DOI: 10.24996/ijs.2020.SI.1.33

ISSN: 0067-2904

On the Construction of d-Koszul algebras

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Received: 6/1/ 2020 Accepted: 15/ 3/2020

Abstract
An algebra $\boldsymbol{B}$ has been constructed from a (D, A)-stacked algebra $\boldsymbol{A}$, under the conditions that $D = dA$, A≥ 1 and $d \geq 2$. It is shown that when the construction of algebra $\boldsymbol{B}$ is built from a (D, A)-stacked monomial algebra $\boldsymbol{A}$ then $\boldsymbol{B}$ is a d-Koszul monomial algebra.

Keywords: Koszul Algebra, d-Koszul monomial algebra, and (D, A)-stacked monomial algebra.

حول بناء جبر الدي كاوزل

رؤى يوسف جواد
معهداعدادالمدربين التقنين,الجامعه التقنيه الوسطى, بغداد, العراق.

الخلاصه
ليكن الجبر Bالمبني من الجبرال مرصوص (D,A) تحت الشروط التاليه: ,D = dA.A≥ 1,d ≥ 2
.ولقد برهنا ان هذا البناء يعطي لنا جبر من نوع الدي كازول اذا كان الجبر اللذي صنع منه من نوع جبرأحادي الحدودالمرصوص.

1-Introduction

In this paper, we take a (D, A)-stacked algebra $\boldsymbol{A} = K\Gamma$/I, where D=dA, A≥ 1 and $d \geq 2$. We use that to construct an algebra B. The aim of this paper is to investigate the question whether each d-Koszul algebra has been derived from a (D, A)-stacked algebra where $D = dA$ by using our construction. Leader [1] gives a construction of an algebra $\widetilde{\boldsymbol{\Lambda}}$ from a d-Koszul algebra $\boldsymbol{\Lambda}$ and she shows that if $\boldsymbol{\Lambda}$ is a d-Koszul algebra then $\widetilde{\boldsymbol{\Lambda}}$ is a (D, A)-stacked algebra, where $D = dA$. Moreover, Jawad and Snashall [2] generalized this construction where they started with a finite dimensional algebra $\boldsymbol{\Lambda} = KQ/I$, for $A \geq 1$ and used it to construct a stretched algebra $\widetilde{\boldsymbol{\Lambda}}$. In this paper, the opposite of the above statement is investigated.

Furthermore, we show how the algebra $\boldsymbol{B}$ is constructed from a (D, A)-stacked algebra $\boldsymbol{A}$, and we prove that if $\boldsymbol{A}$ is an (D, A)-stacked monomial algebra, then $\boldsymbol{B}$ is d-Koszul where $D = dA$.

The paper begins with a background, while in section 3 we provide our construction where we begin with a (D, A)-stacked algebra $\boldsymbol{A} = K\Gamma/I$, we consider $D = dA$, A≥ 1 and $d \geq 2$, and we use the quiver Γ and ideal I to construct an algebra $B = KQ/\tilde{I}$.
Followed by detailed example of an algebra $\boldsymbol{A} = K\Gamma/I$, we construct the algebra B from an algebra $\boldsymbol{A}$. This algebra will be shown to be a d-Koszul algebra.

The main idea of section 3 is Theorem 3.3, in which we prove that the finite dimensional algebra $\boldsymbol{B}$ is a d-Koszul monomial algebra.

Throughout this paper, we use a number of notations. We let K be a field and let $\boldsymbol{A} = K\Gamma$/I be a finite dimensional (D, A)-stacked algebra, where Γ is a finite quiver and I is an admissible ideal [3].

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We denote the set of vertices of Γ by Γ_0 and the set of arrows of Γ by Γ_1 , while we let $rad\boldsymbol{A}$ denotes the Jacobson radical of $\boldsymbol{A}$. An arrow α begins with the vertex $o(\alpha)$ and ends with the vertex $t(\alpha)$; the way to write the arrow in the path is from left side to right side. A path is a sequence $\alpha_1\alpha_2\cdots\alpha_n$ of arrows $\alpha_{1,}\alpha_2,..,\alpha_n$ in Q_1, and the length of path p = $\alpha_1\alpha_2\cdots\alpha_n$ is n, which is denoted by $l(p)$. If the generating set of I is from a set of paths in $K\Gamma$, then we say that $\boldsymbol{A} = K\Gamma$/I is a monomial algebra [4].

