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Analytical Study on Approximate ϵ-Birkhoff-James Orthogonality

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Abstract

In this paper, we obtain a complete characterization for the norm and the minimum norm attainment sets of bounded linear operators on a real Banach spaces at a vector in the unit sphere, using approximate ϵ-Birkhoff-James orthogonality techniques. As an application of the results, we obtained a useful characterization of bounded linear operators on a real Banach spaces. Also, using approximate ϵ-Birkhoff -James orthogonality proved that a Banach space $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ is a reflexive if and only if for any closed hyperspace $\mathbb{H}$ of $\mathbb{X}$, there exists a rank one linear operator $\Theta \neq \mathcal{T} \in \mathcal{B}(\mathbb{X})$ such that $x \in M_{\mathcal{T}}$, for some vectors x in $\mathbb{S}_{\mathbb{X}}$ and $m_{\mathcal{T}} = \mathbb{H} \cap \mathbb{S}_{\mathbb{X}}$ such that $x \perp_{BJC}^{\epsilon} \mathbb{H}$.Mathematics subject classification (2010): 46B20, 46B04, 47L05.

Keywords: Linear operator, Norm attainment, approximate ϵ-Birkhoff-James orthogonality, reflexivity.

دراسة تحليلية حول تعامد بيركوف – جيمس – ϵ التقريبي

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الخلاصه

في هذا البحث حصلنا على تمثيل كامل لمجموعة تحقيق المعيار والحد الأدنى لمجموعة تحقيق المعيار للمؤثرات الخطية المقيدة التقريبيϵ .في فضاء بناخ لمتجهات من الكرة الواحدية باستخدام مفهوم تعامد بيركوف – جيمس – ϵ وكتطبيق لهذه النتائج حصلنا على تمثيل للمؤثرات الخطية المقيدة على فضاء بناخ الحقيقي. وباستخدام تعامد بيركوف – جيمس –التقريبي, برهنا أن فضاء بناخ الحقيقي هو فضاء انعكاسي أذا وفقط أذا كان لكل فضاء زائدي مغلق يوجد مؤثر خطي مقيد غير صفري على فضاء بناخ بحيث أن مجموعة تحقيق المعيار تكافئ مجموعة المتجهة وسالب المتجه لبعض متجهات الكرة الواحدية بالإضافة الى أن تقاطع الكرة الواحدية مع الفضاء الزائدي يكافئ الحد الأدنى لمجموعة تحقيق المعيار.

1. Introduction

An impressive growth occurred in the applications of the Birkhoff-James orthogonality that was first introduced in 1935 and used to solve particular problems in the study of geometry of a Banach spaces. In
recent times, several authors explored this topic[1-5] and obtained many interesting results involving orthogonality of a bounded linear operators. Several recent papers $[6-9]$ were devoted to the description and classification of the following types of orthogonality in a real normed space $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$.

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In this paper, we focus on a specific type of orthogonality which was submitted by Chmielinski . in 2005 and called approximate ϵ-B-$\mathcal{J}$-orthogonality. The other types are seen as tools to enable us to resolve some of the outstanding issues in this work. In 2017, Chmielinski et al characterized the approximate ϵ-B-$\mathcal{J}$- orthogonality and obtained a sufficient condition for using linear functional on a real normed space $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$. In 2018, Kallol P. et al characterized the approximate ϵ-B-$\mathcal{J}$-orthogonality of bounded linear operators on a reflexive real Banach space $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ using the norm attainment set. While, a complete characterization of approximate ϵ-B-$\mathcal{J}$- orthogonality of bounded linear operators on an infinite dimensional real Banach spaces was obtained [10]. It will also be interesting to conduct an analogous study for an approximate ϵ-B-$\mathcal{J}$-orthogonality in other types of spaces such as modular spaces [11] and general fuzzy normed spaces [12]. To proceed in details, we fix some notations and terminologies.

Throughout this paper, we will be working with real normed spaces.

Let $\mathbb{B}_{\mathbb{X}} = \{x \in \mathbb{X} : \| x \|_{\mathbb{X}} \leq 1\}$ and $\mathbb{S}_{\mathbb{X}} = \{x \in \mathbb{X} : \| x \|_{\mathbb{X}} = 1\}$ be the unit ball and unit sphere, respectively, of $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$. Let $\mathcal{B}(\mathbb{X}, \mathbb{Y})$ $(\mathcal{K}(\mathbb{X}, \mathbb{Y}))$ denote the set of all bounded (compact) linear operators from $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ to $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ which is the normed space with the supremum norm.

In this paper, as an application of the approximate ϵ-B-$\mathcal{J}$-orthogonality, we obtain a complete characterization of reflexive Banach spaces in terms of the norm and the minimum norm attainment sets of rank one bounded linear operators on the space.

2. Preliminaries and set background material

In this section, we recall some concepts and results related to the B-$\mathcal{J}$- orthogonality that will be used in the sequel of approximate ϵ-B-$\mathcal{J}$-orthogonality.

The following definition is necessary to obtain the desired characteristics required in this paper.

Definition 2.1. $[13 - 15]$: Let $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$, $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ be two real Banach spaces and $\mathcal{T} \in \mathcal{B}(\mathbb{X}, \mathbb{Y})$. Then:

i. $\mathcal{T}$ is said to be attains norm at a vector x in $\mathbb{S}_{\mathbb{X}}$, if $\| \mathcal{T}(x) \|_{\mathbb{Y}} = \| \mathcal{T} \|_{\mathcal{B}(\mathbb{X},\mathbb{Y})}$.

ii. Let $M_{\mathcal{T}}$ denotes the set of all vectors x in $\mathbb{S}_{\mathbb{X}}$ at which $\mathcal{T}$ attains norm, i.e.,

$M_{\mathcal{T}} = \{x \in \mathbb{S}_{\mathbb{X}} : \| \mathcal{T}(x) \|_{\mathbb{Y}} = \| \mathcal{T} \|_{\mathcal{B}(\mathbb{X},\mathbb{Y})}\}$.

