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Hille and Nehari Type Oscillation Criteria for Conformable Fractional Differential Equations

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Abstract
In this paper, we develop the Hille and Nehari Type criteria for the oscillation of all solutions to the Fractional Differential Equations involving Conformable fractional derivative. Some new oscillatory criteria are obtained by using the Riccati transformations and comparison technique. We show the validity and effectiveness of our results by providing various examples.

Keywords: Courant minimum principle, Oscillation, Eigen value, Eigen function, Conformable derivative, Riccati Technique.

Introduction

Fractional calculus turned out to be very attractive to Mathematicians as well as Physicists, Biologists, Engineers and Economists. The first application of fractional calculus was due to Abel in his solution of the Tautocrone problem. It likewise has applications in Biophysics, Quantum mechanics, Wave theory, Polymers, Continuum mechanics, Lie theory, Field theory, Spectroscopy, and group theory, among other applications [1, 2, 3].

Fractional differential equations are important tools in the modeling for many physical phenomena in many fields of science and engineering, such as electromagnetic waves, viscoelastic system etc, and can be described with very high accuracy. Recently, fractional derivative and associated integral have been freshly defined by Khalil [4, 5]. It is a natural extension of usual derivative and it is named as Conformable, because this operator preserves basic properties of classical derivative (see [6- 9]). Since conformable fractional derivative (CFD) is a local and limit based operator, it quickly takes a place in application problems [10- 16].

Comparison principles of Sturm’s type will be derived for self-adjoint differential equations. The construction of the main result is given in a very general and novel form in terms of eigenvalues associated with boundary problems for the differential operators. The proof is established as an easy consequence of Courant’s variational principle for the quadratic functional associated with an eigen value problem [17], self-adjoint problem [18], differential equations [19, 20], non-oscillation theorems [21], oscillation stability [22] and comparison theorems [23]. In 2016, Pospisil and Skripkova [24] introduced the Sturm’s comparison principles for conformable fractional differential equations (CFDE’s).

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Motivated by the above papers, the objective of this paper is to establish more general nonoscillation criteria which will contain Nehari criteria as special cases. The essential concept used is the fact that there exists a direct connection between the oscillation problems for the equation

$$T_\alpha[T_\alpha x(t)] + q(t)x(t) = 0 \quad (1.1)$$

and the eigen value problem for the equation

$$T_\alpha[T_\alpha x(t)] + \lambda q(t)x(t) = 0 \quad (1.2)$$

with suitable boundary conditions. Our main concern will be to obtain nonoscillation criteria for the equation (1.1).

The nonoscillatory solutions of Equation (1.1)in $(a,\infty)(a \geq 0)$ if every nontrivial solution has at most one zero in (a,∞); it is called nonoscillatory if there exists a number a such that it is nonoscillatory in (a,∞). The equation (1.1) is said to be oscillatory if it has a proper solution which has an infinite number of zeros in $(0,\infty)$.

This work is organized as follows: Section 2 is devoted to providing essential preliminaries and properties of CFD. In Section 3, we present Nehari type oscillation criteria by using Courant minimum principle. In Section 4, we consider Hille type oscillation criteria by the method of Riccati technique.

II. PRELIMINARIES

In this section, we introduce some standard definitions and essential lemmas on CFD. First we shall start with the definition.

Definition 2.1

Define a function $f:[0,\infty) \to \mathbb{R}$. Then the CFD of f of order α is defined by

$$T_\alpha(f)(t) = \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}$$

for all $t > 0, \alpha \in (0,1]$. If f is α-differentiable in some $(0,a)$, $a > 0$, and $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exists, then define

$$f^{(\alpha)}(0) = lim_{t\to 0^+} f^{(\alpha)}(t).$$

We will sometimes write $f^{(\alpha)}(t)$ for $T_\alpha(f)(t)$to denote the CFD of f of order α.

Definition 2.2

$I_\alpha^a(f)(t) = I_1^0(t^{\alpha-1}f) = \int_a^t \frac{f(x)}{x^{1-\alpha}}dx$, where I is the improper Riemann integral, and $\alpha \in (0,1)$.

So, $I_{\frac{1}{2}}^0(\sqrt{t}\cos t) = \int_0^t \cos x\, dx = \sin t$, and $I_{\frac{1}{2}}^0(\cos 2\sqrt{t}) = \sin 2\sqrt{t}$.

Theorem 2.1

$T_\alpha {I_\alpha}^a(f)(t) = f(t)$, for $t \geq a$, where f is any continuous function in the domain of I_α.

III. MAIN RESULTS

In this section, we prove the Comparison theorems for CFDE's.