Section 2: Background

We recall briefly the definition of Koszul algebra and we refer the reader to an earlier publication [5]. A graded algebra $\boldsymbol{\Lambda} = \boldsymbol{\Lambda_0} \oplus \boldsymbol{\Lambda_1} \oplus ...$ is a Koszul algebra if $\boldsymbol{\Lambda_0} = \boldsymbol{\Lambda}/rad\boldsymbol{\Lambda}$ has a linear resolution; in other words, every projective module P^nin the minimal graded projective resolution (P^n, d^n) of $\boldsymbol{\Lambda_0}$ can be generated in degree n. It was shown that if $\boldsymbol{\Lambda} = KQ/I$ is Koszul, then I is quadratic [5]. In another study [6], Berger introduced d-Koszul algebras. He was interested in this class of algebras to study Artin-Schelter regular algebras.

Definition 2.1 [6]. Let $\boldsymbol{\Lambda} = \boldsymbol{\Lambda_0} \oplus \boldsymbol{\Lambda_1} \oplus ...$ be a graded K-algebra generated in degrees 0 and 1. Assume that $\boldsymbol{\Lambda_0} = \boldsymbol{\Lambda}/rad\boldsymbol{\Lambda}$ is a finitely generated semisimple K-algebra, $\boldsymbol{\Lambda_1}$ is a finitely generated K-module and that (P^n, d^n) is a minimal graded

$\boldsymbol{A}$ – module projective resolution of $\boldsymbol{A}/rad\boldsymbol{A}$. Let $d \geq 2$. We call $\boldsymbol{\Lambda}$ a d-Koszul algebra if, for all $n \geq 0$, P^n is generated in single degree, δ(n), and

$$\delta(n) = \begin{cases} \frac{n}{2}d & if\ n\ even \\ \frac{n-1}{2}d + 1 & if\ n\ odd. \end{cases}$$

It can be seen that each Koszul algebra is a 2-Koszul algebra.

Green and Snashall [7] introduced the (D, A)-stacked monomial algebra. The case of non monomial (D, A)-stacked algebras was extended by Leader and Snashall [8].

Definition 2.2 [8, Definition 1.1]. Let $\boldsymbol{\Lambda} = KQ/I$ be a finite dimensional algebra.

Then $\boldsymbol{\Lambda}$ is a (D, A)-stacked algebra if there are natural numbers $D \geq 2, A \geq 1$ such that, for all $0 \leq n \leq gldim\,\boldsymbol{\Lambda}$, the projective module P^n in a minimal projective resolution of $\boldsymbol{\Lambda}/rad\boldsymbol{\Lambda}$ is generated in degree $\delta(n)$, where

$$\delta(n) = \begin{cases} 0 & if\ n = 0 \\ 1 & if\ n = 1 \\ \frac{n}{2}D & if\ n\ even \\ \frac{n-1}{2}D + A & if\ n\ odd. \end{cases}$$

It can be seen that, if $A = 1$, then the finite dimensional (D, A)-stacked algebras are D-Koszul algebras, as defined by Berger.

Green and Marcos [9] introduced the δ-resolution determined algebras. Let $\boldsymbol{\Lambda} = KQ/I$ be a finite dimensional algebra, then $\boldsymbol{\Lambda}$ is δ-resolution determined, if there is a map $\delta: N \to N$ such that, for all $n > 0$ with $n \leq gldim\,\boldsymbol{\Lambda}$, the projective module P^n in a minimal projective resolution $\boldsymbol{\Lambda}/rad\boldsymbol{\Lambda}$ can be generated in one degree $\delta(n)$.

Section 3: Constructing d-Koszul algebras

Now we take a (D, A)-stacked algebra $\boldsymbol{A} = K\Gamma$/I and we consider $D = dA$ and d≥ 2,

where the set $\tilde{\rho}$ is a minimal generating set of homogeneous uniform relations of I and each element of $\tilde{\rho}$ has a length D. The elements of $\tilde{\rho}$ are labelled as $\tilde{\rho}_1, ..., \tilde{\rho}_m$. We are now ready to describe our construction using the quiver Γ and ideal I, and we show that the algebra

$\boldsymbol{B}$ is related to the algebra $\boldsymbol{A}$.

