On y-closed Rickart Modules

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Abstract
In a previous work, Ali and Ghawi studied closed Rickart modules. The main purpose of this paper is to define and study the properties of y-closed Rickart modules. We prove that, Let M and N be two R-modules such that N is singular. Then M is N-y-closed Rickart module if and only if Hom(M,N) = 0. Also, we study the direct sum of y-closed Rickart modules.

Keywords: y-closed submodule, y-closed simple, y-closed Rickart modules.

1. INTRODUCTION
A module M is called closed Rickart if for any \( f \in \text{End}(M) \), \( \text{ann}_M(f) = \text{Ker}(f) \) is closed submodule of M [1]. Recall that a submodule A of an R-module M is called a y-closed submodule of M if \( M/A \) is nonsingular [2]. It is known that every y-closed submodule is closed.

In this paper, we give some results on the y-closed Rickart modules.

In §2, we give the definition of the y-closed Rickart modules with some examples and basic properties. For example, we prove that for two R-modules M and N such that N is nonsingular module, then M is N-y-closed Rickart module, see proposition (2.3).

In section 3, we study the direct sum of y-closed Rickart module. For example, we prove that for two R-modules M and N such that \( M = A \oplus B \), where A and B are submodules of M. If M is N-y-closed Rickart module, then A is N-y-closed Rickart module, see Theorem (3.1).

Throughout this article, R is a ring with identity and M is a unitary left R-module. \( S = \text{End}_R(M) \) will denote the endomorphism ring of M.

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§2: Y-Closed Rickart Modules

In this section, we introduce the definition of y-closed Rickart module. Also we give some basic properties of this concept.

**Definition 2.1:** Let M and N be two R-modules. We say that M is N-y-closed Rickart module if for each \( f \in \text{End}(M, N) \), \( \text{ann}_M(f) = \text{Ker}f \) is a y-closed submodule of M.

For a module M, if M is M-y-closed Rickart module, then we say that M is y-closed Rickart module.

**Examples 2.2:**
1- Consider the modules \( Z \) and \( Q \) as Z-modules. Then \( Z \) is Q-y-closed Rickart module. To show that, let \( f: Z \to Q \) be an R-homomorphism, by the first isomorphism theorem, \( \frac{Z}{\text{Ker}f} \cong \text{Im}f \). Since Q is nonsingular, then \( \text{Im}f \) is nonsingular. Therefore \( \text{Ker}f \) is a y-closed submodule of \( Z \). Thus \( Z \) is Q-y-closed Rickart module.

2- Consider the modules \( Z_4 \) and \( Z_2 \) as Z-modules and let \( f: Z_4 \to Z_2 \) be a map defined by \( f(x) = 3x, \forall x \in Z_4 \). Hence \( \text{Ker}f = \{ x \in Z_4, f(x) = 0 \} = \{ 0, 2 \} \). But \( \frac{Z_{\{0,2\}}}{(0,2)} \cong Z_2 \) and \( Z_2 \) singular as Z-module. Thus \( Z_4 \) is not Z-y-closed Rickart module.

**Note:** A Rickart (closed Rickart) module needs not to be a y-closed Rickart module. For example, the module \( Z_6 \) as Z-module is a Rickart (closed Rickart) module, where \( Z_6 \) is semisimple. We claim that \( Z_6 \) is not y-closed Rickart module. To verify this, let \( f: Z_6 \to Z_6 \) be a map defined by \( f(x) = 3x, \forall x \in Z_6 \). Clearly, \( f \) is an R-homomorphism and \( \text{Ker}f = \{ x \in Z_4, f(x) = 0 \} = \{ 0, 2, 4 \} \). By the first isomorphism theorem, \( \frac{Z_6}{(0,2,4)} \cong Z_2 \) and \( Z_2 \) singular as Z-module. Thus \( Z_6 \) is not y-closed Rickart module.

**Proposition 2.3:** Let M and N be two R-modules such that N is nonsingular module. Then M is N-y-closed Rickart module.

**Proof:** Let \( f: M \to N \) be an R-homomorphism. Since N is nonsingular and \( \text{Im}f \) is a submodule of N, then \( \text{Im}f \) is nonsingular module. By the first isomorphism theorem, \( \frac{M}{\text{Ker}f} \cong \text{Im}f \). Therefore \( \frac{M}{\text{Ker}f} \) is nonsingular. Hence \( \text{Ker}f \) is a y-closed of M. Thus M is N a y-closed Rickart module.

