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Absolutely Self Neat Modules

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Abstract

An R-module A is called absolutely self neat if whenever f is a map from a maximal left ideal of R, with kernel in the filter is generated by the set of annihilator left ideals of elements in A into A, then f is extendable to a map from R into A. The concept is analogous to the absolute self purity, while it properly generalizes quasi injectivity and absolute neatness and retains some of their properties. Certain types of rings are characterized using this concept. For example, a ring R is left max-hereditary if and only if the homomorphic image of any absolutely neat R-module is absolutely self neat, and R is semisimple if and only if all R-modules are absolutely self neat.

Keywords: self neat submodule, absolutely self neat module, m-injective module, quasi injective module.

المقاسات مطلقة الأناقة ذاتياً

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الخلاصة

يقال للمقاس A على الحلقة R أنه مقاس مطلق الأناقة ذاتياً إذا كان لكل تشاكل f من مثالي أيسر أعظم في الحلقة R ، بحيث تكون نواته في المرشحة المتولدة بالمثاليات اليسرى التالفة لعناصر في A ، إلى A فإن f قابل للتوسيع إلى تشاكل من R إلى A . المفهوم مناظر لمفهوم مطلق النقاوة ذاتياً ويعمم بشكل فعلي شبه الإغمار ومطلق الأناقة ويحافظ على بعض خواصها . مثلاً الحلقة R تكون وراثية أعظمية يسرى إذا وفقط إذا كانت الصورة التشاكلية لأي مقاس مطلق الأناقة على R مقاساً مطلق الأناقة ذاتياً ، وتكون R شبه بسيطة إذا وفقط إذا كانت كل المقاسات على R مطلقة الأناقة ذاتياً .

INTRODUCTION

Modules and R-modules are, unless otherwise stated, always left unital over an associative ring R with identity. For two modules M and N, M is called N-*injective* if any homomorphism from a submodule of N into M has an extension to a homomorphism from N into M. The module M is called *injective* if it is N-inective for all modules N, if and only if it is R-injective. The module M is called *quasi injective* if it is M-injective. Every module M is embedded in a minimal (quasi) injective module called the (quasi) injective envelope of M. The injective and quasi injective envelopes of a module M are denoted $\mathrm{E}(M)$ and $\mathrm{Q}(M)$, respectively. For details, see the following references [1, 2]. A

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submodule A of a module B is called *pure* submodule if every finitely presented module is projective with respect to the sequence $0 \to A \to B \to B/A \to 0$, if and only if the sequence remains exact when tensored with any finitely presented right R-module [3]. If the words 'finitely presented' are replaced by 'simple' in the previous two equivalent statements, we get the (nonequivalent) concepts of neat and coneat submodules, respectively [4]. Although (co)neatness is defined for commutative rings in [4], we do not assume commutativity of R unless stated otherwise. The sequence above is called *pure exact* (*neat exact* or *coneat exact*, respectively) if A is pure (neat or coneat, respectively) in B. Absolutely pure (neat or coneat, respectively) modules are modules that are pure (neat or coneat, respectively) in every module containing them and, equivalently, when they are pure (neat, coneat) in their injective envelopes [4, 5, 3]. A question that arises is what happens if the module is pure, neat or coneat in its quasi injective envelope? To answer this, for the purity case, the concept of absolutely self pure modules was studied in [6]. Analogously, for the neatness case, we introduce here the concept of absolute self neatness.

Recall that the module A is neat in B, precisely when every diagram

$$\begin{array}{ccc} I & \hookrightarrow & R \\ f\downarrow & & \downarrow \\ A & \hookrightarrow & B \end{array}$$

with I being a maximal left ideal of R, there is a map $R \to A$ extending f, and that A is absolutely neat exactly when it is injective with respect to all inclusions $I \hookrightarrow R$ for maximal left ideals I of R [4]. For our purposes, we will adopt this definition for (absolute) neatness. Absolutely neat modules are called *m-injective* [7], where it is shown that every module M has an m-injective envelope denoted $\mathrm{E_m}(M)$.

For a given module M, the set of all left ideals I of R, such that $I \supseteq \mathrm{ann}(m)$ for some $m \in M$, is denoted $\Omega(M)$. The filter (= dual ideal) generated by this set in the lattice of all left ideals of the ring R is denoted $\overline{\Omega}(M)$. L. Fuchs [8] proved that a module M is quasi injective if and only if every map f from a left ideal of R into M, such that $\ker f \in \overline{\Omega}(M)$, can be extended to $R \to M$.