Definition 3.1:

1- We keep the notation above, and let $x \in K\Gamma$ where $x = \sum_k c_k\alpha_{k,1}\cdots\alpha_{k,D}$ is a linear combination of paths of length D with $D = dA$, $0 \neq c_k \in K$ and the $\alpha_{k,j} \in \Gamma_1$. We define the paths $\alpha_{k,rA+1}\cdots\alpha_{k,(r+1)A}$ to be the A-subpaths of x for some k, and $0 \leq r \leq d-1$.

2- We fix a minimal generating set $\tilde{\rho}$ for I. We define the A-subpaths of $\boldsymbol{A}$ to be the set of A-subpaths of x, for all $x \in \tilde{\rho}$, denoted as S_A, considering that a set S_A is a set with no repeats. We define now a quiver Q and an ideal $\hat{I}$ of KQ and we let $B = KQ/\hat{I}$.

Definition 3.2: Let $\boldsymbol{A} = K\Gamma/I$ be a (D, A)-stacked algebra and consider that $\mathrm{D} = \mathrm{dA}$ for some $\mathrm{d} \geq 2$, where the set $\tilde{\rho} = \{\tilde{\rho}_1, \dots, \tilde{\rho}_m\}$ is a minimal generating set of uniform relations of a length D of I. So for each $\mathrm{i} = \{1, \dots, \mathrm{m}\}$, we write $\tilde{\rho} = \sum_k c_k\, \alpha_{i,k,1} \cdots \alpha_{i,k,D}$ where $0 \neq c_k \in K$ and $\alpha_{i,k,j} \in \Gamma_1$, for all $1 \leq \mathrm{j} \leq \mathrm{D}$. Then

1- We define the vertex set of Q to be the set $\{o(\tilde{\rho}_i), t(y)$ for all $y \in S_A$ and all $i = \{1, \dots, m\}$. We remark that $t(\alpha_{i,k,dA}) = t(\alpha_{i,k,D}) = t(\tilde{\rho}_i)$. In this set we do not have any repeated vertices, so if $t(\alpha_{i,k,rA}) = t(\alpha_{j,i,sA})$ as vertices of Γ or some $\mathrm{i, j, k, l, r, s}$, then we classify $t(\alpha_{i,k,rA})$ and $t(\alpha_{i,l,sA})$ as the same vertex in Q.

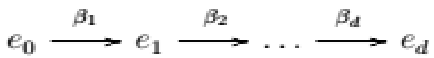
2- The arrows of Q are constructed in the following way. Each $y \in S_A$ corresponds to an arrow β_y in Q. The diagram below illustrates this process.

Consider the path $\alpha_{i,k,1} \cdots \alpha_{i,k,D}$. Then $e_0 = o(\tilde{\rho}_i), e_1 = t(\alpha_{i,k,A}), \dots, e_d = t(\alpha_{i,k,dA})$ are vertices in Q and $\beta_1, \dots, \beta_d$ are arrows in Q corresponding to the A-subpaths

$$\alpha_{i,k,1} \dots \alpha_{i,k,A}, \dots, \alpha_{i,k,(d-1)A+1} \cdots \alpha_{i,k,dA}$$

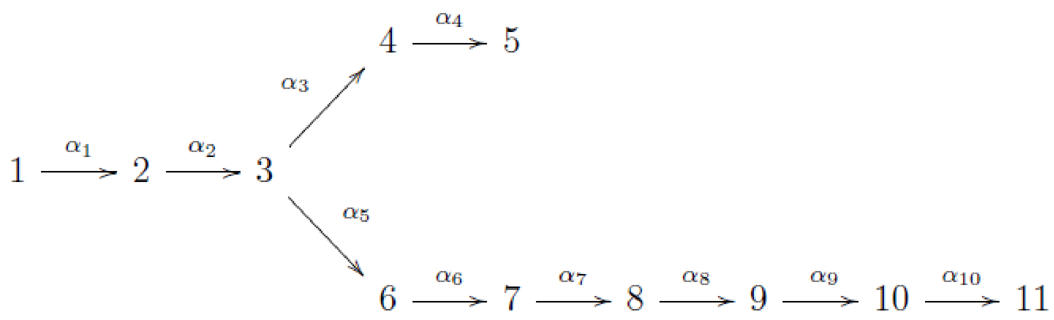
respectively. Then the $\alpha_{i,k,1} \cdots \alpha_{i,k,D}$ may be considered as the path of the length D in $K\Gamma$

which corresponds to the path $\beta_1 \dots \beta_d$ of the length d in KQ.