**Corollary 2.4:** Let R be an integral domain and let M be torsion free R-module. Then M is a y-closed Rickart module.

No, we give the following characterization.

**Propositions 2.5:** Let M and N be two R-modules. Then M is N-y-closed Rickart module if and only if, for every R-homomorphism \( f: M \to N \), \( \text{Im}f \) is a nonsingular module.

**Proof:** Let M be N-y-closed Rickart module and let \( f: M \to N \) be an R-homomorphism. Since M is N-y-closed Rickart module, then \( \text{Ker}f \) is a y-closed submodule of M and hence \( \frac{M}{\text{Ker}f} \) is nonsingular. By the first isomorphism theorem, \( \frac{M}{\text{Ker}f} \cong \text{Im}f \). Thus \( \text{Im}f \) is nonsingular.

Conversely, let \( f: M \to N \) be an R-homomorphism. Since \( \text{Im}f \) is nonsingular and \( \frac{M}{\text{Ker}f} \cong \text{Im}f \), then \( \frac{M}{\text{Ker}f} \) is nonsingular. Therefore \( \text{Ker}f \) is a y-closed submodule of M. Thus M is N-y-closed Rickart module.

Recall that a module M is said to be K-nonsigular if for every homomorphism \( f: M \to M \) such that \( \text{Ker}f \) is essential in M, implies \( f = 0 \) [1].

**Proposition 2.6:** Every y-closed Rickart module is K-nonsigular.

**Proof:** Suppose that M is a y-closed Rickart module and let \( f: M \to M \) be an R-homomorphism such that \( \text{Ker}f \) is essential in M. Then \( \frac{M}{\text{Ker}f} \) is singular, by [2]. But M is a y-closed Rickart module, therefore \( \text{Ker}f \) is a y-closed submodule of M, which implies that \( \text{Ker}f = M \) and so \( f = 0 \). Thus M is K-nonsigular.

**Propositions 2.7:** Let M and N be two R-modules such that N is singular. Then M is N-y-closed Rickart module if and only if \( \text{Hom}(M, N) = 0 \).
Proof: Assume that \( M \) is \( N \)-y-closed Rickart module and let \( f: M \to N \) be an R-homomorphism. Then \( \ker f \) is a \( y \)-closed submodule of \( M \) and hence \( \frac{M}{\ker f} \) is nonsingular. So \( \text{Im } f \) is nonsingular. But \( N \) is singular, therefore \( \text{Im } f = 0 \). Thus \( \text{Hom}(M, N) = 0 \).

The converse is clear.

Corollary 2.8: Let \( A \) be a proper essential submodule of a module \( M \). Then \( M \) is not \( \frac{M}{A} \)-y-closed Rickart module.

Proof. Since \( A \) is an essential submodule of \( M \), then by [2], \( \frac{M}{A} \) is a singular module. Let \( \pi: M \to \frac{M}{A} \) be the natural epimorphism. It is clear that \( 0 \neq \pi \in \text{Hom} \left( M, \frac{M}{A} \right) \). Thus by Proposition (2.7) \( M \) is not \( \frac{M}{A} \)-y-closed Rickart module.

§3 DIRECT SUM OF Y-CLOSED RICKART MODULES

In this section, we study the direct sum of the \( y \)-closed Rickart modules. We begin with the following theorem.

Theorem 3.1: Let \( M \) and \( N \) be two R-modules such that \( M = A \oplus B \), where \( A \) and \( B \) are submodules of \( M \). If \( M \) is \( N \)-y-closed Rickart module, then \( A \) is \( N \)-y-closed Rickart module.

Proof. Let \( \psi: A \to N \) be an R-homomorphism and let \( p: M \to A \) be the projection map. Consider the map \( \psi \circ p: M \to N \). Since \( M \) is \( N \)-y-closed Rickart module, then \( \ker (\psi \circ p) \) is a \( y \)-closed submodule of \( M \). But

\[
\ker (\psi \circ p) = \{ x \in M \mid \psi \circ p(x) = 0 \} = \{ a + b \in A \oplus B \mid (\psi(p(a + b))) = 0, \ a \in A, b \in B \}
\]

\[
= \{ a + b \in A \oplus B \mid \psi(a) = 0, \ a \in A, b \in B \} = \ker \psi \oplus B
\]

Therefore \( \frac{M}{\ker \psi \oplus B} = \frac{A \oplus B}{\ker \psi \oplus B} \cong \frac{A}{\ker \psi} \) is nonsingular. So \( \ker \psi \) is a \( y \)-closed submodule of \( A \). Thus \( A \) is \( N \)-y-closed Rickart module.

