submodule is closed.

In this paper, we give some results on the y-closed Rickart modules .

In §2, we give the definition of the y-closed Rickart modules with some examples and basic properties. For example, we prove that for two R-modules M and N such that N is nonsingular module, then M is N-y-closed Rickart module, see proposition (2.3).

In section 3, we study the direct sum of y-closed Rickart module. For example, we prove that for two R-modules M and N such that $M = A \oplus B$, where A and B are submodules of M. If M is N-y-closed Rickart module, then A is N-y-closed Rickart module, see Theorem (3.1).

Throughout this article, R is a ring with identity and M is a unitary left R-module. $S = \mathrm{End}_R(M)$ will denote the endomorphism ring of M.

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§2: Y-Closed Rickart Modules

In this section, we introduce the definition of y-closed Rickart module. Also we give some basic properties of this concept.

Definition 2.1: Let M and N be two R-modules. We say that M is N-y-closed Rickart module if for each $f \in End(M, N)$, $ann_M(f) = Kerf$ is a y-closed submodule of M.
For a module M, if M is M-y-closed Rickart module, then we say that M is y-closed Rickart module.

Examples 2.2:
1- Consider the modules Z and Q as Z-modules. Then Z is Q-y-closed Rickart module.To show that, let $f: Z \to Q$ be an R-homomorphism, by the first isomorphism theorem $\frac{Z}{Kerf} \simeq Imf$. Since Q is nonsingular, then Imf is nonsingular. Therefore Kerf is a y-closed submodule of Z. Thus Z is Q-y-closed Rickart module.
2- Consider the modules Z_4 and Z_2 as Z-modules and let $f: Z_4 \to Z_2$ be a map defined by $f(x) = 3x$, $\forall x \in Z_4$. Hence $Kerf = \{ x \in Z_4, \ f(x) = \bar{0}\} = \{\bar{0}, \bar{2}\}$. But $\frac{Z}{\{\bar{0},\bar{2}\}} \cong Z_2$ and Z_2 singular as Z-module. Thus Z_4 is not Z_2-y-closed Rickart module.

Note : A Rickart (closed Rickart) module needs not to be a y-closed Rickart module. For example, the module Z_6 as Z-module is a Rickart (closed Rickart) module, where Z_6 is semisimple. We claim that Z_6 is not y-closed Rickart module. To verify this, let $f: Z_6 \to Z_6$ be a map defined by $f(x) = 3x, \forall x \in Z_6$. Clearly, f is an R-homomorphism and $Kerf = \{ x \in Z_4, \ f(x) = 0\} = \{\bar{0}, \bar{2}, \bar{4}\}$. By the first isomorphism theorem, $\frac{Z_6}{\{\bar{0},\bar{2},\bar{4}\}} \cong Z_2$ and Z_2 singular as Z-module. Thus Z_6 is not y-closed Rickart module.

Proposition 2.3: Let M and N be two R-modules such that N is nonsingular module. Then M is N-y-closed Rickart module.

Proof: Let $f: M \to N$ be an R-homomorphism. Since N is nonsingular and Imf is a submodule of N, then Imf is nonsingular module. By the first isomorphism theorem, $\frac{M}{Kerf} \cong Imf$. Therefore $\frac{M}{Kerf}$ is nonsingular. Hence Kerf is a y-closed of M. Thus M is N a y-closed Rickart module.

Corollary 2.4: Let R be an integral domain and let M be torsion free R-module. Then M is a y-closed Rickart module.

No, we give the following characterization.

Propositions 2.5: Let M and N be two R-modules. Then M is N-y-closed Rickart module if and only if, for every R-homomorphism $f: M \to N$, Imf is a nonsingular module.

Proof: Let M be N-y-closed Rickart module and let $f: M \to N$ be an R-homomorphism. Since M is N-y-closed Rickart module, then Kerf is a y-closed submodule of M and hence $\frac{M}{kerf}$ is nonsingular. By the first isomorphism theorem, $\frac{M}{kerf} \cong Imf$. Thus Imf is nonsingular.

Conversely, let $f: M \to N$ be an R-homomorphism. Since Imf is nonsingular and $\frac{M}{kerf} \cong Imf$, then $\frac{M}{kerf}$ is nonsingular. Therefore Kerf is a y-closed submodule of M. Thus M is N-y-closed Rickart module.

Recall that a module M is said to be K-nonsingular if for every homomorphism $f: M \to M$ such that kerf is essential in M, implies $f = 0$ [1].

Proposition 2.6: Every y-closed Rickart module is K-nonsigular.