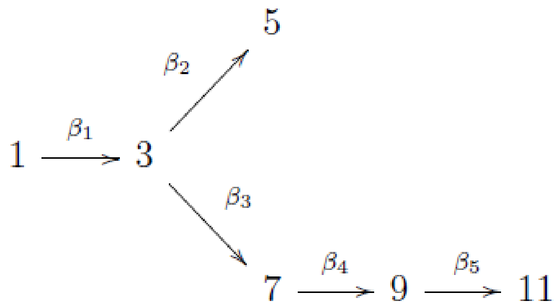


3- The ideal $\hat{I}$ of KQ can be constructed as follows. For each $i = \{1, \dots, m\}$, we define $\rho_i = \sum_k c_k \beta_{k1} \cdots \beta_{kd}$ in KQ where β_j is the arrow in KQ which corresponds to a path $\alpha_{j,k,rA+1} \dots \alpha_{j,k(r+1)A}$ for all $j = 1, \dots, d$ and $r = 0, \dots, d-1$. Now we define $\hat{I}$ to be the ideal of KQ that is generated by the set $\rho = \{\rho_1, \dots, \rho_m\}$. We remark that ρ is certainly a minimal generating set for $\hat{I}$, since $\tilde{\rho}$ is a minimal generating set for $\hat{I}$. Let $\boldsymbol{B} = KQ/\hat{I}$.We illustrate this construction in the following example.

Example 3.3: Let $\boldsymbol{A} = K\Gamma/I$ be the (4, 2)-stacked algebra that is illustrated in the following quiver



and with $I = \langle \alpha_1\alpha_2\alpha_3\alpha_4, \ \alpha_1\alpha_2\alpha_5\alpha_6, \alpha_5\alpha_6\alpha_7\alpha_8, \alpha_7\alpha_8\alpha_9\alpha_{10} \rangle$. We can see that the sets $g^n$are given as follows: $g^0 = \{e_1 \dots e_{11}\}$, $g^2 = \{\alpha_1\alpha_2\alpha_3\alpha_4, \alpha_1\alpha_2\alpha_5\alpha_6, \alpha_5\alpha_6 \alpha_7\alpha_8, \alpha_7\alpha_8\alpha_9\alpha_{10}\}$, $g^3 = \{\alpha_1\alpha_2\alpha_5\alpha_6\alpha_7\alpha_8, \alpha_5\alpha_6\alpha_7\alpha_8\alpha_9\alpha_{10}\}$, and $g^4 = \{\alpha_1\alpha_2 \alpha_5\alpha_6\alpha_7\alpha_8\alpha_9\alpha_{10}\}$. We see that the length of the elements $g_i^n \in g^n$ is δ(n) where $D = 4$ and $A = 2$. So every projective P^n is generated in one degree δ(n). Hence, $\boldsymbol{A}$ is a (D, A)-stacked monomial algebra with $D = 4, A = 2$ *and* $d = 2$. So $S_A = \{\alpha_1\alpha_2, \alpha_3\alpha_4, \alpha_5\alpha_6, \alpha_7\alpha_8, \alpha_9\alpha_{10}\}$. Then, by using the construction above, Q has six vertices $o(\alpha_1\alpha_2), t(\alpha_3\alpha_4), t(\alpha_5\alpha_6), t(\alpha_7\alpha_8), t(\alpha_9\alpha_{10})$ and five arrows $\{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6\}$, where $\beta_1$corresponds to $\alpha_1\alpha_2$, β_2 corresponds to $\alpha_3\alpha_4$, $\beta_3$corresponds to $\alpha_5\alpha_6$, β_4 corresponds to $\alpha_7\alpha_8$, and β_5 corresponds to $\alpha_9\alpha_{10}$. Therefore, $B = KQ/\hat{I}$ is given by the quiver



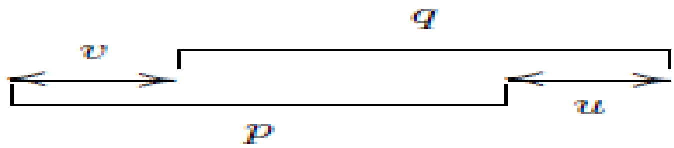
and $\hat{I} = \langle \beta_1\beta_2, \beta_1\beta_3, \beta_3\beta_4, \beta_4\beta_5 \rangle$. So ***B*** is a Koszul algebra (see [10]). More examples can be found in a previous article [11].