Extending the above ideas, we say that a module M is absolutely self neat if every map f from a maximal left ideal of R into M, such that $\ker f \in \overline{\Omega}(M)$, can be extended to $R \to M$. Max-hereditary and semisimple rings are characterized using this concept.

1. SELF NEATNESS AND ABSOLUTE SELF NEATNESS

Definition 1.1: A submodule M of an R-module N is called *self neat* submodule (denoted $M \leq^{sn} N$) if, for every maximal left ideal I of R and any commutative diagram

$$\begin{array}{ccc} I & \hookrightarrow & R \\ f\downarrow & & \downarrow \\ M & \hookrightarrow & N \end{array}$$

such that $\ker f \in \overline{\Omega}(M)$, there is a map $g: R \to M$ extending f.

The analogy with self purity [6] (by replacing the words 'maximal left ideal' by 'finitely generated left ideal') is obvious. However, as we will see later (Theorem 2.3), the two concepts are related as much as purity and neatness are. (In Theorem 2.3, the equivalence (6) ⇔ (7) shows that pure submodules are neat exactly when self pure submodules are self neat.) Obviously, neat submodules are self neat but not vice-versa. For example, quasi injective modules that are not absolutely neat must clearly be self neat in their injective envelopes but of course not neat. It is easily seen that if a module contains the ring R then it must be neat in any module containing it if and only if it is self neat in it.

Proposition 1.2: *Let $L \subseteq M \subseteq N$ be R-modules.*

1. *If $L \leq^{sn} M \leq^{sn} N$ then $L \leq^{sn} N$.*
2. *If $L \leq^{sn} N$ then $L \leq^{sn} M$.*

Proof. 1. Consider the commutative diagram

$$\begin{array}{ccc} I & \hookrightarrow & R \\ f\downarrow & & \downarrow \\ L & \hookrightarrow & N \end{array}$$

with I being a maximal left ideal of R and $\ker f \in \overline{\Omega}(L)$. Hence, $\ker f \in \overline{\Omega}(M)$. By considering f as a map $I \to M$ and since $M \leq^{sn} N$, we see, for the above diagram with L replaced by M, that there is a map $g: R \to M$ making the upper triangle commutative. So we have the commutative diagram

$$\begin{array}{ccc} I & \hookrightarrow & R \\ f\downarrow & & g\downarrow \\ L & \hookrightarrow & M \end{array}$$

But $L \leq^{sn} M$ gives the existence of a map $R \to M$ that extends f, hence we get the result.
2. Now consider any commutative diagram like the last one above with $\ker f \in \bar{\Omega}(L)$. As $M \subseteq N$, we can consider g as a map $R \to N$. By assumption, there is an $h: R \to L$ extending f, as desired. □

Now we are ready to introduce absolute self neatness.

Definition 1.3: A module is called *absolutely self neat* if it is neat in every module containing it.

Absolute self neatness is a form of injectivity, as in the following theorem.

Theorem 1.4: *A module M is absolutely self neat if and only if for every map $f: I \to M$, where I is a maximal left ideal of R and* $\ker f \in \bar{\Omega}(M)$, *has an extension $R \to M$.*

Proof. For any map $f: I \to M$ as above, there is a map $g: R \to \mathrm{Q}(M)$ making the following diagram commutative

$$\begin{array}{ccc} I & \hookrightarrow & R \\ f \downarrow & & g \downarrow \\ M & \hookrightarrow & \mathrm{Q}(M) \end{array}$$

But M is absolutely self neat if and only if it is neat in $\mathrm{Q}(M)$ if and only if there is a map $h: R \to M$ that extends f. □

The above proof would proceed similarly if, instead of $\mathrm{Q}(M)$, we use $\mathrm{E_m}(M)$ or $\mathrm{E}(M)$.

Example 1.5:

(1) Any absolutely neat module is absolutely self neat.
(2) Any quasi injective module is absolutely self neat.
(3) Suppose M is self neat in $\mathrm{Q}(M)$ or in $\mathrm{E_m}(M)$, and let N be any module that contains M as a submodule. As $\mathrm{Q}(M) \leq^{sn} \mathrm{Q}(N)$ and $\mathrm{E_m}(M) \leq^{sn} \mathrm{E_m}(N)$, we have by proposition 1.2 that $M \leq^{sn} N$.
(4) (A special case of (3).) Modules that are neat in their quasi injective envelopes must be absolutely self neat.

Therefore, we see that a module is absolutely self neat if and only if it is self neat in some (quasi) injective, m-injective, or absolutely self neat module, if and only if it is self neat in its (quasi) injective or m-injective envelope. In particular, direct summands of absolutely self neat modules are absolutely self neat.