Propositions 3.2: Let \( M = \bigoplus_{i \in I} M_i \) and \( N = \bigoplus_{i \in I} N_i \) be two R-modules, such that for every \( f \in \text{Hom}(M, N), f(M_i) \subseteq N_i, \forall i \in I \). If \( M_i \) is \( N_i \)-y-closed Rickart module, \( \forall i \in I \), then \( M \) is \( N \)-y-closed Rickart module.

Proof. Assume that \( M_i \) is \( N_i \)-y-closed Rickart module, \( \forall i \in I \), and let \( f: M \to N \) be an R-homomorphism. We want to show that \( \ker f \) is a \( y \)-closed submodule of \( M \). By our assumption,

\[
f |_{M_i}: M_i \to N_i, \forall i \in I.
\]

It is clear that \( \ker f |_{M_i} = \ker f \cap M_i \), for each \( i \in I \). We claim that

\[
\ker f = \bigoplus_{i \in I} (\ker f |_{M_i}).
\]

To show that, let \( x \in \ker f \). Then \( x = \sum_{i \in I} x_i \), where \( x_i \in M_i \), for each \( i \in I \) and \( x_i \neq 0 \) for at most a finite number of \( i \in I \) and \( f(x) = 0 \). Then \( f(x) = f(\sum_{i \in I} x_i) = \sum_{i \in I} f(x_i) = 0 \), where \( f(x_i) \in N_i \). But \( N = \bigoplus_{i \in I} N_i \). Therefore \( f(x_i) = 0, \forall i \in I \). So \( x_i \in (\ker f \cap M_i), \forall i \in I \) and hence \( x = \sum_{i \in I} x_i \in \bigoplus_{i \in I} \ker (f |_{M_i}) \). Thus \( \ker f = \bigoplus_{i \in I} \ker (f |_{M_i}) \). Since \( M_i \) is \( N_i \)-y-closed Rickart module for each \( i \in I \), then \( \ker (f |_{M_i}) \) is a \( y \)-closed submodule of \( M_i \). Therefore \( \ker f = \bigoplus_{i \in I} \ker (f |_{M_i}) \) is a \( y \)-closed submodule of \( M \), by [3]. Thus \( M \) is \( N \)-y-closed Rickart module.

Let \( M \) be an R-module, then \( M \) is called a \( y \)-closed simple if \( M \) and \( 0 \) are the only \( y \)-closed submodules of \( M \).

Example 3.3:

1. The module \( Z \) as \( Z \)-module is a \( y \)-closed simple module, where \( \frac{Z}{nZ} \cong Z_n, \forall n \geq 2 \) and \( Z_n \) is singular as \( Z \)-module. Thus \( nZ \) is not \( y \)-closed submodule of \( Z, \forall n \geq 2 \).

2. The module \( Z_6 \) as \( Z \)-module is not \( y \)-closed simple module, where \( \frac{Z_6}{\{0\}} \cong Z_6 \) and \( Z_6 \) as \( Z \)-module is singular. Hence the submodule \( \{0\} \) of \( Z_6 \) is not \( y \)-closed submodule.
Propositions 3.4: Let $M$ be a $y$-closed simple $R$-module and let $N$ be an $R$-module. If $M$ is $N$-$y$-closed Rickart, then either

1. $\text{Hom}(M,N)=0$ or
2. Every nonzero $R$-homomorphism from $M$ to $N$ is a monomorphism.

**Proof.** Assume that $\text{Hom}(M,N) \neq 0$ and let $f : M \to N$ be a non-zero $R$-homomorphism. Since $M$ is $N$-$y$-closed Rickart, then $\ker f$ is $y$-closed submodule of $M$. But $M$ is $y$-closed simple, therefore $\ker f = \{0\}$ and $f$ is a monomorphism.

Recall that an $R$-module $M$ is called a Quasi-Dedekind $R$-module if every nonzero endomorphism of $M$ is a monomorphism [4, Th(1.5), CH2].