Section 4: d-Koszul monomial algebra

In this section, the algebra ***B***, which has been constructed from a (D, A)-stacked monomial algebra, will be shown to be d-Koszul. Before that, the idea of overlaps has been used to prove the main theorem in this section. Green and Zacharia [10] have used the concept of overlaps to describe a basis of the Ext algebra of a monomial algebra.

We consider that $\boldsymbol{A} = KQ/I$ is a monomial algebra.

Definition 4.1 [7]. Let q and p be paths. Then we say that a path q overlaps a path p with an overlap , if there are paths u and v such that $pu = vq$ and $1 \leq l(u) \leq l(q)$. The definition can be shown in the following diagram:



Thus, the two paths allow $l(v)$ to be zero.

2- A path q properly overlaps a path p with an overlap pu if q overlaps p and $l(v) \leq 1$.

3- A path p has no overlaps with a path q if p does not properly overlap q and q does not properly overlap p.

Definition 4.2 [12]. A path p is a prefix of a path q if there is some path p such that $q = pp'$.

The minimal projective $resolution$ (P^n, d^n) of $A/radA$ has been constructed by Green, Happel and Zacharia [4], by using overlaps. Moreover, it has been further described by other authors [12,10]. The same previously demonstrated notation [12] has been used in this paper. The sets $\mathcal{R}^n$ can be defined as follows:

$\mathcal{R}^0 =$ the vertices set of Q,

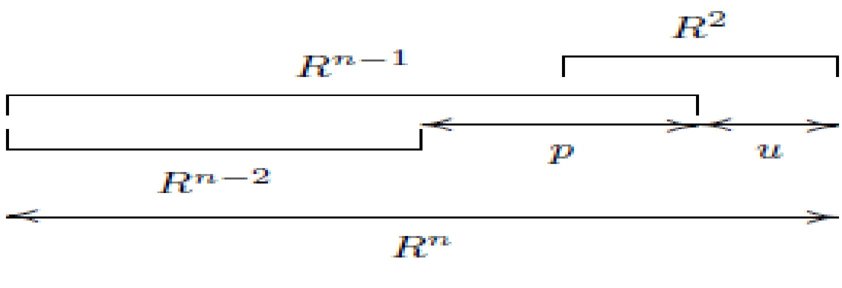
$\mathcal{R}^1 =$the arrows set of Q

$\mathcal{R}^2 =$ the minimal set of monomials in the generating set of I.

For all $n \geq 3, R^2 \in \mathcal{R}^2$ maximally overlaps $R^{n-1} \in \mathcal{R}^{n-1}$ with overlap $R^n = R^{n-1}u$ for some $u \in KQ$, if it satisfies the following conditions:

1- $R^{n-1} = R^{n-2}p$, for some path p in KQ;
2- R^2 overlaps p with overlap pu;
3- There is no element in $\mathcal{R}^2$ which overlaps p, with the overlap being a proper prefix of pu.

The set $\mathcal{R}^n$ is the set of all overlaps R^n. We illustrate R^n with the following diagram:



Now we define the minimal projective resolution (P^n, d^n) of $\Lambda/rad\,\Lambda$ using the construction of Green, Happel and Zacharia [4]. For all $n \geq 0$, let $P^n = \bigoplus_{R^n \in \mathcal{R}^n} t(R^n)\Lambda$. Define, for $n \geq 1$ and $R^n \in \mathcal{R}^n$, the map $d^n: P^n \rightarrow P^{n-1}$ via $t(R^n) \rightarrow (0, \dots, 0, p, 0, \dots)$, where $R^n = R^{n-1}p$ and p occurs in the component of P^{n-1}corresponding to R^{n-1}.

Now we prove the main theorem in this section.

Theorem 4.3: Let $\boldsymbol{A} = K\Gamma/I$ be a (D, A)-stacked monomial algebra with $gldim\,\boldsymbol{A} \geq 4$, so $D = dA$, for some $d \geq 2$. Let $\boldsymbol{B}$ be the algebra constructed from $\boldsymbol{A}$ using Definition 3.2. Then $\boldsymbol{B}$ is a d-Koszul monomial algebra.