Proof: Suppose that M is a y-closed Rickart module and let $f: M \to M$ be an R-homomorphism such that kerf is essential in M. Then $\frac{M}{kerf}$ is singular, by [2]. But M is a y-closed Rickart module, therefore kerf is a y-closed submodule of M, which implies that $kerf = M$ and so $f = 0$. Thus M is K-nonsigular.

Propositions 2.7: Let M and N be two R-modules such that N is singular. Then M is N-y-closed Rickart module if and only if $Hom(M, N) = 0$.

Proof: Assume that M is N-y-closed Rickart module and let $f: M \to N$ be an R-homomorphism. Then Kerf is a y-closed submodule of M and hence $\frac{M}{kerf}$ is nonsingular. So Imf is nonsingular. But N is singular, therefore $Imf = 0$. Thus $Hom(M, N) = 0$.

The converse is clear .

Corollary 2.8: Let A be a proper essential submodule of a module M. Then M is not $\frac{M}{A}$ –y-closed Rickart module.

Proof. Since A is an essential submodule of M, then by [2], $\frac{M}{A}$ is a singular module. Let $\pi: M \to \frac{M}{A}$ be the natural epimorphism. It is clear that $0 \neq \pi \in Hom\left(M, \frac{M}{A}\right)$ .Thus by Proposition (2.7) M is not $\frac{M}{A}$ –y-closed Rickart module.

§3 DIRECT SUM OF Y-CLOSED RICKART MODULES

In this section, we study the direct sum of the y-closed Rickart modules. We begin with the following theorem .

Theorem 3.1: Let M and N be two R-modules such that $M = A \oplus B$, where A and B are submodules of M. If M is N-y-closed Rickart module, then A is N-y-closed Rickart module.

Proof. Let $\psi: A \to N$ be an R-homomorphism and let $p: M \to A$ be the projection map. Consider the map $\psi \circ p: M \to N$. Since M is N-y-closed Rickart module, then $Ker(\psi \circ p)$ is a y-closed submodule of M. But

$$\begin{aligned} ker(\psi \circ p) &= \{x \in M,\ \psi \circ p(x) = 0\} \\ &= \{a + b \in A \oplus B,\ (\psi(p(a + b))) = 0,\ a \in A, b \in B\} \\ &= \{a + b \in A \oplus B,\ \psi(a) = 0,\ a \in A, b \in B\} \\ &= ker\psi \oplus B \end{aligned}$$

Therefore $\frac{M}{ker\psi \oplus B} = \frac{A \oplus B}{ker\psi \oplus B} \cong \frac{A}{ker\psi}$ is nonsingular. So $Ker\psi$ is a y-closed submodule of A. Thus A is N-y-closed Rickart module.

Propositions 3.2: Let $M = \bigoplus_{i \in I} M_i$ and $N = \bigoplus_{i \in I} N_i$ be two R-modules, such that for every $f \in Hom(M, N), f(M_i) \subseteq N_i, \forall i \in I$. If M_i is N_i -y-closed Rickart module, $\forall i \in I$, then M is N-y-closed Rickart module.

Proof. Assume that M_i is N_i -y-closed Rickart module, $\forall i \in I$, and let $f: M \to N$ be an R-homomorphism. We want to show that kerf is a y-closed submodule of M. By our assumption, $f|_{M_i}: M_i \to N_i$, $\forall i \in I$. It is clear that $kerf|_{M_i} = kerf \cap M_i$, for each $i \in I$. We claim that $kerf = \bigoplus_{i \in I} (kerf|_{M_i})$. To show that, let $x \in Kerf$. Then $x = \sum_{i \in I} x_i$, where $x_i \in Mi$, for each $i \in I$ and $x_i \neq 0$ for at most a finite number of $i \in I$ and $f(x) = 0$. Then $f(x) = f(\sum_{i \in I} x_i) = \sum_{i \in I} f(x_i) = 0$, where $f(x_i) \in N_i$. But $N = \bigoplus_{i \in I} N_i$. Therefore $f(x_i) = 0$, $\forall i \in I$. So $x_i \in (Kerf \cap M_i)$, $\forall i \in I$ and hence $x = \sum_{i \in I} x_i \in \bigoplus_{i \in I} Ker(f|_{M_i})$. Thus $Kerf = \bigoplus_{i \in I} Ker(f|_{M_i})$. Since M_i is N_i-y-closed Rickart module for each $i \in I$, then $Ker(f|_{M_i})$is a y-closed submodule of M_i. Therefore $Kerf = \bigoplus_{i \in I} Ker(f|_{M_i})$ is a y-closed submodule of M, by [3]. Thus M is N-y-closed Rickart module.