Apart from those previously reported [16], some results on the characterization of were also obtained in another work [16]. The set $M_{\mathcal{T}}$ plays an important role in characterizing B-$\mathcal{J}$-orthogonal of bounded linear operators and was obtained in an earlier work [16].

iii. Following similar procedure, the notation of the minimum norm attainment set m_T for $T \in \mathcal{B}(\mathbb{X}, \mathbb{Y})$ is defined in the following way:

$m_{\mathcal{T}} = \{x \in \mathbb{S}_{\mathbb{X}} : \| \mathcal{T}(x) \|_{\mathbb{Y}} = m(\mathcal{T})\}$, where $m(\mathcal{T}) = inf\{\| \mathcal{T}(x) \|_{\mathbb{Y}} : \| x \|_{\mathbb{X}} = 1\}$.

Also, the set $m_{\mathcal{T}}$ plays a very crucial role in determining the geometry of $\mathcal{B}(\mathbb{X}, \mathbb{Y})$.

In this paper, we obtain a complete characterization of the $m_{\mathcal{T}}$ by applying the concept of approximate ϵ-B-$\mathcal{J}$-orthogonality. We further study the relative position of $M_{\mathcal{T}}$ and $m_{\mathcal{T}}$ within this concept. The existence of at least one such vector is guarantee by Hahn Banach Theorem.

Definition 2.2 $[6, 16, 17]$: For any two vectors x and y in a normed linear space $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$:

i. x is said to be orthogonal to y in the sense of Birkhoff-James (for brief, B-$\mathcal{J}$-orthogonality) and written as $x \perp_{BJ} y$, if the following is true:

$\| x + \mu y \|_{\mathbb{X}} \geq \| x \|_{\mathbb{X}}$ for all $\mu \in \mathbb{R}$.

ii. x is said to be orthogonal to y in the sense of Robert (for brief, $\mathcal{R}$- orthogonality) and written as $x \perp_R y$, if the following is true:

$\| x + \mu y \|_{\mathbb{X}} = \| x - \mu y \|_{\mathbb{X}}$ for all $\mu \in \mathbb{R}$.

iii. x is said to be approximate ϵ-Birkhoff-James orthogonal to y in the sense of Chmielinski, (for brief, approximate ϵ-B-$\mathcal{J}$-orthogonality) and written as $x \perp_{BJC}^{\epsilon} y$, if the following is true:

$\| x + \mu y \|_{\mathbb{X}}^2 \geq \| x \|_{\mathbb{X}}^2 - 2\epsilon \| x \|_{\mathbb{X}} \| \mu y \|_{\mathbb{X}}$ for all $\mu \in \mathbb{R}$.

Otherwise, x is not approximate ϵ-B-$\mathcal{J}$-orthogonality to y and has the symbol $x \not\perp_{BJC}^{\epsilon} y$. The relation between these concepts and several of their properties can be found in the literature $[6, 7, 8, 9, 18]$. Obviously, the relationship between notations is given as follows $\perp_R \Rightarrow \perp_{BJ} \Rightarrow \perp_{BJC}^{\epsilon}$. Later on, Chmielinski J. [3], introduced another notion $x^{\perp_{BJC}^{\epsilon}}$:

Definition 2.3 [9]: For any vector x in a normed linear space $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$, a set $x^{\perp_{BJC}^{\epsilon}} = \{ y \in \mathbb{X} : x \perp_{BJC}^{\epsilon} y \}$ is said to be an approximate ϵ-B-$\mathcal{J}$-orthogonal complement of x.

Before going ahead with $x^{\perp_{BJC}^{\epsilon}}$, we are getting some results which appear to show some sufficient and necessary conditions about reaching the desired result, which states that $\mathcal{T}(x^{\perp_{BJC}^{\epsilon}}) \subseteq (\mathcal{T}x)^{\perp_{BJC}^{\epsilon}}$ for every $x \in m_{\mathcal{T}}$.

Let us also note that the following concepts are important in this paper:

Definition 2.4 [19]: A subspace $\mathbb{H}$ of linear space $\mathbb{X}$ is said to be a hyperspace, if $\mathbb{H}$ is a maximal subspace with co-dimension 1.

Theorem 2.5 [19]:

i. For any linear functional $\Theta \neq \psi$ on a normed linear space $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$, the null space of ψ denoted as $ker\psi$ is a hyperspace.

ii. A hyperspace $\mathbb{H}$ of $\mathbb{X}$ is closed if and only if it is a kernel of a linear functional $\Theta \neq \psi$ on $\mathbb{X}$.

iii. A subspace $\mathbb{H}$ of $\mathbb{X}$ is hyperspace if and only if $\mathbb{X} = Lin\{x_0 + \mathbb{H}\}$ for some vectors x_0 in $\mathbb{X}$.

Definition 2.6. [20] : Let $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$, $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ be two Banach spaces and $\mathcal{T} \in \mathcal{B}(\mathbb{X}, \mathbb{Y})$. Then:

i. $\mathcal{T}$ is said to be a finite rank operator, if it is a linear operator whose range is finite dimensional (i.e. $\mathcal{T}$ $(\mathbb{X})$ has a finite dimension).

ii. $\mathcal{T}$ is said to be a compact linear operator, if the image under $\mathcal{T}$ for any bounded subset K of $\mathbb{X}$ is relatively compact (has compact closure) of $\mathbb{Y}$.

Remark 2.7. [20]:

Any bounded linear operator of a finite-rank is compact.

Among others, there are constructions of a rank one linear operator ($\mathcal{T} \in \mathcal{B}(\mathbb{X}, \mathbb{Y})$ that have one rank, if dimension of $\mathcal{T}(\mathbb{X}) = 1$). We connect this concept with several notions to study some properties of a reflexive Banach space $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$. We discuss many of these results and give proofs. Let us finish the introduction with the needed definition.

Definition 2.8 [21]: A normed linear space $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ is a strictly convex, if for any two vectors x and y in $\mathbb{S}_{\mathbb{X}}$ with $x \neq y$, then $\| x + y \|_{\mathbb{X}} < 2$.