When taking finite direct sums (or products) of copies of a module satisfying the property of absolute self neatness, the property is preserved.

Theorem 1.6: *A module M is absolutely self neat if and only if $M \oplus M$ is absolutely self neat.*

Proof. If $\mathrm{M} \oplus \mathrm{M}$ is absolutely self neat then M is also absolutely self neat, since it is a direct summand of $\mathrm{M} \oplus \mathrm{M}$. Conversely, if M is absolutely self neat then for any map $\mathrm{f}: \mathrm{I} \to \mathrm{M} \oplus \mathrm{M}$, where I is a maximal left ideal of R and $\ker \mathrm{f} \in \bar{\Omega}(\mathrm{M} \oplus \mathrm{M})$, we have $\ker \mathrm{f} \supseteq \bigcap_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{ann}(\mathrm{a_i}, \mathrm{b_i}) = \bigcap_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{ann}(\mathrm{a_i}) \cap \bigcap_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{ann}(\mathrm{b_i})$, for some $\mathrm{a_i}, \mathrm{b_i} \in \mathrm{M}$, $\mathrm{i} = 1, \cdots, \mathrm{n}$. This means that $\ker \mathrm{f} \in \bar{\Omega}(\mathrm{M})$. Having $\mathrm{f} = \mathrm{f_1} \oplus \mathrm{f_2}$, where $\mathrm{f_1}$ and $\mathrm{f_2}$ are obtained by following f by the natural projections of $\mathrm{M} \oplus \mathrm{M}$ onto $\mathrm{M} \oplus 0$ and $0 \oplus \mathrm{M}$, respectively, we see that each of $\ker \mathrm{f_1}$ and $\ker \mathrm{f_2}$ contains $\ker \mathrm{f}$ and, therefore, they must be in $\bar{\Omega}(\mathrm{M})$. By absolute self neatness of M, there are maps $\mathrm{g_1}$ and $\mathrm{g_2}: \mathrm{R} \to \mathrm{M}$ extending $\mathrm{f_1}$ and $\mathrm{f_2}$, respectively. Now $\mathrm{g_1} \oplus \mathrm{g_2}$ is the desired extension of f. □

2. CHARACTERIZATIONS OF CERTAIN TYPES OF RINGS

Following [4], a ring R is called left *max-hereditary* if every maximal left ideal is projective. Such rings are characterized in [7] by the property that the homomorphic image of any absolutely neat R-module is absolutely neat. For the next theorem, we will use the following lemma (lemma 3.4, [6]), which gives a weaker condition for a left ideal to be projective.

Lemma 2.1 [6]: *A left ideal I of R is projective if and only if, for any epimorphism $A \to B$, any homomorphism $I \to B$ whose kernel contains* $\mathrm{ann}(b)$ *for some $b \in B$ can be lifted to a homomorphism $I \to A$.* □

Theorem 2.2: *The following are equivalent for a ring R.*

(1) *R is left max-hereditary.*
(2) *The homomorphic image of any absolutely neat R-module is absolutely neat.*
(3) *The homomorphic image of any absolutely neat R-module is absolutely self neat.*
(4) *Every maximal left ideal of R is projective.*

Proof. (1) ⇔ (2) ⇔ (4) are given in (theorem 10, [7]). (2) ⇒ (3) is trivial. For (3) ⇒ (4), we consider the diagram:

$$\begin{array}{ccc} R & \longleftarrow & I \\ & & f\downarrow \\ M & \longrightarrow & N \end{array}$$

where I is a maximal left ideal of R, the homomorphism $I \to R$ is the inclusion map i, and $f: I \to N$ is any map whose kernel contains the annihilator of some $n \in N$, and $M \to N$ is any epimorphism α from an injective module M into N. By assumption, N is absolutely self neat, therefore, there is a map $g: R \to N$ extending f. But R is projective. Hence, there is a lifting $h: R \to M$ of g. This means that $\alpha hi = gi = f$ and I is projective by lemma 2.1. □

Recall that a module M is called *finitely R-injective* [9] if every homomorphism f from a finitely generated left ideal of R into M has an extension $R \to M$. A module is called *absolutely self pure* [6] if it is self pure in every extension. If all maximal left ideals of R are finitely generated then it is easy to see that all absolutely (self) pure modules over R are absolutely (self) neat. In fact, this last condition characterizes such rings as follows.