3.1 Comparison Theorems For Eigenfunctions :

In 1957, Nehari [20] discovered a connection between the oscillatory behavior of the solution of

$$u'' + c(x)u = 0$$

and the eigenvalue problem

$$u'' + \lambda c(x)u = 0, \;\; u(\alpha) = u'(\beta) = 0.$$

Now, consider the solution of CFDE's of the form

$$T_\alpha[T_\alpha x(t)] + q(t)x(t) = 0 \quad (3.1)$$

and the eigenvalue problem

$$T_\alpha[T_\alpha x(t)] + \lambda q(t)x(t) = 0, \quad x(a) = T_\alpha(x(b)) = 0.$$

Actually, this is the same as the problem

$$lx = \lambda x\,, \qquad P_a[x] = P_b[x] = 0,$$

where the homogeneous boundary conditions

$$P_a[x] = \beta x(a) - m(a)T_\alpha[x(a)] = 0,$$
$$P_b[u] = \gamma x(b) - m(b)T_\alpha[x(b)] = 0, \quad (3.2)$$

for some number β and γ.Consider the differential operator l, defined by

$$lx = \frac{1}{K(t)}\{-T_\alpha[m(t)T_\alpha(x(t))] + q(t)x(t)\},$$

where m , q and K are real-valued continuous functions on$[a,b]$, m and K are positive on $[a,b]$, and $m \in C^\alpha(a,b\,)$.

The differential equation $lx = \lambda x$ satisfies the boundary conditions (3.2).

$$K(t)\lambda x(t) = -T_\alpha\left[m(t)T_\alpha\big(x(t)\big)\right] + q(t)x(t),$$

$$T_\alpha[m(t)T_\alpha(x(t))] + [K(t)\lambda - q(t)]x(t) = 0,\ P_a[x] = P_b[x] = 0. \tag{3.3}$$

In the special case that $m(t),\ K(t),\ q(t),\ P_a(x)$ and $P_b(x)$ in (3.3) are replaced by 1, $q(t)$, 0, $x(a)$ and $T_\alpha[x(b)]$ respectively, we get

$$T_\alpha[T_\alpha x(t)] + \lambda q(t)x(t) = 0, \quad x(a) = T_\alpha\big(x(b)\big) = 0. \tag{3.4}$$

Let $j[x]$ be the quadratic function defined by

$$j[x] = \int_a^b \left[m\left|T_\alpha\big(x(t)\big)\right|^2 + q|x(t)|^2\right]dt + \gamma|x(b)|^2 + \beta|x(a)|^2$$

with domain $D = C^\alpha[a,b]$. The analog of above equation in the case $\beta = \infty$ and γ is finite,

$$j_a[x] = \int_a^b \left[m\left|T_\alpha\big(x(t)\big)\right|^2 + q|x(t)|^2\right]dt + \gamma|x(b)|^2, \tag{3.5}$$

and the associated quadratic functional of (3.5) becomes

$$j_a[x] = I_\alpha\left(\int_a^b [T_\alpha|x(t)|^2]\,dt\right) = \int_a^b t^{\alpha-1}\left(t^{2(1-\alpha)}[x'(t)]^2\right)dt \quad \text{(by Definition 2.2)}$$

$$j_a[x] = \int_a^b t^{1-\alpha}\,[x'(t)]^2\ dt\,,\ \ x \in D_a, \tag{3.6}$$

where $D_a = \{x \in C^\alpha[a,b],\ x(a) = 0\}$.

Theorem 3.1

If λ_0 is the smallest eigenvalue of (3.1), then

$$\lambda_0 \parallel y \parallel^2 \le j_a[y] \text{ for all real } y \in D_a\,. \tag{3.7}$$

In fact, this is even true for all real $y \in C^\alpha[\beta,\gamma]$ satisfying the weaker condition $lim_{t\to a^+} \frac{y^2(t)}{t-a} = 0$.

Proof :

Consider $(T_\alpha\,[y(t)] - \frac{T_\alpha[x(t)]\ \ y(t)}{x(t)})^2$. The proof follows from the identity

$$I_\alpha(T_\alpha\,[y(t)] - \frac{T_\alpha[x(t)]\ \ y(t)}{x(t)})^2 \ge 0.$$

$$0 \le \int_{a+\varepsilon}^b t^{\alpha-1}(T_\alpha\, y(t) - \frac{T_\alpha[x(t)]\ \ y(t)}{x(t)})^2 dt\,.$$

$$\le \int_{a+\varepsilon}^b t^{\alpha-1}(T_\alpha[y(t)])^2\,dt - 2\int_{a+\varepsilon}^b t^{\alpha-1}T_\alpha[y(t)]\ \frac{T_\alpha\big(x(t)\big)\,y(t)}{x(t)}dt + \int_{a+\varepsilon}^b t^{\alpha-1}\frac{\left[T_\alpha\big(x(t)\big)\right]^2 y^2(t)}{x^2(t)}dt$$