Let M be an R-module, then M is called a y-closed simple if M and 0 are the only y-closed submodules of M.

Example 3.3:

1- The module Z as Z-module is a y-closed simple module, where $\frac{Z}{nZ} \simeq Z_n$, $\forall n \geq 2$ and Z_n is singular as Z-module. Thus nZ is not y-closed submodule of Z, $\forall n \geq 2$.

2- The module Z_6 as Z-module is not y-closed simple module, where $\frac{Z_6}{\{\bar{0}\}} \simeq Z_6$ and Z_6 as Z-module is singular. Hence the submodule $\{\bar{0}\}$ of Z_6 is not y-closed submodule.

Propositions 3.4: Let M be a y-closed simple R-module and let N be an R-module. If M is N-y-closed Rickart, then either
(1) $\mathrm{Hom}(M, N)=0$ or
(2) Every nonzero R-homomorphism from M to N is a monomorphism.
Proof. Assume that $\mathrm{Hom}(M, N) \neq 0$ and let $f: M \rightarrow N$ be a non-zero R-homomorphism. Since M is N-y-closed Rickart, then kerf is y-closed submodule of M. But M is y-closed simple, therefore $\mathrm{kerf} = \{0\}$ and f is a monomorphism.

Recall that an R-module M is called a Quasi-Dedekind R-module if every nonzero endomorphism of M is a monomorphism [4 , Th(1.5) , CH2].
Corollary 3.5: Let M be a y-closed simple R-module and let N be any R-module such that $\mathrm{Hom}(M, N) \neq 0$. If M is N-y-closed Rickart module, then M is Quasi-Dedekind. In particular, if M is y-closed Rickart, then M is Quasi-Dedekind.
Proof. By Proposition (3.4), there is a monomorphism $f: M \rightarrow N$. Assume that M is not Quasi-Dedekind R-module. So there exists a homomorphism $g: M \rightarrow M$ such that $\mathrm{Kerg} \neq 0$. Since f is a monomorphism, then $\mathrm{Ker}(f \circ g) = \mathrm{Kerg} \neq 0$. But M is N-y-closed Rickart module, therefore $\mathrm{Kerf} \circ g = \mathrm{Kerg}$ is a y-closed submodule of M. So $\mathrm{Kerg} = M$, where M is a y-closed simple. Thus $g = 0$, which is a contradiction. Thus M is a Quasi-Dedekind R-module.
Proposition 3.6: Let M be an R-module. If R is M-y-closed Rickart module, then every cyclic submodule of M is projective. In particular, if R is y-closed Rickart ring, then every principal ideal is projective, i.e. , R is a principal projective ring.
Proof. Let M be an R-module such that R is M-y-closed Rickart module and let $m \in M$. Now consider the following short exact sequence

$$0 \longrightarrow \mathrm{kerf} \xrightarrow{i} R \xrightarrow{f} Rm \longrightarrow 0$$

where i is the inclusion homomorphism and f is a map defined by $f(r) = rm, \forall r \in R$. It is clear that f is an epimorphism. Let $i_2: Rm \rightarrow M$ be the inclusion map. Since R is M-y-closed Rickart module and $i_2 \circ f: R \rightarrow M$, then $\mathrm{Ker}(i_2 \circ f)$ is a y-closed ideal of R. But i_2 is a monomorphism, therefore $\mathrm{Ker}(i_2 \circ f) = \mathrm{kerf}$ is a y-closed ideal of R. Hence $\frac{R}{\mathrm{Kerf}}$ is nonsingular. By the first isomorphism theorem, $\frac{R}{\mathrm{Kerf}} \simeq Rm$. So Rm is nonsingular, by [2,corollary(1.25),p35]. Thus Rm is projective.

Recall that an R-module M is called dualizable if $\mathrm{Hom}(M, R) \neq 0$ [5].
Corollary 3.7: Let M be a y-closed simple dualizable R-module. If M is R-y-closed Rickart module, then M is isomorphic to an ideal of R. Hence, if R has nonzero nilpotent elements, then End(M) is commutative.
Proof. Since $\mathrm{Hom}(M, R) \neq 0$, then by Proposition (3.4), M is isomorphic to an ideal I of R and hence $\mathrm{End}(M) \cong \mathrm{End}(I)$. For the second part, since R has no nonzero elements and I is an ideal in R, then End(I) is commutative [6, propositon(2.1),CH1]. Thus End(M) is commutative.