Theorem 2.9 [22]: A normed linear space $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ is a strictly convex if and only if for any vector x in $\mathbb{S}_{\mathbb{X}}$ there exists $\mathcal{T} \in \mathcal{B}(\mathbb{X}, \mathbb{Y})$, which attains norm only at vectors of the form ηx with $|\eta| = 1$.

As an application of this definition, we obtain a complete characterization of reflexive Banach spaces in terms of the sets $M_{\mathcal{T}}$ and $m_{\mathcal{T}}$ of the rank one linear operators and the concept of approximate ϵ-B-$\mathcal{J}$- orthogonality.

In order to prove the desired results, we make use of the following easy proposition, stated previously [6].

Proposition 2.10 [6]: For any two vectors x and y in a normed linear space $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$, some properties of approximate ϵ-B-$\mathcal{J}$- orthogonality are:

i. $x \perp_{BJC}^{\epsilon} \Theta$ and $\Theta \perp_{BJC}^{\epsilon} x$ for all x in $\mathbb{X}$ and $x \perp_{BJC}^{\epsilon} x$, if and only if $x = \Theta$.

ii. Note that the relation $\perp_{BJC}^{\epsilon}$ is homogenous, but neither symmetric nor additive.

iii. For any non-zero vectors x and y in , if $x \perp_{BJC}^{\epsilon} y$, then x and y are linearly independent.

Theorem 2.11 [23]: For any two vectors x and y in a normed linear space $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$, $x \perp_{BJC}^{\epsilon} y$ if and only if there exists $z \in Lin\{x, y\}$ such that $x \perp_{BJ} z$ and $\| y - z \|_{\mathbb{X}} \leq \epsilon \| y \|_{\mathbb{X}}$.

Theorem 2.12. [11]: Let $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ be a normed linear space. For any two non-zero linear functional φ, ψ on $\mathbb{X}$, $\varphi \perp_{BJ} \psi$ if and only if either of the conditions in (i) or in (ii) holds:

i. There exists $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{S}_{\mathbb{X}}$ such that $\| \varphi(x_n) \|_{\mathbb{F}} \longrightarrow \| \varphi \|_{\mathbb{X}^*}$ and $\psi(x_n) \longrightarrow 0$.

ii. There exist sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in $\mathbb{S}_{\mathbb{X}}$ such that:

(a) $|\varphi(x_n)| \longrightarrow \| \varphi \|_{\mathbb{X}^*}$ and $|\varphi(y_n)| \longrightarrow \| \varphi \|_{\mathbb{X}^*}$ as $n \longrightarrow \infty$.

(b) $\varphi(x_n) \cdot \psi(x_n) \geq 0$ and $\varphi(y_n) \cdot \psi(y_n) \leq 0$ for all $n \in \mathbb{N}$.

Theorem 2.13. [24]**:** For any linear functional $\Theta \neq \varphi$ on a Banach space $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$, $|\varphi(x)| = \| \varphi \|_{\mathbb{X}^*} \| x \|_{\mathbb{X}}$ if and only if $x \perp_{BJ} \mathbb{H}$, where $\mathbb{H}$ is any closed hyperspace of $\mathbb{X}$ with $\varphi(h) = 0$ for all $h \in \mathbb{H}$.

The following theorem [24 , 21] gives necessary and sufficient conditions of a Banach space $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ to be reflexive.

Theorem 2.14. [24, 21]**:** A Banach space $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ is reflexive if and only if one of the following conditions is satisfied:

i. For any linear functional φ on $\mathbb{X}$ attains norm on $B_{\mathbb{X}}$.

ii. For any $\mathcal{T} \in \mathcal{K}(\mathbb{X}, \mathbb{Y})$ attains norm, where $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ is any Banach space.

iii. For any closed hyperspace $\mathbb{H}$ of $\mathbb{X}$, there exists a unit vector x in $\mathbb{X}$ such that $x \perp_{BJ} \mathbb{H}$.

3. Main Results

In this section, we first obtain a complete characterization of $M_{\mathcal{T}}$ and $m_{\mathcal{T}}$ within approximate ϵ-B-$\mathcal{J}$-orthogonality. We need the terminology in the following remark to be relevant in this paper.

Remark 3.1: For any two vectors x and y in $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ with $\epsilon \in [0,1)$, and $y \in x^{\perp_{BJC}^{\epsilon}}$, two subsets of $x^{\perp_{BJC}^{\epsilon}}$ will be defined, that is:

$y \in x^{+(\epsilon)}$, if $\| x + \mu y \|_{\mathbb{X}}^2 \geq \| x \|_{\mathbb{X}}^2 - 2\epsilon \| x \|_{\mathbb{X}} \| \mu y \|_{\mathbb{X}}$ for all $\mu \geq 0$.

and

$y \in x^{-(\epsilon)}$, if $\| x + \mu y \|_{\mathbb{X}}^2 \geq \| x \|_{\mathbb{X}}^2 - 2\epsilon \| x \|_{\mathbb{X}} \| \mu y \|_{\mathbb{X}}$ for all $\mu \leq 0$.

We will state some obvious but useful properties of this notion which would be used later on in this work, without giving an explicit proof.

Proposition 3.2: For any two vectors x and y in a normed linear space $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$, the following statements are satisfying:

i. Either $y \in x^{+(\epsilon)}$ or $y \in x^{-(\epsilon)}$.

ii. If $y \in x^{+(\epsilon)}$ ($y \in x^{-(\epsilon)}$), implies that $\delta y \in (\eta x)^{+(\epsilon)}$ ($\delta y \in (\eta x)^{-(\epsilon)}$) for all $\delta, \eta > 0$.

iii. If $y \in x^{+(\epsilon)}$ ($y \in x^{-(\epsilon)}$), implies that $-y \in x^{-(\epsilon)}$ ($-y \in x^{+(\epsilon)}$) and $y \in (-x)^{-(\epsilon)}$ ($y \in (-x)^{+(\epsilon)}$).