$$= \int_{a+\varepsilon}^b t^{\alpha-1}(t^{1-\alpha}[y'(t)])^2\,dt - 2\int_{a+\varepsilon}^b t^{\alpha-1}t^{1-\alpha}[y'(t)]\ \frac{t^{1-\alpha}\big(x'(t)\big)\ y(t)}{x(t)}dt + \int_{a+\varepsilon}^b t^{\alpha-1}t^{2(1-\alpha)}\frac{[x'(t)]^2}{x^2(t)}y^2(t)dt$$

$$= \int_{a+\varepsilon}^b t^{(1-\alpha)}\,[y'(t)]^2 dt - 2\int_{a+\varepsilon}^b \frac{t^{1-\alpha}\big(x'(t)\big)y'(t)\,y(t)}{x(t)}dt + \int_{a+\varepsilon}^b t^{1-\alpha}\frac{[x'(t)]^2}{x^2(t)}y^2(t)dt. \tag{3.8}$$

Consider

$$T_\alpha\left[\frac{T_\alpha[x(t)]\,y^2(t)}{x(t)}\right] = \frac{x(t)T_\alpha[t^{1-\alpha}x'(t)y^2(t)] - t^{1-\alpha}x'(t)y^2(t)T_\alpha[x(t)]}{[x(t)]^2}$$

$$= \frac{t^{2(1-\alpha)}x(t)x''(t)y^2(t)}{[x(t)]^2} + \frac{(1-\alpha)t^{1-2\alpha}x(t)x'(t)y^2(t)}{[x(t)]^2} + \frac{2t^{2(1-\alpha)}x(t)y(t)x'(t)y'(t)}{[x(t)]^2} - \frac{t^{2(1-\alpha)}[x'(t)]^2y^2(t)}{[x(t)]^2}$$

$$= \frac{t^{2(1-\alpha)}x''(t)y^2(t)}{x(t)} + \frac{(1-\alpha)t^{1-2\alpha}x'(t)y^2(t)}{x(t)} + \frac{2t^{(1-\alpha)}y(t)x'(t)y'(t)}{x(t)}t^{1-\alpha} - \frac{t^{2(1-\alpha)}[x'(t)]^2y^2(t)}{[x(t)]^2}$$

$$t^{1-\alpha}\left[\frac{x'(t)y(t)}{x(t)}\right]^2 - 2\frac{t^{1-\alpha}x'(t)y(t)y'(t)}{x(t)} = \frac{t^{1-\alpha}\,x''(t)y^2(t)}{x(t)} + \frac{(1-\alpha)t^{-\alpha}x'(t)y^2(t)}{x(t)} - \left[\frac{t^{1-\alpha}x'(t)y^2(t)}{x(t)}\right]' \tag{3.9}$$

By substituting (3.9) in (3.8), we have

$$0 \le \int_{a+\varepsilon}^{b} t^{(1-\alpha)}[y'(t)]^2\, dt + \int_{a+\varepsilon}^{b} t^{(1-\alpha)} \frac{x''(t)\, y^2(t)}{x(t)} dt + (1-\alpha)\int_{a+\varepsilon}^{b} \frac{t^{-\alpha}x'(t)y^2(t)}{x(t)} dt - \int_{a+\varepsilon}^{b} t^{(1-\alpha)} \left[\frac{x'(t)y^2(t)}{x(t)}\right]' dt.$$

Integratinglast term of the above inequality by parts valid for all $x \in c^{\alpha}(a,b)$ which do not vanish in the interval $(a,b)$and $0 < \varepsilon < b-a$. If x is an eigenfunction of (3.4) corresponding to the smallest eigenvalue λ_0 and hence free of zeros in (a,b), then the $limit\ \epsilon \longrightarrow 0$:

$$0 \le \int_{a}^{b} t^{(1-\alpha)}[y'(t)]^2\, dt + \int_{a}^{b} t^{(1-\alpha)} \frac{x''(t)\, y^2(t)}{x(t)} dt + (1-\alpha)\int_{a}^{b} \frac{t^{-\alpha}x'(t)y^2(t)}{x(t)} dt - \int_{a}^{b} t^{(1-\alpha)} \left[\frac{x'(t)y^2(t)}{x(t)}\right]' dt. \quad (3.10)$$

If u is an eigenfunction of (3.4) corresponding to the smallest eigen value λ_0 , then

$$T_\alpha\big[T_\alpha\big(x(t)\big)\big] + \lambda_0\, q(t)x(t) = 0\,, \quad x(a) = T_\alpha\big(x(b)\big) = 0.$$

From this, we have

$$T_\alpha[t^{1-\alpha}x'(t)] = -\lambda_0\, q(t)x(t),$$
$$t^{1-\alpha}T_\alpha[x'(t)] + x'(t)T_\alpha[t^{1-\alpha}] = -\lambda_0\, q(t)x(t),$$
$$t^{2(1-\alpha)}x''(t) + x'(t)(1-\alpha)t^{1-2\alpha} = -\lambda_0\, q(t)x(t). \quad (3.11)$$