Recall that an R-module M is called a multiplication module if for each submodule N of M there exists an ideal I of R such that $N = IM$, [6] .
Corollary 3.8: Let M be a y-closed simple projective R-module and R has no nonzero nilpotent element. If M is R-y-closed Rickart module and $\mathrm{Hom}(M, R) \neq 0$, then M is a multiplication module.
Proof. By the same argument of the proof of Corollary (3.7), End(M) is a commutative
and hence M is a multiplication [7].
Proposition 3.9: Let M be an R-module with the property that the intersection of any two y-closed submodules of M is a y-closed submodule of M. Then the following statements are equivalent.
(a) M is a y-closed Rickart module,
(b) The left annihilator in M of every left finitely generated ideal $I = (f_1, \ldots, f_n)$ of $\mathrm{End}_R(M)$ is a y-closed submodule of M.
Proof. (**a**) ⇒ (**b**) Let $I = (f_1, \ldots, f_n)$ be a left finitely generated ideal of the $\mathrm{End}_R(M)$. Since M is a y-closed Rickart module, then $\mathrm{ann}_M(f_j)$ is a y-closed submodule of M, $\forall\, 1 \leq j \leq n$. Hence

$\bigcap_{j=1}^{n} ann_M(f_j)$ is a y-closed submodule of M, by [3]. But $ann_M(I) = ann_M(Sf_1 + \cdots + Sf_n) = \bigcap_{j=1}^{n} ann_M(Sf_j)$. Therefore $ann_M(I)$ is y-closed submodule of M.
(**b**) ⇒ (**a**) Clear.

Now, we give the following characterization.

Theorem 3.10: Let M_1 and M_2 be two R-modules. Then the following statements are equivalent.
(1) M_1 is M_2-y-closed Rickart module;
(2) For every submodule N of M_2, every direct summand K of M_1 is N-y-closed Rickart;
(3) For every direct summand K of M_1, every y-closed submodule L of M_2 and every $f \in Hom_R(M, L)$. The kernel of the restricted map $f|_K$ is a y-closed submodule of K.

Proof. (1) ⇒ (2) Let N be submodule of M_2. Let K be a direct summand of M_1 and let $f: K \to N$ be an R-homomrphism. Then $M_1 = K \oplus K_1$, for some submodule K_1 of M. Let $g: M_1 \to M_2$ be a map defined by

$$g(x) = \begin{cases} f(x), & \text{if } x \in K \\ 0 & \text{if } x \in K_1 \end{cases}$$

It is clear that g is an R-homomrphism. Since M_1 is M_2-y-closed Rickart module, then Kerg is a y-closed submodule of M_1. But

$$\begin{aligned} Kerg &= \{a + b \in M_1,\ g(a + b) = 0,\quad a \in K, b \in K_1\} \\ &= \{a + b \in M_1,\ f(a) = 0\quad , a \in K, b \in K_1\} \\ &= kerf \oplus K_1 \end{aligned}$$

Therefore $kerf \oplus K_1$ is a y-closed submodule of M_1 and hence $\frac{M_1}{kerf \oplus K_1}$ is nonsingular. But $\frac{M_1}{kerf \oplus K_1} = \frac{K \oplus K_1}{kerf \oplus K_1} \cong \frac{K}{kerf}$, so Kerf is a y-closed submodule of K. Thus K is N-y-closed Rickart module.

(2) ⇒ (3) Let K be a direct summand of M_1 and L be a submodule of M_2. Let $f: M_1 \to L$ be an R-homomrphism. Consider the map $f|_K: K \to L$. Since K is L-y-closed Rickart module, then $Kerf|_K$ is a y-closed submodule of K.

(3) ⇒ (1) Let $f: M_1 \to M_2$ be an R-homomrphism. Take $L = M_2$ and $K = M_1$. Since $f|_K: K \to L$ and K is L-y-closed Rickart module, therefore Kerf is a y-closed submodule of M_1. Thus M_1 is M_2-y-closed Rickart module.

Remark 3.11: Let M and N be two R-modules and $f: M \to N$ be an R-homomorphism. Let $A_M = M \oplus 0$, $B_N = 0 \oplus N$, $\bar{f}: A_M \to B_N$ be a map defined by $\bar{f}(m, 0) = (0, f(m))$, for every $m \in M$ and
$T_f = \{x + \bar{f}(x), x \in A_M\}$. Then :
1- $M \oplus N = A_M \oplus B_N$
2- $\bar{f}$ is an R-homomorphism
3- $ker\bar{f} = kerf \oplus 0$
4- T_f is a submodule of $M \oplus N$
5- $A_M + T_f = A_M \oplus Im\bar{f}$.