Notation 3.3: Let $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$, $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ be two normed linear spaces and $\Theta \neq \mathcal{T} \in \mathcal{L}(\mathbb{X}, \mathbb{Y})$. For any vector x in $\mathbb{X}$ with $y \in x^{\perp_{BJC}^{\epsilon}}$. Then $\mathcal{T}(x^{\perp_{BJC}^{\epsilon}}) = \{ \mathcal{T}(y) \subseteq \mathbb{Y} : x \perp_{BJC}^{\epsilon} y \}$.

Remark 3.4: The notation $x \perp\!\!\perp y$ signals to the following cases $x \perp_{BJC}^{\epsilon} y$ and $x \perp_R y$ for any two vectors x and y in $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$, which we will use in the following theorem that plays an important role in this work.

We strive to obtain a necessary condition for $\Theta \neq \mathcal{T} \in \mathcal{B}(\mathbb{X}, \mathbb{Y})$ to attain norm at x in $\mathbb{S}_{\mathbb{X}}$ as the first case.

Theorem 3.5: Let $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$, $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ be two Banach spaces and $\Theta \neq \mathcal{T} \in \mathcal{B}(\mathbb{X}, \mathbb{Y})$, be such that there exists $x \in M_{\mathcal{T}}$ with $\mathcal{T}x \perp\!\!\perp \mathcal{T}y$. Then for any vector y in $\mathbb{X}$, implies that $x \perp_{BJC}^{\epsilon} y$.

Proof:

If possible, suppose that $x \not\perp_{BJC}^{\epsilon} y$ (clearly $y \neq \Theta$), then there exists $0 \neq \mu_o \in \mathbb{R}$ such that:

$0 < \| x + \mu_o y \|_{\mathbb{X}}^2 < \| x \|_{\mathbb{X}}^2 - 2\epsilon \| x \|_{\mathbb{X}} \| \mu_o y \|_{\mathbb{X}}$;

and assume that $\mu_o < 0$. It is easy to show that for any $\mu \in [\mu_o, 0)$, we note that $\| x + \mu y \|_{\mathbb{X}}^2 > 0$ for any two different values of μ.

Let $z = \frac{x + \mu_o y}{\|x + \mu_o y\|_{\mathbb{X}}}$. Then $\| z \|_{\mathbb{X}} = 1$ and z can be written as the form $z = \eta x + \gamma y$, with $\eta = \frac{1}{\|x + \mu_o y\|_{\mathbb{X}}} > 1$ and $\gamma = \frac{\mu_o}{\|x + \mu_o y\|_{\mathbb{X}}} < 0$. Also, $\| \mathcal{T}x \|_{\mathbb{Y}} > 0$, since $\mathcal{T} \neq \Theta$ and $x \in M_{\mathcal{T}}$. Now, we have:

$\| \mathcal{T} \|_{\mathcal{B}(\mathbb{X},\mathbb{Y})}^2 = \| \mathcal{T}z \|_{\mathbb{Y}}^2 = \| \mathcal{T}(\eta x + \gamma y) \|_{\mathbb{Y}}^2 = |\eta|^2 \| \mathcal{T}(x + \frac{\gamma}{\eta} y) \|_{\mathbb{Y}}^2 > |\eta|^2 \| \mathcal{T}x \|_{\mathbb{Y}}^2 > \| \mathcal{T}x \|_{\mathbb{Y}}^2 =$
$\| \mathcal{T} \|_{\mathcal{B}(\mathbb{X},\mathbb{Y})}^2$;

Which is a contradiction, since $\| z \|_{\mathbb{X}} = 1$.

Remark 3.6:

i. In this work, $\Theta \neq \mathcal{T} \in \mathcal{B}(\mathbb{X}, \mathbb{Y})$, where $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ and $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ are Banach spaces, if $M_{\mathcal{T}} \neq \phi$.
i.e. there exists $x \in M_{\mathcal{T}}$ and $\mathcal{T}x \perp_{BJC}^{\epsilon} \mathcal{T}y$, then it needs not to be $x \perp_{BJC}^{\epsilon} y$ for any vector y in $\mathbb{X}$. Here, $\mathcal{R}$-orthogonality needs to be used to overcome this obstacle.

ii. Certainly, Theorem (3.5) sends a clear message; if we omit the condition "$\perp_R$" in the theorem mentioned, with $\mathcal{T}x \perp_{BJC}^{\epsilon} \mathcal{T}y$ remaining, then it needs not to be $x \perp_{BJC}^{\epsilon} y$. We will also give some results of procedure used to construct the counterexample using the real normed space:
$c_o = \{(x_n)_{n\in\mathbb{N}} : x_n \in \mathbb{F}, x_n \to 0 \; as \; n \to \infty\}$.
Example 3.7: Let $\mathbb{X} = (c_0)^*$ and $\mathcal{T} : \mathbb{X} \longrightarrow \mathbb{R}$, defined by $\mathcal{T}(\varphi) = \varphi(e_n)$, for all $\varphi \in c_o^*$ and we consider $e_n = (0, 0, \dots, 1, 0, \dots)$, with 1 in the n-th position and 0 in anywhere else .
Let $\varphi : c_o \longrightarrow \mathbb{R}, x \longmapsto \sum_{k=1}^{\infty} \frac{x_k}{k}$ and $\psi : c_o \longrightarrow \mathbb{R}, x \longmapsto \sum_{k=1}^{\infty} 2^{-k+1} x_k$ for all $x \in c_o$.
Then $\varphi(e_n) = (1 - \frac{1}{n}) \leq 1$, implies that $\| \varphi \|_{c_o{}^*} = 1$.
i.e. $\varphi \in M_{\mathcal{T}}$ and $\psi(e_n) = \sum_{n=1}^{\infty} 2^{-n+1} = 2(\frac{1}{1-2^n}) \longrightarrow 2$ as $n \longrightarrow \infty$. It is clear that $\mathcal{T}\varphi \perp_{BJC}^{\epsilon} \mathcal{T}\psi$ and by applying Theorem (2.12. i), this implies that $\varphi \not\perp_{BJC}^{\epsilon} \psi$.

Now, we are ready to prove the following necessary condition for the set $M_{\mathcal{T}}$.