**Corollary 3.5:** Let $M$ be a $y$-closed simple $R$-module and let $N$ be any $R$-module such that $\text{Hom}(M,N) \neq 0$. If $M$ is $N$-$y$-closed Rickart module, then $M$ is Quasi-Dedekind. In particular, if $M$ is $y$-closed Rickart, then $M$ is Quasi-Dedekind.

**Proof.** By Proposition (3.4), there is a monomorphism $f : M \to N$. Assume that $M$ is not Quasi-Dedekind $R$-module. So there exists a homomorphism $g : M \to N$ such that $\ker g \neq 0$. Since $f$ is a monomorphism, then $\ker (f \circ g) = \ker g \neq 0$. But $M$ is $N$-$y$-closed Rickart module, therefore $\ker f \circ g = \ker g$ is a $y$-closed submodule of $M$. So $\ker g = M$, where $M$ is a $y$-closed simple. Thus $g = 0$, which is a contradiction. Thus $M$ is a Quasi-Dedekind $R$-module.

**Proposition 3.6:** Let $M$ be an $R$-module. If $R$ is $M$-$y$-closed Rickart module, then every cyclic submodule of $M$ is projective. In particular, if $R$ is $y$-closed Rickart ring, then every principal ideal is projective, i.e., $R$ is a principal projective ring.

**Proof.** Let $M$ be an $R$-module such that $R$ is $M$-$y$-closed Rickart module and let $m \in M$. Now consider the following short exact sequence

$$0 \longrightarrow \ker f \xrightarrow{i} R \xrightarrow{f} Rm \longrightarrow 0$$

where $i$ is the inclusion homomorphism and $f$ is a map defined by $f(r) = rm, \forall r \in R$. It is clear that $f$ is an epimorphism. Let $i_2 : Rm \to M$ be the inclusion map. Since $R$ is $M$-$y$-closed Rickart module and $i_2 \circ i : R \to M$, then $\ker (i_2 \circ i)$ is a $y$-closed ideal of $R$. But $i_2$ is a monomorphism, therefore $\ker (i_2 \circ f) = \ker f$ is a $y$-closed ideal of $R$. Hence $R_{\ker f}$ is nonsingular. By the first isomorphism theorem, $R_{\ker f} \cong Rm$. So $Rm$ is nonsingular, by [2, corollary(1.25), p35]. Thus $Rm$ is projective.

Recall that an $R$-module $M$ is called dualizable if $\text{Hom}(M,R) \neq 0$ [5].

**Corollary 3.7:** Let $M$ be a $y$-closed simple dualizable $R$-module. If $M$ is $R$-$y$-closed Rickart module, then $M$ is isomorphic to an ideal of $R$. Hence, if $R$ has nonzero nilpotent elements, then $\text{End}(M)$ is commutative.

**Proof.** Since $\text{Hom}(M,R) \neq 0$, then by Proposition (3.4), $M$ is isomorphic to an ideal $I$ of $R$ and hence $\text{End}(M) \cong \text{End}(I)$. For the second part, since $R$ has no nonzero elements and $I$ is an ideal in $R$, then $\text{End}(I)$ is commutative [6, proposition(2.1), CH1]. Thus $\text{End}(M)$ is commutative.

Recall that an $R$-module $M$ is called a multiplication module if for each submodule $N$ of $M$ there exists an ideal $I$ of $R$ such that $N = IM$, [6].

**Corollary 3.8:** Let $M$ be a $y$-closed simple projective $R$-module and $R$ has no nonzero nilpotent element. If $M$ is $R$-$y$-closed Rickart module and $\text{Hom}(M,R) \neq 0$, then $M$ is a multiplication module.

**Proof.** By the same argument of the proof of Corollary (3.7), $\text{End}(M)$ is a commutative and hence $M$ is a multiplication module [7].

**Proposition 3.9:** Let $M$ be an $R$-module with the property that the intersection of any two $y$-closed submodules of $M$ is a $y$-closed submodule of $M$. Then the following statements are equivalent.

(a) $M$ is a $y$-closed Rickart module,
(b) The left annihilator in $M$ of every left finitely generated ideal $I = (f_1, ..., f_n)$ of $\text{End}_R(M)$ is a $y$-closed submodule of $M$.