In the following theorem by A_M, B_M, $\bar{f}$, T_f, we mean the same concepts in the previous above Remark.

Now, we give another characterization for the relative y-closed Rickart module.

Theorem 3.12: Let M and N be two R-modules. Then M is N-y-closed Rickart module if and only if for every homomorphism $f: M \to N$, $A_M \cap T_f$ is y-closed submodule of A_M.

Proof. Let $f: M \to N$ be an R-homomorphism. Since M is N-y-closed Rickart module, then Kerf is a y-closed submodule of M and hence $\frac{M}{Kerf}$ is nonsingular. Then $\frac{A_M}{Ker\bar{f}} = \frac{M \oplus 0}{Kerf \oplus 0} \simeq \frac{M}{kerf}$ is nonsingular. So $Ker\bar{f}$ is a y-closed submodule of A_M. By the same argument of the proof of the [8,Theorem(2.2)], $Ker\bar{f} = A_M \cap T_f$.

For the converse, let $f: M \to N$ be an R-homomorphism. Then by our assumption, $A_M \cap T_f$ is a y-closed submodule of A_M. Since $Ker\bar{f} = A_M \cap T_f$, then $Ker\bar{f}$ is a y-closed submodule of A_M and hence $\frac{A_M}{Ker\bar{f}}$ is nonsingular. Therefore $\frac{M \oplus 0}{Kerf \oplus 0} \cong \frac{M}{Kerf}$ is nonsingular. So kerf is a y-closed submodule of M. Thus M is N-y-closed Rickart module.

But, we have the following.

Theorem 3.13: Let M and N be two R-modules and let $f: M \to N$ be an R-homomorphism.Then M is N-y-closed Rickart module if and only if T_f is y-closed submodule of $A_M + T_f$.

Proof. Let $f: M \to N$ be an R-homomorphism. Now consider the following short exact sequences:

$$0 \longrightarrow A_M \cap T_f \xrightarrow{i_1} A_M \xrightarrow{\pi_1} \frac{A_M}{A_M \cap T_f} \longrightarrow 0$$

$$0 \longrightarrow T_f \xrightarrow{i_2} A_M + T_f \xrightarrow{\pi_2} \frac{A_M + T_f}{T_f} \longrightarrow 0$$

where i_1, i_2 are the inclusion homomorphisms and π_1, π_2 are the natural epimorphisms. Since M is N-y-closed Rickart, then kerf is y-closed submodule of M and hence $\frac{M}{\mathrm{Kerf}}$ is nonsingular. So $\frac{A_M}{\mathrm{Ker}\bar{f}} = \frac{M \oplus 0}{\mathrm{Kerf} \oplus 0} \simeq \frac{M}{\mathrm{kerf}}$ is nonsingular. Thus $\mathrm{Ker}\bar{f} = A_M \cap T_f$ is a y-closed submodule of A_M. Hence $\frac{A_M}{A_M \cap T_f}$ is nonsingular. By the second isomorphism theorem, $\frac{A_M}{A_M \cap T_f} \simeq \frac{A_M + T_f}{T_f}$ is nonsingular. Thus T_f is a y-closed submodule of $A_M + T_f$.

For the converse, let $f: M \to N$ be an R-homomorphism. Consider the following short exact sequences:

$$0 \longrightarrow A_M \cap T_f \xrightarrow{i_1} A_M \xrightarrow{\pi_1} \frac{A_M}{A_M \cap T_f} \longrightarrow 0$$

$$0 \longrightarrow T_f \xrightarrow{i_2} A_M + T_f \xrightarrow{\pi_2} \frac{A_M + T_f}{T_f} \longrightarrow 0$$

where i_1, i_2 are the inclusion homomorphisms and π_1, π_2 are the natural epimorphisms. By the second isomorphism theorem, $\frac{A_M}{A_M \cap T_f} \simeq \frac{A_M + T_f}{T_f}$. Since T_f is y-closed submodule of $A_M + T_f$, then $\frac{A_M + T_f}{T_f}$ is nonsingular, therefore $\frac{A_M}{A_M \cap T_f}$ is nonsingular. Hence $A_M \cap T_f$ is a y-closed submodule of A_M. So $\mathrm{Ker}\bar{f} = \mathrm{Kerf} \oplus 0$ is a y-closed submodule of $A_M = M \oplus 0$. Thus kerf is y-closed submodule of M.

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