Theorem 3.8: Let $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ and $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ be two Banach spaces. Let $\Theta \neq \mathcal{T} \in \mathcal{B}(\mathbb{X}, \mathbb{Y})$ be such that there exists $x \in M_{\mathcal{T}}$ with $\mathcal{T}x \perp\!\!\!\perp \mathcal{T}y$ for any vector y in $\mathbb{X}$. Then:

i. $\mathcal{T}(x^{+(\epsilon)} \backslash x^{\perp_{BJC}^{\epsilon}}) \subseteq (\mathcal{T}x)^{+(\epsilon)} \backslash (\mathcal{T}x)^{\perp_{BJC}^{\epsilon}}$.

ii. $\mathcal{T}(x^{-(\epsilon)} \backslash x^{\perp_{BJC}^{\epsilon}}) \subseteq (\mathcal{T}x)^{-(\epsilon)} \backslash (\mathcal{T}x)^{\perp_{BJC}^{\epsilon}}$.

Proof: Let us first prove (i). Let $\Theta \neq y \in x^{+(\epsilon)} \backslash x^{\perp_{BJC}^{\epsilon}}$, implies that $x \not\perp_{BJC}^{\epsilon} y$, there exists $0 \neq \mu_o \in \mathbb{R}$ such that:
$\| x + \mu_o y \|_{\mathbb{X}}^2 < \| x \|_{\mathbb{X}}^2 - 2\epsilon \| x \|_{\mathbb{X}} \; \| \mu_o y \|_{\mathbb{X}}$.
Clearly, $y \in x^{+(\epsilon)}$, then we must have $\mu_o < 0$. Also, by the contradiction to our hypothesis in the proof of the Theorem (3.5), this implies that: $\| x + \mu_o y \|_{\mathbb{X}} > 0$.
Following the same motivation, as in the proof of Theorem (3.5), consider that $z = \frac{x+\mu_o y}{\|x+\mu_o y\|_{\mathbb{X}}} = \eta x + \gamma y$, where $\eta = \frac{1}{\|x+\mu_o y\|_{\mathbb{X}}} > 1$ and $\gamma = \frac{\mu_o}{\|x+\mu_o y\|_{\mathbb{X}}} < 0$. Also $\| \mathcal{T}x \|_{\mathbb{Y}} > 0$, since $\mathcal{T} \neq \Theta$ and $x \in M_{\mathcal{T}}$. We assume that $\| x + \mu_o y \|_{\mathbb{Y}} > 0$. Now, we have:
$\| \mathcal{T} \|_{\mathcal{B}(\mathbb{X},\mathbb{Y})}^2 = \| \mathcal{T}z \|_{\mathbb{Y}}^2 = \| \mathcal{T}(\eta x + \gamma y) \|_{\mathbb{Y}}^2 = |\eta|^2 \; \| \mathcal{T}(x + \frac{\gamma}{\eta} y) \|_{\mathbb{Y}}^2 > \| \mathcal{T}(x + \frac{\gamma}{\eta} y) \|_{\mathbb{Y}}^2$. Since $|\alpha| > 1$ and
$\| \mathcal{T}(x + \frac{\gamma}{\eta} y) \|_{\mathbb{Y}}^2 = \frac{1}{|\eta|^2} \| \mathcal{T}(\eta x + \gamma y) \|_{\mathbb{Y}}^2 = \frac{1}{|\eta|^2 \|x+\mu_o y\|_{\mathbb{X}}^2} \| \mathcal{T}(x + \mu_o y) \|_{\mathbb{Y}}^2 > 0$. Claim that $\mathcal{T}y \notin (\mathcal{T}x)^{-(\epsilon)}$.
Suppose that $\mathcal{T}y \in (\mathcal{T}x)^{-(\epsilon)}$. Since $\frac{\gamma}{\eta} < 0$, it follows from Proposition (3.2. iii), implies that $\frac{\gamma}{\eta}\mathcal{T}y \in (\mathcal{T}x)^{+(\epsilon)}$. Therefore, we have $\| \mathcal{T}z \|_{\mathbb{Y}}^2 > \| \mathcal{T}x + \frac{\gamma}{\eta}\mathcal{T}y \|_{\mathbb{Y}}^2 \geq \| \mathcal{T}x \|_{\mathbb{Y}}^2 = \| \mathcal{T} \|_{\mathcal{B}(\mathbb{X},\mathbb{Y})}$. This is a contradiction; we must have $\mathcal{T}y \notin (\mathcal{T}x)^{-(\epsilon)}$. It now follows from Proposition (3.2. i) that $\mathcal{T}y \in (\mathcal{T}x)^{+(\epsilon)} \backslash \mathcal{T}x^{\perp_{BJC}^{\epsilon}}$.

Corollary 3.9: Let $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ be a Banach space and $\Theta \neq \mathcal{T} \in \mathcal{B}(\mathbb{X})$ be such that there exists $x \in M_{\mathcal{T}}$ and $\mathcal{T}x \perp\!\!\!\perp \mathcal{T}y$ for any vector y in . Then $ker\, \mathcal{T} \subseteq \bigcap_{x\in M_{\mathcal{T}}} x^{\perp_{BJC}^{\epsilon}}$.

Proof:

Let us assume that $M_{\mathcal{T}} \neq \phi$. Let $z \in ker\, \mathcal{T}$ and for any $x \in M_{\mathcal{T}}$, we have, $\mathcal{T}z \in (\mathcal{T}x)^{\perp_{BJC}^{\epsilon}}$. Allaying Theorem (3.5) implies that $z \notin x^{+(\epsilon)} \backslash x^{\perp_{BJC}^{\epsilon}}$ and $z \notin x^{-(\epsilon)} \backslash x^{\perp_{BJC}^{\epsilon}}$. Since for any vector y in $\mathbb{X}$, we have: $y \in x^{+(\epsilon)} \backslash x^{\perp_{BJC}^{\epsilon}} \cup x^{-(\epsilon)} \backslash x^{\perp_{BJC}^{\epsilon}} \cup x^{\perp_{BJC}^{\epsilon}}$.