We get

$$t^{1-\alpha}\frac{x''(t)}{x(t)} = -\lambda_0\, q(t)t^{\alpha-1} - (1-\alpha)t^{-\alpha}\frac{x'(t)}{x(t)}.$$

Now, (3.10) can be written as,

$$0 \le \int_{a}^{b} t^{(1-\alpha)}[y'(t)]^2\, dt + \int_{a}^{b}\left[-\lambda_0 q(t)t^{\alpha-1} - (1-\alpha)t^{-\alpha}\frac{x'(t)}{x(t)}\right] y^2(t)dt + (1-\alpha)\int_{a}^{b} t^{-\alpha}\frac{x'(t)y^2(t)}{x(t)} dt$$

which implies that

$$\lambda_0 \int_{a}^{b} t^{\alpha-1}q(t)y^2(t)dt \le \int_{a}^{b} t^{(1-\alpha)}[y'(t)]^2\, dt,$$

which implies that

$$\lambda_0\ \|y\|^2 \le j_a[y] \text{ for all } y \in D_a,$$

where $\|y\|^2 = I_\alpha\left[\int_a^b |y|^2 q(t)dt = \int_a^b t^{\alpha-1}q(t)y^2(t)\, dt\right]$.

Theorem 3.2

Let λ_0 denote the smallest eigenvalue of (3.4). Then equation (3.1) is nonoscillatory in (a,∞) if and only if $\lambda_0\ > 1$ for all b satisfying $b > a$.

Proof

If (3.1) is nonoscillatory in (a,∞) and $y(t)$ is a solution of (3.1) such that $y(a) = 0$ and $T_\alpha y(a) > 0$.

Claim: $T_\alpha y(a) > 0$ for all $t \ge a$.

$$T_\alpha[T_\alpha(y(t))] + q(t)y(t) = 0, y(a) = T_\alpha(y(b)) = 0. \quad (3.12)$$

By taking $I_\alpha$of both sides fromt_1 to t_2, we get

$$I_\alpha^t T_\alpha[T_\alpha(y(t))] = -\int_{t_1}^{t_2} q(t)y(t)d_\alpha t$$

$$T_\alpha(y(t_2)) - T_\alpha\big(y(t_1)\big) = -\int_{t_1}^{t_2} q(t)y(t)d_\alpha t < 0$$

$$T_\alpha(y(t_2)) - T_\alpha\big(y(t_1)\big) < 0 \text{ for } a \le t_1 < t_2 < \infty.$$

Then $T_\alpha(y(t))$ is never increasing for $a < t$ and the graph of $y = y(t)$ is concave downwards, it follows that $T_\alpha(y(t)) > 0\ \forall\ t > a$.

Let x be a positive eigenfunction of (3.4) in $[a,b]$ corresponding to λ_0.Then by multiplying$y(t)$ by (3.11), we have

$$t^{2(1-\alpha)}x''(t)y(t)+(1-\alpha)x'(t)t^{1-2\alpha}y(t)=-\lambda_0 q(t)x(t)y(t),$$

which implies that

$$\lambda_0 x(t)[-q(t)y(t)]-t^{2(1-\alpha)}x''(t)y(t)-(1-\alpha)x'(t)t^{1-2\alpha}y(t)=0.$$

By using (3.11), $\lambda_0>1$, we get

$$x(t)t^{2(1-\alpha)}y''(t)-t^{2(1-\alpha)}x''(t)y(t)-(1-\alpha)t^{1-2\alpha}[x'(t)y(t)-y'(t)x(t)]>0,$$

$$t^{2(1-\alpha)}[x(t)y''(t)-x''(t)y(t)]-(1-\alpha)t^{1-2\alpha}[x'(t)y(t)-y'(t)x(t)]>0.$$

By integrating the above inequality from a to b, and using (3.11),we get

$$\int_a^b t^{2(1-\alpha)}[x(t)y''(t)-x''(t)y(t)]dt-(1-\alpha)\int_a^b t^{1-2\alpha}[x'(t)y(t)-y'(t)x(t)]dt>0.$$

By taking I_α from a to b, we obtain

$$I_\alpha\left[\int_a^b t^{2(1-\alpha)}[x(t)y''(t)-x''(t)y(t)]dt-(1-\alpha)\int_a^b t^{1-2\alpha}[x'(t)y(t)-y'(t)x(t)]dt\right]>0$$

$$\int_a^b t^{1-\alpha}[x(t)y''(t)-x''(t)y(t)]dt-(1-\alpha)\int_a^b t^{-\alpha}[x'(t)y(t)-y'(t)x(t)]dt>0,$$

which gives

$$\int_a^b t^{\alpha-1}[\lambda_0 x(t)q(t)y(t)-q(t)y(t)x(t)]dt+(1-\alpha)\int_a^b x'(t)t^{-\alpha}y(t)dt$$
$$-(1-\alpha)\int_a^b t^{-\alpha}x(t)y'(t)dt-(1-\alpha)\int_a^b t^{-\alpha}x'(t)y(t)dt+$$
$$(1-\alpha)\int_a^b t^{-\alpha}x(t)y'(t)dt>0$$