Proof: Assume that the algebra $\boldsymbol{B} = KQ/\hat{I}$ is constructed from $\boldsymbol{A} = K\Gamma/I$. Then $\hat{I}$ is monomial. We set

$\tilde{\mathcal{R}}^0 =$ the vertices set of Γ,

$\tilde{\mathcal{R}}^1 =$ the arrows set of Γ,

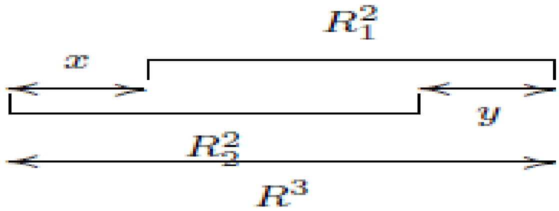
$\tilde{\mathcal{R}}^2 =$ the minimal set of monomials in the generating set of I, (denoted $\tilde{\rho}$ in Definition 3.2) and

$\mathcal{R}^0 =$ the vertices set of Q,

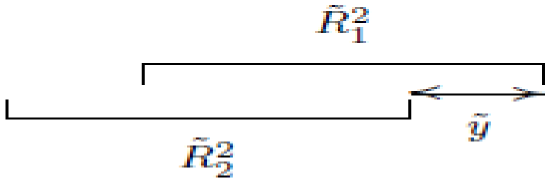
$\mathcal{R}^1 =$ the arrows set of Q,

$\mathcal{R}^2 =$ the minimal set of monomials in the generating set of $\hat{I}$, (denoted as ρ in Definition 3.2).

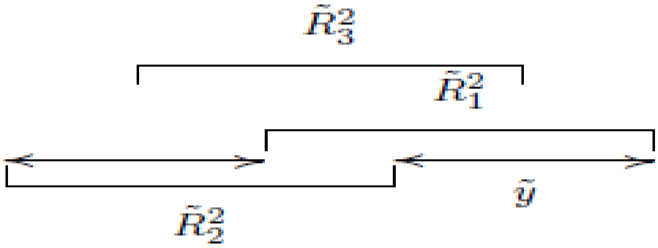
It can be seen that $l(R^0) = 0$, $l(R^1) = 1$, and $l(R^2) = d$, for all $R^0 \in \mathcal{R}^0$, $R^1 \in \mathcal{R}^1$, and $R^2 \in \mathcal{R}^2$. We denote by $\tilde{\mathcal{R}}^n$ (respectively $\mathcal{R}^n$) the set of overlaps in $\boldsymbol{A}$ (respectively $\boldsymbol{B}$), for all $n \geq 3$. Since $\boldsymbol{A}$ is a (D, A)-stacked monomial algebra, then the n<u>th</u> projective module in a minimal resolution of $\boldsymbol{A}/rad\,\boldsymbol{A}$ is $\tilde{P}^n = \bigoplus_{R^n \in \tilde{R}^n} t(\tilde{R}^n)A$, and we can see that it is generated in single degree $\delta(n)$ (see Definition 1.3), where $n \geq 0$. We now consider $\mathcal{R}^3$. An element $R^3 \in \mathcal{R}^3$ is constructed from R_1^2 which maximally overlaps R_2^2 of the form $R^3 = R_2^2\,y$ as follows:



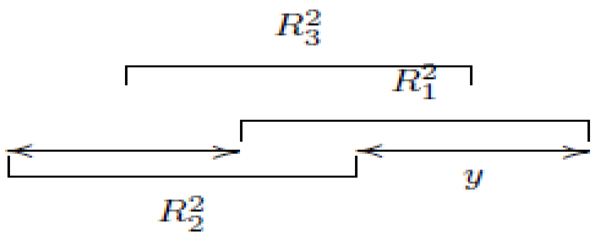
By Definition 2.2, R_1^2 (respectively R_2^2) corresponds to $\tilde{R}_1^2$ (respectively $\tilde{R}_2^2$) in the minimal generating set $\tilde{\mathcal{R}}^2$ for I. This element gives an overlap of $\tilde{R}_1^2$ with $\tilde{R}_2^2$:



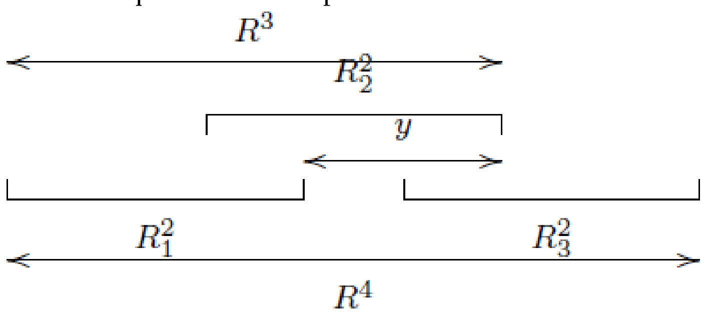
We want to show that $\tilde{R}_2^2\tilde{y}$ is in $\tilde{R}^3$. If $\tilde{R}_1^2$ does not maximally overlap $\tilde{R}_2^2$, then we have $\tilde{R}_3^2 \in \tilde{R}^2$ which maximally overlaps $\tilde{R}_2^2$:



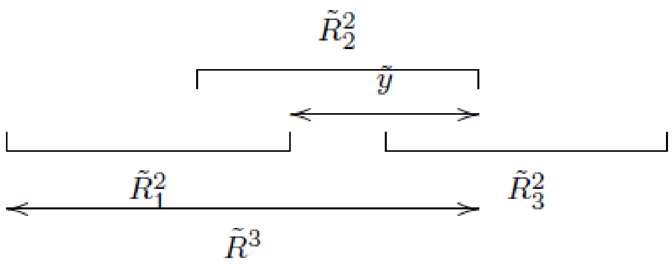
But $\tilde{R}_3^2$ corresponds to some $R_3^2 \in \mathcal{R}^3$ and so R_3^2 overlapsR_2^2 :



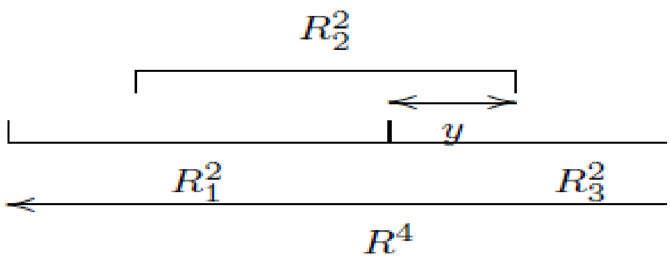
This is a contradiction, as R_1^2 maximally overlaps R_2^2. Therefore, $\tilde{R}_1^2$ does maximally overlap $\tilde{R}_2^2$ and so $\tilde{R}_2^2\tilde{y} \in \tilde{\mathcal{R}}^3$. Now we write $\tilde{R}_2^2\tilde{y} = \tilde{R}^3 \in \tilde{\mathcal{R}}^3$. Since $l(\tilde{R}^3) = D + A$ and $l(\tilde{R}_2^2) = D$, then $l(\tilde{y}) = A$. However, $\tilde{y}$ is a suffix of $\tilde{R}_1^2$ and so $\tilde{y} \in S_A$. So $\tilde{y}$ corresponds to an arrow in B. Hence, $l(y) = 1$ and $l(R^3) = d + 1$. The same argument has been used for the elements of $\mathcal{R}^4$. An element $R^4 \in \mathcal{R}^4$ is constructed from a sequence of overlaps as follows:



From Definition 3.2, R_1^2 (respectively R_2^2, R_3^2) corresponds to $\tilde{R}_1^2$ (respectively $\tilde{R}_2^2$, $\tilde{R}_3^2$) in the minimal generating set $\tilde{\mathcal{R}}^2$ for I. Using the above argument for $\tilde{R}^3$, we get:



and $l(\tilde{y}) = A$. From our construction of R^4, we get $l(y) = 1$. However, R_3^2 overlaps y and so y must be a prefix of R_3^2. Thus $t(R_1^2) = o(R_3^2)$ and we have



Then, $R^4 = R_1^2R_3^2$ and $l(R^4) = 2d$. By continuing in the same manner, by the induction for all $n \geq 0$ and all $R^n \in \mathcal{R}^n$, we have

$$l(R^n) = \begin{cases} \frac{n}{2}d & if\ n\ even \\ \frac{n-1}{2}d + 1 & if\ n\ odd \end{cases}$$

Thus, $\boldsymbol{B}$ is a d-Koszul.

Acknowledgement
I would like to deliver a special thank you to my supervisor, Professor Nicole Snashall, for her guidance.

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