It follows that $z \in x^{\perp_{BJC}^{\epsilon}}$. As this is true for every $z \in ker\, \mathcal{T}$ and for every $x \in M_{\mathcal{T}}$, we must have:

$ker\mathcal{T} \subseteq \bigcap_{x\in M_{\mathcal{T}}} x^{\perp_{BJC}^{\epsilon}}$.

Let us now prove the useful necessary condition for the set $m_{\mathcal{T}}$.

Theorem 3.10: Let $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$, $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ be two Banach spaces and $\Theta \neq \mathcal{T} \in \mathscr{B}(\mathbb{X}, \mathbb{Y})$, $x \in m_{\mathcal{T}}$. Then:

i. $\mathcal{T}(x^{+(\epsilon)}) \subseteq (\mathcal{T}x)^{+(\epsilon)}$.
ii. $\mathcal{T}(x^{-(\epsilon)}) \subseteq (\mathcal{T}x)^{-(\epsilon)}$.
iii. $\mathcal{T}(x^{\perp_{BJC}^{\epsilon}}) \subseteq (\mathcal{T}x)^{\perp_{BJC}^{\epsilon}}$.

Proof:

(i) Let $y \in x^{+(\epsilon)}$. Then $\| x + \mu y \|_{\mathbb{X}}^2 \geq \| x \|_{\mathbb{X}}^2 - 2\epsilon \| x \|_{\mathbb{X}} \| \mu y \|_{\mathbb{X}}$ for all $\mu \geq 0$. Since $x \in m_{\mathcal{T}}$, it follows that $\| x + \mu y \|_{\mathbb{X}} \geq 1$ and for any $\mu \geq 0$, we must have $\| \mathcal{T}x \|_{\mathbb{Y}} \leq \| \mathcal{T}(\frac{x+\mu y}{\|x+\mu y\|_{\mathbb{X}}}) \|_{\mathbb{Y}}$. Now,

$$\| \mathcal{T}x \|_{\mathbb{Y}}^2 \leq \| \mathcal{T}\left(\frac{x+\mu y}{\|x+\mu y\|_{\mathbb{X}}}\right) \|_{\mathbb{Y}}^2 = \frac{1}{\|x+\mu y\|_{\mathbb{X}}^2} \| \mathcal{T}(x+\mu y) \|_{\mathbb{Y}}^2 \leq \| \mathcal{T}(x+\mu y) \|_{\mathbb{Y}}^2 =$$

$\| \mathcal{T}x + \mu\mathcal{T}y \|_{\mathbb{Y}}^2$.

It follows from this description that $\| \mathcal{T}x + \mu\mathcal{T}y \|_{\mathbb{Y}}^2 \geq \| \mathcal{T}x \|_{\mathbb{Y}}^2 \geq \| \mathcal{T}x \|_{\mathbb{Y}}^2 - 2\epsilon \| \mathcal{T}x \|_{\mathbb{Y}} \| \mu \mathcal{T}y \|_{\mathbb{Y}}$. This implies that $\mathcal{T}(y) \in (\mathcal{T}x)^{+(\epsilon)}$, but $\mathcal{T}(y) \in \mathcal{T}(x^{+(\epsilon)})$.

The last parts, (ii) and (iii), can be proved similarly.

Remark 3.11: The other direction in Theorem (3.10) is generally not achieved, as the following example shows:

We denote by $l_\infty(\mathbb{R}^2)$ the space $\mathbb{R}^2$ endowed with the $\|\cdot\|_\infty$ norm.

Let $l_\infty(\mathbb{R}^2)$ whose $\mathbb{S}_{l_\infty(\mathbb{R}^2)}$ is given by the regular hexagon with vertices at $\pm(1,0), \pm(\frac{1}{2},\frac{\sqrt{3}}{2}), \pm(-\frac{1}{2},\frac{\sqrt{3}}{2})$. Let

$$\mathcal{T} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

It follows immediately that $\| \mathcal{T} \|_{\mathcal{B}(l_\infty(\mathbb{R}^2))} = 1$ and $m(\mathcal{T}) = 0$. It is also easy to check that $m_{\mathcal{T}} = \{\pm(0,\frac{\sqrt{3}}{2})\}$.

We indeed have $(1,0) \perp_{BJC}^{\epsilon} (0,\frac{\sqrt{3}}{2})$ and $(1,0) = (1,0) \perp_{BJC}^{\epsilon} \mathcal{T}(0,\frac{\sqrt{3}}{2}) = (0,\frac{3}{4})$, while $(1,0) \notin m_{\mathcal{T}}$.

Remark 3.12: It is interesting to observe that, in the Proposition (3.10.iii), if we assume that $x \in M_{\mathcal{T}}$ instead of assuming $x \in m_{\mathcal{T}}$, then we don't necessarily reach the desired result, as the following example shows:

Let $\Theta \neq \mathcal{T} \in \mathcal{B}(l_\infty(\mathbb{R}^2))$, which is defined by:

$$\mathcal{T}(1,1) = (0,1) \text{ and } \mathcal{T}(-1,1) = (-1,0).$$

Then $M_{\mathcal{T}} = \{(1,1), (-1,1), (-1,-1), (1,-1)\}$. Hence, we have $(1,1) \in M_{\mathcal{T}}$ with $(1,1) \perp_{BJC}^{\epsilon} (0,1)$, but $\mathcal{T}(1,1) = (0,1) \not\perp_{BJC}^{\epsilon} \mathcal{T}(0,1) = (-\frac{1}{2},\frac{1}{2})$.

4. Reflexivity and rank one bounded linear operator

In this section, we study the sets $M_{\mathcal{T}}$ and $m_{\mathcal{T}}$ of a rank one bounded linear operator on a reflexive Banach space (strictly convex). As we will observe, this will lead us to an interesting characterization of reflexivity, in terms of these two sets.

Remark 4.1: To create a more fertile environment for the next theorems in this section, we consider Roberts orthogonality verification.