$$\int_a^b t^{\alpha-1}[\lambda_0 q(t)x(t)y(t)-q(t)x(t)y(t)]\,dt>0.$$

Finally, we have

$$\int_a^b t^{\alpha-1}(\lambda_0-1)x(t)q(t)y(t)dt>0, \quad (3.13)$$

since q,x and y are positive in (a,b), $\lambda_0>1$.
Conversely, if $\lambda_0>1$, then (3.13) shows that y cannot have a zero to the right of a, if b is the first zero,then (3.13) implies that $y'(b)>0$ an impossibility.

Theorem 3.3

If (3.1) is non-oscillatory in (a,∞), then

$$(t-a)^{1-l}\int_a^t (s-a)^l q(s)ds+(t-a)^{1-m}\int_t^\infty (s-a)^m q(s)ds\le \frac{l-m}{4}\left[1+\frac{[1+4\alpha(\alpha-1)}{(l-2\alpha+1)(2\alpha-1-m)}\right], \quad (3.14)$$

where l,m are arbitrary numbersand satisfying $l>2\alpha-1$, $0\le m<2\alpha-1$.

Proof

Since (3.1) is nonoscillatory, then $\lambda_0>1$ by theCourant's minimum principle (3.7), which yields

$$\int_a^b q(s)y^2(s)ds<\int_a^b [T_\alpha(y(s))]^2\,ds\,,b>a \quad (3.15)$$

for all $y\in\mathcal{C}^\alpha[a,b]$ such that $lim(t-b)^{-1}y^2(t)=0$ $(t\to a^+)$ with

$$y(t)=\begin{cases}(t-a)^{l/2}(t_0-a)^{-l/2} & if\ a\le t\le t_0\\ (t-a)^{m/2}(t_0-a)^{-m/2} & if\ t\ge t_0\end{cases} \quad (3.16)$$

$$T_\alpha(y(t))=\begin{cases}\frac{l}{2}(t-a)^{\frac{l}{2}-\alpha}(t_0-a)^{-l/2} & if\ a\le t\le t_0\\ \frac{m}{2}(t-a)^{\frac{m}{2}-\alpha}(t_0-a)^{-m/2} & if\ t\ge t_0\end{cases} \quad (3.17)$$

where $a<t_0<b$. (3.14) becomes

$$\int_a^b q(s)y^2(s)ds<\int_a^b [T_\alpha y(s)]^2\,ds\,,\qquad b>a,$$

which gives

$$\int_a^{t_0} q(s)y^2(s)ds + \int_{t_0}^b q(s)y^2\,(s)ds \; < \int_a^{t_0} [T_\alpha y(s)]^2\,ds + \int_{t_0}^b [T_\alpha y(s)]^2\; ds.$$

From (3.16) and (3.17), we get

$$\int_a^{t_0} q(s)\left[(s-a)^{\frac{l}{2}}(t_0-a)^{\frac{-l}{2}}\right]^2 ds + \int_{t_0}^b q(s)\left[(s-a)^{\frac{m}{2}}(t_0-a)^{\frac{-m}{2}}\right]^2 ds$$
$$< \int_a^{t_0}\left[\frac{l}{2}(s-a)^{\frac{l}{2}-\alpha}(t_0-a)^{\frac{-l}{2}}\right]^2 ds + \int_{t_0}^b\left[\frac{m}{2}(s-a)^{\frac{m}{2}-\alpha}(t_0-a)^{\frac{-m}{2}}\right]^2 ds. \qquad (3.18)$$

Consider

$$\int_a^{t_0}\left[\frac{l}{2}(s-a)^{\frac{l}{2}-\alpha}(t_0-a)^{\frac{-l}{2}}\right]^2 ds \;=\; \int_a^{t_0}\frac{l^2}{4}(s-a)^{2(\frac{l}{2}-\alpha)}(t_0-a)^{\frac{-2l}{2}}ds$$
$$= \frac{l^2}{4(t_0-a)^l}\int_a^{t_0}(s-a)^{l-2\alpha}\,ds.$$
$$\int_a^{t_0}\left[\frac{l}{2}(s-a)^{\frac{l}{2}-\alpha}(t_0-a)^{\frac{-l}{2}}\right]^2 ds = \frac{l^2}{4(l-2\alpha+1)(t_0-a)^{2\alpha-1}} \qquad (3.19)$$

and

$$\int_{t_0}^b\left[\frac{m}{2}(s-a)^{\frac{m}{2}-\alpha}(t_0-a)^{\frac{-m}{2}}\right]^2 ds = \int_{t_0}^b\frac{m^2}{4}(s-a)^{m-2\alpha}(t_0-a)^{-m}ds.$$
$$= \frac{m^2}{4(t_0-a)^m}\int_{t_0}^b(s-a)^{m-2\alpha}\,ds.$$
$$\int_{t_0}^b\left[\frac{m}{2}(s-a)^{\frac{m}{2}-\alpha}(t_0-a)^{\frac{-m}{2}}\right]^2 ds \;=\; \frac{m^2\left[(b-a)^{m-2\alpha+1}-(t_0-a)^{m-2\alpha+1}\right]}{4(m-2\alpha+1)(t_0-a)^m}$$