Theorem 2.3: *For a ring R, the following are equivalent.*
(1) *Every maximal left ideal of R is finitely generated.*
(2) *Every finitely R-injective module over R is absolutely neat.*
(3) *Every absolutely pure R-module is absolutely neat.*
(4) *Every absolutely pure R-module is absolutely self neat.*
(5) *Every absolutely self pure R-module is absolutely self neat.*
(6) *Pure submodules are neat.*
(7) *Self pure submodules are self neat.*

Proof. (1) ⇒ (2) ⇒ (3) ⇒ (4), (1) ⇒ (6) ⇒ (3), and (1) ⇒ (7) ⇒ (5) ⇒ (4) are trivial. (4) ⇒ (3) Let M be an absolutely pure R-module. Hence, $M \oplus \mathrm{E}(R)$ is also an absolutely pure R-module which must be absolutely self neat by assumption, and as it contains R, it must be absolutely neat. (3) ⇒ (1) Every absolutely pure R-module is injective with respect to the sequence $0 \to I \hookrightarrow R$, where I is a maximal left ideal of R. So I must be finitely generated by [10]. □

Corollary 2.4: *Consider the following conditions on a ring R.*
(1) *Every absolutely neat R-module is injective.*
(2) *Every absolutely neat R-module is quasi injective.*
(3) *Every absolutely self neat R-module is quasi injective.*
Suppose that every maximal left ideal of R is finitely generated, then any one of the above statements implies that R is left noetherian.

Proof. It is easily seen that (1) ⇒ (2) and (3) ⇒ (2). Moreover, (2) ⇒ (1): Let M be absolutely neat. Therefore, $M \oplus \mathrm{E}(R)$ is also absolutely neat which must then be quasi injective. Hence M is $\mathrm{E}(R)$-injective and M is injective. So, we only need to show that (1) implies that R is left noetherian, provided that every maximal left ideal is finitely generated. But then, by the above Theorem, absolutely pure R-modules are absolutely neat and, therefore, from (1) we see that all absolutely pure R-modules are injective. By (theorem 3, [5]), this means that R is left noetherian. □

Example 2.5:
(1) Suppose that R is not noetherian but all its maximal left ideals are finitely generated. Then there must exist, by the above corollary, an absolutely self neat R-module M that is not quasi injective.
(2) We know that $\mathbb{Z}_2$ is a quasi injective $\mathbb{Z}$-module and, hence, absolutely self neat. But it can be easily verified that $\mathbb{Z}_2$ is not absolutely neat.
(3) Let R and M be as in (1) above. Using an argument similar to that of (example 3.3, [6]), one can show that, over the ring $\begin{pmatrix} R & 0 \\ 0 & \mathbb{Z} \end{pmatrix}$, the module $\begin{pmatrix} M \\ \mathbb{Z}_2 \end{pmatrix}$ is absolutely self neat which is neither quasi injective nor absolutely neat.

Theorem 2.6: *The following are equivalent for a ring R.*
(1) *R is semisimple.*
(2) *Every maximal left ideal of R is a direct summand of R.*
(3) *Every maximal left ideal of R is neat in R.*
(4) *Every R-module is absolutely neat.*

(5) *Every left ideal of R is absolutely neat.*
(6) *Every maximal left ideal of R is absolutely neat.*
(7) *Every exact sequence of R-modules is neat exact.*
(8) *Every R-module is absolutely self neat.*

Proof. (1) ⇒ (2) ⇔ (3), (1) ⇒ (4) ⇒ (5) ⇒ (6), (4) ⇔ (7) and (4) ⇒ (8) are obvious. (2) ⇒ (1) It is easy to see that (2) is equivalent to the fact that every simple left R-module is projective. Hence by (corollary 8.2.2, [11]) we get (1). (6) ⇒ (2). For every maximal left ideal I of R there is a map $R \to I$ completing the following diagram.

$$\begin{array}{ccc} I & \hookrightarrow & R \\ \| & & \\ I & & \end{array}$$

This means that I is a direct summand of R. (8) ⇒ (4) For every R-module M we have that $M \oplus R$ is absolutely self neat and, therefore, so is M. □

M. F. Hamid [12] proved that R is a right *SF-ring* (= every simple right R-mdoule is flat) if and only if all exact sequences of left R-modules are coneat exact (i.e. all left R-modules are absolutely coneat). Now, if R is a commutative ring whose all maximal ideals are principal, then neatness and coneatness are equivalent concepts (Theorem 2.1, [4]). Therefore, we have the following.

Corollary 2.7: *Suppose that R is a commutative ring with all its maximal ideals being principal, then it must be semisimple provided that it is an SF-ring.*

Proof. If R is an SF-ring with the given assumption then by (Theorem 3.16, [12]), all exact sequences of left R-modules are (co)neat exact. Therefore, by Theorem 2.6, R is semisimple.□

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