Theorem 4.2: Let $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$, $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ be two Banach spaces and $\Theta \neq \mathcal{T} \in \mathcal{B}(\mathbb{X},\mathbb{Y})$. Let $x \in M_{\mathcal{T}}$ and $y = \mathcal{T}(x)$. Then there exist a closed hyperspaces $\mathbb{H}_x$ and $\mathbb{H}_y$ of $\mathbb{X}$ and $\mathbb{Y}$, respectively, such that $x \perp_{BJC}^{\epsilon} \mathbb{H}_x$ and $y \perp_{BJC}^{\epsilon} \mathbb{H}_y$ with $\mathcal{T}(\mathbb{H}_x) \subseteq \mathbb{H}_y$.

Proof:

Without loss of generality, we assume that $\| \mathcal{T} \|_{\mathcal{B}(\mathbb{X},\mathbb{Y})} = 1$. Let $x \in M_{\mathcal{T}}$. By applying Hahn Banach Theorem and Theorem (2.13), implies that for any $y = \mathcal{T}(x)$, there exists a vector ψ in $\mathbb{S}_{\mathbb{Y}^*}$ such that:

$$\psi(\mathcal{T}(x)) = \| \psi \|_{\mathbb{Y}^*} \| \mathcal{T}(x) \|_{\mathbb{Y}} = 1.$$

Let $ker\psi = \mathbb{H}_y$ ($ker\psi$ is a closed hyperspace). Applying Theorem (2.13) again, implies that $\mathcal{T}(x) \perp_{BJC}^{\epsilon} \mathbb{H}_y$.

Now, $\psi \text{o} \mathcal{T} \in \mathbb{S}_{\mathbb{X}^*}$ with $\psi \text{ o} \mathcal{T}(x) = \| \mathcal{T}(x) \|_{\mathbb{Y}} = 1 = \| x \|_{\mathbb{X}}$ and $\| \psi \text{o} \mathcal{T} \|_{\mathcal{B}(\mathbb{X},\mathbb{Y})} \leq \| \psi \|_{\mathcal{B}(\mathbb{X},\mathbb{Y})} \| \mathcal{T} \|_{\mathcal{B}(\mathbb{X},\mathbb{Y})}$, where $\| \mathcal{T} \|_{\mathcal{B}(\mathbb{X},\mathbb{Y})} = 1$. Let $ker(\psi \text{ o} \mathcal{T}) = \mathbb{H}_x$. By applying Theorem (2.13), we have,

$x \perp_{BJC}^{\epsilon} \mathbb{H}_x$. Let $w \in \mathbb{H}_x$. Then $\psi \circ \mathcal{T}(w) = \psi(\mathcal{T}(w)) = 0$, implies that $\mathcal{T}(w) \in \mathbb{H}_y$. Since w is arbitrary, we have $\mathcal{T}(\mathbb{H}_x) \subseteq \mathbb{H}_y$.

Theorem 4.3: Let $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$, $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ be two Banach spaces and $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ be a reflexive. If $\Theta \neq \mathcal{T} \in \mathcal{B}(\mathbb{X}, \mathbb{Y})$ be a rank one linear operator, then $x \in M_{\mathcal{T}}$ for some vectors x in $\mathbb{S}_{\mathbb{X}}$ and $m_{\mathcal{T}} = \mathbb{H}_x \cap \mathbb{S}_{\mathbb{X}}$, where $\mathbb{H}_x$ is a closed hyperspace of $\mathbb{X}$ such that $x \perp_{BJC}^{\epsilon} \mathbb{H}_x$.

Proof:

Assume that $\| \mathcal{T} \|_{\mathcal{B}(\mathbb{X},\mathbb{Y})} = 1$. Since $\mathcal{T}$ is a rank one linear operator on a reflexive space $\mathbb{X}$, and from Remark (2.7), we must have $\mathcal{T} \in \mathcal{K}(\mathbb{X}, \mathbb{Y})$. Now, from reflexivity of $\mathbb{X}$, implies that $\mathcal{T}$ attains norm at some vectors x in $\mathbb{S}_{\mathbb{X}}$ and hence $x \in M_{\mathcal{T}}$. Let $y = \mathcal{T}x$. As we mentioned earlier in Theorem (4.2), there exist closed hyperspaces $\mathbb{H}_x$ and $\mathbb{H}_y$ of $\mathbb{X}$ and $\mathbb{Y}$, respectively, such that $x \perp_{BJC}^{\epsilon} \mathbb{H}_x$, $y \perp_{BJC}^{\epsilon} \mathbb{H}_y$ with $\mathcal{T}(\mathbb{H}_x) \subseteq \mathbb{H}_y$. It is clear that $\mathcal{T}x \neq \Theta$. We claim that $\mathcal{T}z = \Theta$ for all $z \in \mathbb{H}_x$. If not, then as $\mathcal{T}x \perp_{BJC}^{\epsilon} \mathcal{T}z$, and hence from Proposition (2.10.iii), $\{\mathcal{T}x, \mathcal{T}z\}$ is linearly independent in $\mathbb{Y}$. However, this implies that the rank of is more than one, which is a contradiction to our hypothesis. Thus, $\mathcal{T}z = \Theta$ for all $z \in \mathbb{H}_x$ and so $\mathbb{H}_x \cap \mathbb{S}_{\mathbb{X}} \subseteq m_{\mathcal{T}}$. Now, let $w \in m_{\mathcal{T}}$. Then $w = h + \eta x$ for some $\eta \in \mathbb{R}$ and $h \in \mathbb{H}_x$. Clearly, $\Theta = \mathcal{T}w = \mathcal{T}h + \eta\mathcal{T}x = \eta\mathcal{T}x$, and so that $\eta = 0$, hence $w = h \in \mathbb{H}_x \cap \mathbb{S}_{\mathbb{X}}$. Thus $m_{\mathcal{T}} \subseteq \mathbb{H}_x \cap \mathbb{S}_{\mathbb{X}}$.

Theorem 4.4: A Banach space $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ is reflexive if and only if, for any closed hyperspace $\mathbb{H}$ of $\mathbb{X}$, there exists a rank one linear operator $\Theta \neq \mathcal{T} \in \mathcal{B}(\mathbb{X})$ with $x \in M_{\mathcal{T}}$ for some vectors x in $\mathbb{S}_{\mathbb{X}}$ and $m_{\mathcal{T}} = \mathbb{H} \cap \mathbb{S}_{\mathbb{X}}$ such that $x \perp_{BJC}^{\epsilon} \mathbb{H}$.