**Proof.** (a) $\Rightarrow$ (b) Let $I = (f_1, ..., f_n)$ be a left finitely generated ideal of the $\text{End}_R(M)$. Since $M$ is a $y$-closed Rickart module, then $\text{ann}_M(f_i)$ is a $y$-closed submodule of $M$, $\forall \ 1 \leq j \leq n$. Hence
\( \cap_{i=1}^{n} \text{ann}_M(f_i) \) is a y-closed submodule of \( M \), by [3]. But \( \text{ann}_M(I) = \text{ann}_M(Sf_1 + \cdots + Sf_n) = \cap_{i=1}^{n} \text{ann}_M(Sf_i) \). Therefore \( \text{ann}_M(I) \) is y-closed submodule of \( M \).

(b) \( \Rightarrow \) (a) Clear.

Now, we give the following characterization.

**Theorem 3.10:** Let \( M_1 \) and \( M_2 \) be two \( R \)-modules. Then the following statements are equivalent.

1. \( M_1 \) is \( M_2 \)-y-closed Rickart module;
2. For every submodule \( N \) of \( M_2 \), every direct summand \( K \) of \( M_1 \) is \( N \)-y-closed Rickart;
3. For every direct summand \( K \) of \( M_1 \), every y-closed submodule \( L \) of \( M_2 \) and every \( f \in \text{Hom}_R(M,L) \). The kernel of the restricted map \( f|_K \) is a y-closed submodule of \( K \).

**Proof.** (1) \( \Rightarrow \) (2) Let \( N \) be submodule of \( M_2 \). Let \( K \) be a direct summand of \( M_1 \) and let \( f: K \to N \) be an \( R \)-homomorphism. Then \( M_1 = K \oplus K_1 \), for some submodule \( K_1 \) of \( M \). Let \( g: M_1 \to M_2 \) be a map defined by

\[
g(x) = \begin{cases} f(x), & \text{if } x \in K \\ 0, & \text{if } x \in K_1 \end{cases}
\]

It is clear that \( g \) is an \( R \)-homomorphism. Since \( M_1 \) is \( M_2 \)-y-closed Rickart module, then \( \ker g \) is a y-closed submodule of \( M_1 \). But

\[
\ker g = \{ a + b \in M_1, \ g(a + b) = 0, \ a \in K, b \in K_1 \} = \{ a + b \in M_1, \ f(a) = 0, \ a \in K, b \in K_1 \} = \ker f \oplus K_1
\]

Therefore \( \ker f \oplus K_1 \) is a y-closed submodule of \( M_1 \) and hence \( \frac{M_1}{\ker f \oplus K_1} \) is nonsingular. But \( \frac{M_1}{\ker f \oplus K_1} = \frac{K \oplus K_1}{\ker f} \cong \frac{K}{\ker f} \), so \( \ker f \) is a y-closed submodule of \( K \). Thus \( K \) is \( N \)-y-closed Rickart module.

(2) \( \Rightarrow \) (3) Let \( K \) be a direct summand of \( M_1 \) and \( L \) be a submodule of \( M_2 \). Let \( f: M_1 \to L \) be an \( R \)-homomorphism. Consider the map \( f|_K: K \to L \). Since \( K \) is \( L \)-y-closed Rickart module, then \( \ker f|_K \) is a y-closed submodule of \( K \).

(3) \( \Rightarrow \) (1) Let \( f: M_1 \to M_2 \) be an \( R \)-homomorphism. Take \( L = M_2 \) and \( K = M_1 \). Since \( f|_K: K \to L \) and \( K \) is \( L \)-y-closed Rickart module, therefore \( \ker f \) is a y-closed submodule of \( M_1 \). Thus \( M_1 \) is \( M_2 \)-y-closed Rickart module.

**Remark 3.11:** Let \( M \) and \( N \) be two \( R \)-modules and \( f: M \to N \) be an \( R \)-homomorphism. Let \( A_M = M \oplus 0 \), \( B_N = 0 \oplus N \), \( \tilde{f}: A_M \to B_N \) be a map defined by \( \tilde{f}(m,0) = (0,f(m)) \), for every \( m \in M \) and

\[
T_f = \{ x + f(x), x \in A_M \}. Then :
1. \( M \oplus N = A_M \oplus B_N \)
2. \( T_f \) is an \( R \)-homomorphism
3. \( \ker \tilde{f} = \ker f \oplus 0 \)
4. \( T_f \) is a submodule of \( M \oplus N \)
5. \( A_M + T_f = A_M \oplus \text{Im} f \).

In the following theorem by \( A_M, B_M, \tilde{T}_f, T_f \), we mean the same concepts in the previous above Remark.