(3.20)

By substituting (3.19) and (3.20) in (3.18), we obtain

$$\int_a^{t_0} \mathrm{q(s)}\left[(s-a)^{\frac{l}{2}}(\mathrm{t}_0-a)^{\frac{-l}{2}}\right]^2 \mathrm{ds} + \int_{\mathrm{t}_0}^b \mathrm{q(s)}\left[(s-a)^{\frac{m}{2}}(\mathrm{t}_0-a)^{\frac{-m}{2}}\right]^2 \mathrm{ds}$$
$$< \frac{l^2}{4(l-2\alpha+1)(t_0-a)^{2\alpha-1}} + \frac{m^2[(b-a)^{m-2\alpha+1}-(t_0-a)^{m-2\alpha+1}]}{4(m-2\alpha+1)(t_0-a)^m}$$
$$< \frac{l^2}{4(l-2\alpha+1)(t_0-a)^{2\alpha-1}} + \frac{m^2[(t_0-a)^{m-2\alpha+1}-(b-a)^{m-2\alpha+1}]}{4(2\alpha-m-1)(t_0-a)^m}$$
$$(t_0-a)^{-l}\int_a^{t_0} q(s)(s-a)^l\,ds + (t_0-a)^{-m}\int_{t_0}^\infty q(s)(s-a)^m\,ds$$
$$< \frac{l^2}{4(l-2\alpha+1)(t_0-a)^{2\alpha-1}} + \frac{m^2\left[(t_0-a)^{m-2\alpha+1}-(b-a)^{m-2\alpha+1}\right]}{4(2\alpha-m-1)(t_0-a)^m}.$$

and (3.14) follows in the limit $b\to\infty$, we get

$$(t-a)^{-l}\int_a^t q(s)(s-a)^l\,ds + (t-a)^{-m}\int_t^\infty q(s)(s-a)^m\,ds$$
$$\le \quad \frac{l^2}{4(l-2\alpha+1)(t-\beta)^{2\alpha-1}} + \frac{m^2(t-a)^{m-2\alpha+1}}{4(2\alpha-m-1)(t-a)^m}$$
$$\le \frac{l^2}{4(l-2\alpha+1)(t-a)^{2\alpha-1}} + \frac{m^2}{4(2\alpha-m-1)(t-a)^{2\alpha-1}}$$
$$\le \frac{1}{4(t-a)^{2\alpha-1}}\left[\frac{l^2}{l-2\alpha+1}+\frac{m^2}{2\alpha-m-1}\right]$$
$$(t-a)^{2\alpha-l-1}\int_a^t q(s)(s-a)^l\,ds + (t-a)^{2\alpha-m-1}\int_t^\infty q(s)(s-a)^m\,ds$$
$$\le \frac{1}{4}\left[\frac{l^2}{l-2\alpha+1}+\frac{m^2}{2\alpha-m-1}\right].$$
$$\le \frac{1}{4}\left[\frac{2\alpha(l^2-m^2)-l^2+m^2-lm(l-m)}{(l-2\alpha+1)(2\alpha-m-1)}\right]$$

$$\leq \frac{(l-m)}{4}\left[\frac{2\alpha(l+m)-(l+m)-lm}{(l-2\alpha+1)(2\alpha-m-1)}\right]$$
$$\leq \frac{(l-m)}{4}\left[1+\frac{(1+4\alpha^2-4\alpha)}{(l-2\alpha+1)(2\alpha-m-1)}\right]$$
$$\leq \frac{(l-m)}{4}\left[1+\frac{(1+4\alpha(\alpha-1))}{(l-2\alpha+1)(2\alpha-m-1)}\right].$$