Proof:

Let us first prove the sufficient part. Let $\mathbb{H}$ be a closed hyperspace of $\mathbb{X}$. Then from the our hypothesis, there exists a rank one linear operator $\Theta \neq \mathcal{T} \in \mathcal{B}(\mathbb{X})$ with $x \in M_{\mathcal{T}}$ for some vectors x in $\mathbb{S}_{\mathbb{X}}$ and $m_{\mathcal{T}} = \mathbb{H} \cap \mathbb{S}_{\mathbb{X}}$. Since the rank of $\mathcal{T}$ is one, implies that $m_{\mathcal{T}} = 0$ and $\mathcal{T}h = \Theta$ for all $h \in \mathbb{H}$. We have that $\mathcal{T}x \perp_{BJC}^{\epsilon} \mathcal{T}h$ for all $h \in \mathbb{H}$, by applying Theorem (3.5), it now follows that $x \perp_{BJ}^{\epsilon} h$ for all $h \in \mathbb{H}$. Thus for any closed hyperspace $\mathbb{H}$ of $\mathbb{X}$ with $x \perp_{BJC}^{\epsilon} \mathbb{H}$. Therefore, It follows form Theorem (2.14.iii) that $\mathbb{X}$ is a reflexive.

The necessary part: Assume that $\mathbb{X}$ is a reflexive, deduce from Theorem (2.14.iii), that for any closed hyperspace $\mathbb{H}$ of $\mathbb{X}$, there exists a vector x in $\mathbb{S}_{\mathbb{X}}$ such that $x \perp_{BJC}^{\epsilon} \mathbb{H}$. Clearly, any vector w in $\mathbb{X}$ can be written as $w = h + \eta x$, where $\eta \in \mathbb{R}$ and $h \in \mathbb{H}$.

Now, define $\mathcal{T}: (\mathbb{X}, \|\cdot\|_{\mathbb{X}}) \to (\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ as follows:

$$\mathcal{T}w = \mathcal{T}(h + \eta x) = \eta y, \text{ where } y \in \mathbb{S}_{\mathbb{X}}.$$

Clearly, $\mathcal{T}$ is well-defined and it is a rank one linear operator. Since $x \perp_{BJC}^{\epsilon} \mathbb{H}$, so from Theorem (4.3), it is deduce that $x \in M_{\mathcal{T}}$ and $m_{\mathcal{T}} = \mathbb{H} \cap \mathbb{S}_{\mathbb{X}}$. This establishes the theorem in its entirety.

Theorem 4.5: Let $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ and $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ be two Banach spaces, where $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ be a strictly convex. Then $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ is reflexive if and only if for any closed hyperspace $\mathbb{H}$ of $\mathbb{X}$, there exists a rank one linear operator $\Theta \neq \mathcal{T} \in \mathcal{B}(\mathbb{X}, \mathbb{Y})$ with $M_{\mathcal{T}} = \{\pm x\}$ for some vector x in $\mathbb{S}_{\mathbb{X}}$ and $m_{\mathcal{T}} = \mathbb{H}_x \cap \mathbb{S}_{\mathbb{X}}$.

Proof:

The necessary part: Assume that $\mathbb{X}$ is a reflexive, deduce from Theorem (4.3), for any closed hyperspace $\mathbb{H}$ of $\mathbb{X}$, there exists a rank one linear operator $\Theta \neq \mathcal{T} \in \mathcal{B}(\mathbb{X})$ with $x \in M_{\mathcal{T}}$ for some vectors x in $\mathbb{S}_{\mathbb{X}}$ and $m_{\mathcal{T}} = \mathbb{H} \cap \mathbb{S}_{\mathbb{X}}$ such that $x \perp_{BJC}^{\epsilon} \mathbb{H}$.

Now, to prove that $M_{\mathcal{T}} = \{\pm x\}$, it is clear that any vector w in $\mathbb{S}_{\mathbb{X}}$ can be written as $w = h + \eta x$, for some $\eta \in \mathbb{R}$, $h \in \mathbb{H}$ and $x \perp_{BJC}^{\epsilon} h$.

By applying Theorem (2.9), (strict convexity of $\mathbb{X}$) deduce from the Proposition (2.10.ii), $\eta x \perp_{BJC}^{\epsilon} h$:

$$1 = \| w \|_{\mathbb{X}}^2 = \| h + \eta x \|_{\mathbb{X}}^2 \geq |\eta|^2 \| x \|_{\mathbb{X}}^2 - 2\epsilon \| \eta x \|_{\mathbb{X}} \| h \|_{\mathbb{X}} = |\eta|^2 - 2\epsilon \| \eta x \|_{\mathbb{X}} \| h \|_{\mathbb{X}}.$$

It is easy to know the following $|\eta| = 1 \Leftrightarrow h = 0$.

And from Theorem (3.10.iii), implies that:

$$\| \mathcal{T}w \|_{\mathbb{Y}}^2 = \| \mathcal{T}(h+\eta x) \|_{\mathbb{Y}}^2 \geq |\eta|^2 \| \mathcal{T}x \|_{\mathbb{Y}}^2 - 2\epsilon \| \eta\mathcal{T}(x) \|_{\mathbb{X}} \| \mathcal{T}(h) \|_{\mathbb{X}} = |\eta|^2 \| \mathcal{T}x \|_{\mathbb{Y}}^2.$$

Since $\| \mathcal{T} \|_{\mathcal{B}(\mathbb{X},\mathbb{Y})} = 1$, $\| \mathcal{T}x \|_{\mathbb{Y}} = 1$ we have $\| \mathcal{T}w \|_{\mathbb{Y}} = 1$ and $|\eta| = 1 \Leftrightarrow h = 0$.

Therefore, we must have $M_T = \{\pm x\}$.

The sufficient part is produced directly from Theorem (4.4).

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