Now, we give another characterization for the relative y-closed Rickart module.

**Theorem 3.12:** Let \( M \) and \( N \) be two \( R \)-modules. Then \( M \) is \( N \)-y-closed Rickart module if and only if for every homomorphism \( f: M \to N \), \( A_M \cap T_f \) is y-closed submodule of \( A_M \).

**Proof.** Let \( f: M \to N \) be an \( R \)-homomorphism. Since \( M \) is \( N \)-y-closed Rickart module, then \( \ker \tilde{f} \) is a y-closed submodule of \( M \) and hence \( \frac{M}{\ker \tilde{f}} \) is nonsingular. Then \( \frac{A_M}{\ker \tilde{f}} = \frac{M \oplus 0}{\ker \tilde{f} \oplus 0} \cong \frac{M}{\ker \tilde{f}} \) is nonsingular. So \( \ker \tilde{f} \) is a y-closed submodule of \( A_M \). By the same argument of the proof of the [8,Theorem(2.2)], \( \ker \tilde{f} = A_M \cap T_f \).

For the converse, let \( f: M \to N \) be an \( R \)-homomorphism. Then by our assumption, \( A_M \cap T_f \) is a y-closed submodule of \( A_M \). Since \( \ker \tilde{f} = A_M \cap T_f \) then \( \ker \tilde{f} \) is a y-closed submodule of \( A_M \) and hence \( \frac{A_M}{\ker \tilde{f}} \) is nonsingular. Therefore \( \frac{M \oplus 0}{\ker \tilde{f} \oplus 0} \cong \frac{M}{\ker \tilde{f}} \) is nonsingular. So \( \ker \tilde{f} \) is a y-closed submodule of \( M \). Thus \( M \) is \( N \)-y-closed Rickart module.
But, we have the following.

**Theorem 3.13:** Let M and N be two R-modules and let \( f: M \rightarrow N \) be an R-homomorphism. Then M is N-\( y \)-closed Rickart module if and only if \( T_f \) is \( y \)-closed submodule of \( A_M + T_f \).

**Proof.** Let \( f: M \rightarrow N \) be an R-homomorphism. Now consider the following short exact sequences:

\[
0 \rightarrow A_M \cap T_f \xrightarrow{i_1} A_M \xrightarrow{\pi_1} A_M/ T_f \xrightarrow{\pi_2} A_M + T_f/ T_f \rightarrow 0
\]

where \( i_1, i_2 \) are the inclusion homomorphisms and \( \pi_1, \pi_2 \) are the natural epimorphisms. Since M is N-\( y \)-closed Rickart, then kerf is \( y \)-closed submodule of M and hence \( \frac{M}{\ker f} \) is nonsingular. So \( A_M/ T_f = \frac{M}{\ker f} \), \( M \oplus 0 \) is nonsingular. Thus \( \text{Ker} f = A_M \cap T_f \) is a \( y \)-closed submodule of \( A_M \). Hence \( \frac{A_M}{A_M \cap T_f} \) is nonsingular. By the second isomorphism theorem, \( \frac{A_M}{A_M \cap T_f} \approx \frac{A_M + T_f}{T_f} \) is nonsingular. Thus \( T_f \) is a \( y \)-closed submodule of \( A_M + T_f \).

For the converse, let \( f: M \rightarrow N \) be an R-homomorphism. Consider the following short exact sequences:

\[
0 \rightarrow A_M \cap T_f \xrightarrow{i_1} A_M \xrightarrow{\pi_1} A_M/ T_f \xrightarrow{\pi_2} A_M + T_f/ T_f \rightarrow 0
\]

where \( i_1, i_2 \) are the inclusion homomorphisms and \( \pi_1, \pi_2 \) are the natural epimorphisms. By the second isomorphism theorem, \( \frac{A_M}{A_M \cap T_f} \approx \frac{A_M + T_f}{T_f} \). Since \( T_f \) is \( y \)-closed submodule of \( A_M + T_f \), then \( \frac{A_M + T_f}{T_f} \) is nonsingular, therefore \( \frac{A_M}{A_M \cap T_f} \) is nonsingular. Hence \( A_M \cap T_f \) is a \( y \)-closed submodule of \( A_M \). So \( \text{Ker} f = \text{Ker} f \oplus 0 \) is a \( y \)-closed submodule of \( A_M = M \oplus 0 \). Thus kerf is \( y \)-closed submodule of M.

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