Since the left side of inequality (3.13) is nonnegative, the following inequalities are obtained when $l = 2\alpha$ and $m = 0$, respectively:

$$(t-a)^{2\alpha-l-1}\int_a^t q(s)(s-a)^l\,ds \leq \frac{l}{4}\left[1+\frac{(1+4\alpha^2-4\alpha)}{(l-2\alpha+1)(2\alpha-1)}\right]$$
$$\leq \frac{l}{4}\left[\frac{2\alpha l-l-4\alpha^2+2\alpha+2\alpha-1+1+4\alpha^2-4\alpha}{(l-2\alpha+1)(2\alpha-1)}\right]$$
$$(t-a)^{2\alpha-l-1}\int_a^t q(s)(s-a)^l\,ds \leq \frac{l^2}{4(l-2\alpha+1)}, \quad l > 2\alpha-1. \tag{3.21}$$

and

$$(t-a)^{2\alpha-m-1}\int_t^\infty q(s)(s-a)^m\,ds \leq \frac{(2\alpha-m)}{4}\left[1+\frac{1+4\alpha^2-4\alpha}{(2\alpha-2\alpha+1)(2\alpha-m-1)}\right]$$
$$\leq \frac{(2\alpha-m)}{4}\left[1+\frac{1+4\alpha^2-4\alpha}{(2\alpha-m-1)}\right]$$
$$(t-a)^{2\alpha-m-1}\int_t^\infty q(s)(s-a)^m\,ds \leq \frac{(2\alpha-m)}{4}\left[\frac{4\alpha^2-2\alpha-m}{2\alpha-m-1}\right]. \tag{3.22}$$

Hence

$$\text{limsup}_{t\to\infty}\, t^{2\alpha-l-1}\int_a^t q(s)s^l\,ds \leq \frac{l^2}{4(l-2\alpha+1)}, \quad l > 2\alpha-1 \tag{3.23}$$
$$\text{limsup}_{t\to\infty}\, t^{2\alpha-m-1}\int_a^\infty q(s)s^m\,ds \leq \frac{(2\alpha-m)}{4}\left[\frac{4\alpha^2-2\alpha-m}{2\alpha-m-1}\right], 0 \leq m < 2\alpha-1. \tag{3.24}$$

In particular, (3.21) and (3.22) show that $t^p q(t)$, $0 \leq m < 2\alpha-1$ is integrable in (a,∞) if (3.1) is non-oscillatory in (a,∞).

IV. HILLE OSCILLATION CRITERIA FOR CONFORMABLE FRACTIONALDIFFERENTIAL EQUATIONS

In this section, we consider that the CFDE's are

$$T_\alpha[T_\alpha x(t)] + q(t)x(t) = 0 \tag{4.1}$$
$$T_\alpha[T_\alpha y(t)] + Q(t)y(t) = 0, \tag{4.2}$$

on the half-open interval $[0,\infty)$, where $q(t), Q(t)$ are positive continuous functions on this interval.
Hille [20] stated his results in terms of the function $h(t)$ defined by

$$h(t) = t\int_t^\infty q(s)ds, \tag{4.3}$$

and the numbersh_* and $h^*$are defined by

$$h_* = \lim_{t\to\infty}\inf\ h(t) \text{ and } h^* = \lim_{t\to\infty}\sup\ h(t) \tag{4.4}$$

If the integral in (4.3) is not finite, then in this case we get $h^* = h_* = \infty$.

4.1. THE HILLE-WINTER COMPARISON THEOREMS

Hille [20] also proved a comparison theorem similar to the Sturm's theorem. Consider the following two CFDE's,

$$T_\alpha[T_\alpha y(t)] + F(t)y(t) = 0 \tag{4.5}$$
$$T_\alpha[T_\alpha x(t)] + f(t)x(t) = 0. \tag{4.6}$$

Define $H(t)$ and $h(t)$ by

$$H(t) = \int_t^\infty F(s)\,ds$$
$$h(t) = \int_t^\infty f(s)ds. \tag{4.7}$$

Hille first proved the following theorem under the additional hypothesis that $F(t)$ and $f(t)$ were both positive. The proof presented here was provided by Winter [23, 24] and requires no such restriction on $F(t)$or $f(t)$. This is known as the Hille-Winter comparison theorem.

Theorem 4.2

Define $H(t)$ and $h(t)$ as in (4.7). Let $F(t)$ and $f(t)$ be continuous on $(0,\infty)$ such that $h(t)$ and $H(t)$ both converge (may be only conditionally) Further, let $0 \le H(t) \le h(t)$ for all $t \ge a > 0$. If (4.6) is nonoscillatory, then (4.5) is nonoscillatory. If (4.5) is oscillatory, then (4.6) is oscillatory.

Lemma 4.1

If (4.1) is nonoscillatory and $x(t)$ is a solution of (4.1) such that $x(t) \neq 0$ for $t \ge a$, then

$$0 < (t+\delta)y(t) \le 1, \quad t \ge a, \tag{4.8}$$

where

$$y(t) = \frac{T_\alpha[x(t)]}{x(t)},\ \delta = -a + {}^{1}\!/_{y(a)}.$$

Proof

If $x(t) > 0$ for $t \ge a$, then we also know that $x'(t) > 0$ for $t \ge a$. Similarly, $y(t) > 0$ if $x(t) < 0$ for $t \ge a$. This proves the left part of the inequality (4.8). Since $x(t)$ satisfies (4.1), then$y(t)$ satisfies the Riccati equation

$$T_\alpha[y(t)] = \frac{x(t)T_\alpha[t^{1-\alpha}x'(t) - t^{1-\alpha}x'(t)[T_\alpha x(t)]}{x^2(t)}$$

$$t^{1-\alpha}y'(t) = \frac{t^{2(1-\alpha)}x''(t)}{x(t)} + \frac{(1-\alpha)t^{2(1-\alpha)}x'(t)}{x(t)} - y^2(t)$$

$$t^{1-\alpha}y'(t) + y^2(t) + q(t) = 0, \quad t \ge a. \tag{4.9}$$

Since $q(t) > 0$, then $t^{1-\alpha}y'(t) + y^2(t) \le 0$ and hence

$$T_\alpha\left[-\frac{1}{y(t)} + t\right] \le 0.$$

Thus

$$-\frac{1}{y(t)} + t \le -\frac{1}{y(a)} + a, \quad a \le t,$$

which is equivalent to (4.8).

Theorem 4.3

Equation (4.1) is nonoscillatoryiff the nonlinear integral equation

$$t^{1-\alpha}y(t) + (1-\alpha)\int_t^\gamma s^{-\alpha}\, y(s)ds = \int_t^\infty y^2(s)\, ds + \int_t^\infty q(s)ds \tag{4.10}$$

has a solution for sufficiently large s.

Proof

If (4.1) is nonoscillatory, then $y(t) = \frac{t^{1-\alpha}x'(t)}{x(t)}$ satisfies (4.9) as pointed out before. Integration of (4.9) from t to γ gives

$$\int_t^\gamma s^{1-\alpha}y'(s)ds + \int_t^\gamma y^2(s)ds + \int_t^\gamma q(s)ds = 0$$

$$[s^{1-\alpha}\, y(s)]_t^\gamma - (1-\alpha)\int_t^\gamma s^{1-\alpha-1}\, y(s)ds + \int_t^\gamma y^2(s)ds + \int_t^\gamma q(s)ds = 0$$

$$\gamma^{1-\alpha}y(\gamma) - t^{1-\alpha}y(t) - (1-\alpha)\int_t^\gamma s^{-\alpha}\, y(s)ds + \int_t^\gamma y^2(s)ds + \int_t^\gamma q(s)ds = 0.$$

Since (4.1) is nonoscillatory, Lemma 4.1 shows that the second integral tends to a finite limit as $\gamma \to \infty$ and also that $y(\gamma) \to 0$ as $\gamma \to \infty$.

It follows that the last integral also tends to a finite limit as $\gamma \to \infty$ and that $y(t)$ satisfies the integral equation (4.10),which implies that

$$t^{1-\alpha}y(t) + (1-\alpha)\int_t^\gamma s^{-\alpha}y(s)ds = \int_t^\infty y^2(s)ds + \int_t^\infty q(s)ds.$$

Conversely, if there exists a finite β such that $T_\alpha[T_\alpha y(t)] + 2H(F)T_\alpha[y(t)] + [H^2(t) + p(t)]y = 0$ has a solution $y(t)$ for $t \ge a$, it follows from the form of the equation that $y^2(t)$ is integrable in (a,∞) and $y(t)$ is positive, monotone decreasing, differentiable function.

Differentiation of (4.10) with respect to t yields

$$[t^{1-\alpha}y'(t) + (1-\alpha)y(t)t^{-\alpha}] + (1-\alpha)\frac{d}{ds}\left[\int_t^\infty s^{-\alpha}y(s)ds\right] = \frac{d}{ds}\left[\int_t^\infty y^2(s)ds\right] + \frac{d}{ds}\left[\int_t^\infty q(s)ds\right]$$

$$t^{1-\alpha}y'(t) + (1-\alpha)t^{-\alpha}y(t) - (1-\alpha)t^{-\alpha}y(t) = -y^2(t) - q(t)$$

$$t^{1-\alpha}y'(t) = -y^2(t) - q(t)$$

$$t^{1-\alpha}y'(t)+y^2(t)+q(t)=0,$$

which shows that $y(t)$ satisfies $y(F)T^{1-\alpha}+\int_t^T y^2(s)ds$. Hence

$$v(t)=\frac{t^{1-\alpha}u'(t)}{u(t)}$$

Taking integral from a to γ

$$\int_a^\gamma t^{1-\alpha}\,y(t)dt=\int_a^\gamma \frac{x'(t)}{x(t)}dt$$

$$exp\left(\int_a^\gamma t^{1-\alpha}\,y(t)dt\right)=x(t),$$

satisfies (4.1) for $t\geq a$, and since $x(t)\geq 1$, (4.1) is nonoscillatory.

V.CONCLUSION AND FUTURE WORK

In this investigation, the aim was to present some oscillatory or nonoscillatorybehaviors of the conformable fractional differential equations through the instrument of the Nehari and Hille type theorems. Since the obtained results are general forms of earlier works, they would assist the investigations in future studies.

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