Iraqi Journal of Science, 2020, Special Issue, pp: 223-232
DOI: 10.24996/ijs.2020.SI.1.30

ISSN: 0067-2904

Dynamics and optimal Harvesting strategy for biological models with Beverton –Holt growth

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Received: 2/1/ 2020 Accepted: 15/ 3/2020

Abstract
In this work, the dynamic behavior of discrete models is analyzed with Beverton- Holt function growth . All equilibria are found . The existence and local stability are investigated of all its equilibria.. The optimal harvest strategy is done for the system by using Pontryagin's maximum principle to solve the optimality problem. Finally numerical simulations are used to solve the optimality problem and to enhance the results of mathematical analysis .

Keywords: Beverton- Holt ; Discrate System; Local Stability; Optimal Harvesting;

ديناميكية و سياسة مثلى لانظمة بايولوجية مع دالة بيفرتون – هلوت للنمو

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الخلاصة :
هدف هذا البحث هو دراسة السلوك الديناميكي لنماذج احيائية متقطعة مع وجود دالة Beverton– Holt للنمو حيث تم ايجاد نقاط الاتزان جميعها وتم دراسة الاستقرار المحلي لكل نقاط الاتزان . تم توسيع النموذج الى مسالة سيطرة مثلى واستعمل مبدا Pontryagin الاعظم للحصول على الحل الامثل للنظام عدديا تم اعطاء امثلة عددية لتاكيد النتائج النظرية.

1- Introduction

For many organisms births occur in reguler times each month or year or each circle. Discrete time function is well used to describe the life of them. Many researchers have analyzed models that described by system of difference eqautions[1-4]. For one dimension model,the general form is governed by first order difference eqaution $x_{t+1} = f(x_t)$, where $x_t$denotes the size of population at year or period t. For two or more dimantions model, the well known model is Lotka – Voltera ,that was first introduced by Lotka and Voltera [5].After their work, more realistic models were introduced and modified by many authors ,we refere to reader for more detials see [6-10].

It is well known that harvesting plays an important role in managing the renew resources , so one should consider useful strategies in order to decrease the risks of extinction as well as to increase the net gains .Scientists and researchers used different harvesting strategies in their models [11-13] . For example, Sanchez and Braner analyzed and investigated the effect of periodic harvesting in periodic environments[14,15].A great deal of attention was given to the discrete as well as continuous logistic models in [16]. Other models are considered in the literatures, for example Ricker model, with constant depletion rate [17] . It was shown numerically that the population exhibits chaotic

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oscillations, which are not necessarily lead to extinction [18,19]. In this paper we will investigate the dynamic and an optimal harvesting of biological models with Beverton-Holt model in one and two dimensions models.

This work is organized as follows: In section 2 we study the dynamics behaviour of a single species with Beverton-Holt growth function with and without harvesting . In section 3, Stability analysis of the prey-predator model is analyzed. In section 4 the model is extended to an optimal harvesting problem. We used the extension of Pontryagin's maximum principle to find the optimal solutions . In section 5, numerical simulations are used to solve the optimality problem and to enhance the results of mathematical analysis. Finally conculsion is provided.

2) Single species

In this section we will study and investigate the dynamic behaviour of the classical Beverton Holt models with and without constant rate harvesting for single population .

The model without harvesting is given by

$$x_{t+1} = \frac{rx_t}{1+ax_t} \tag{1}$$

Where r is the inherent growth rate, and a is the population carrying capacity. The dynamics of (1) are well known ,For any point $x_0 > 0$, r<1 the extinction equilibrium point is globally asymptotically stable ,while if r>1 then $x = 0$ is unstable and the survival equilibrium $x = \frac{r-1}{a}$ is sink point. The last case when $r = 1$ the equilibrium point is nonhyperbolic point . Now if one puts a constant rate harvesting then the model will be as the following :

$$x_{t+1} = \frac{rx_t}{1+ax_t} - qx_t \tag{2}$$

Where q is a constant representing the intensity of harvesting due to fishing or hunting , here q depends on the density of the population , so that we cannot harvest more than the population density. Thus our mathematical analysis is concerned with 0<q $< q_{max} \leq 1$. The model (2) has also two equilibria ,the exinction equilibrium x_0 always exists,and the unique positive equilibrium $x_q = \frac{r-(1+q)}{a(1+q)}$ exists only when $r > 1 + q$.

Next lemma describes the stability of x_0 and x_q .

Lemma 1: For the equilibrium points x_0 and x_q of the model 2 we have the following:

1- x_0 is sink point if $r < 1 + q$.

2- x_0 is source point if $r > 1 + q$.

3- x_0 is nonhyperbolic point if $r = 1 + q$.

4- $x_q = \frac{r-(1+q)}{a(1+q)}$ is always sink point if $r > 1 + q$.

Proof : The proof of 1 , 2, and 3 is easy so it is omitted. It is clear that $|f'(x_q)| < 1$ if and only if $-1 < \frac{(1+q)^2}{r} - q < 1$ if and only if $r > 1 + q$. So that x_q is always sink point.

3- Stability analysis of the prey-predator model.

In this section we will study the dynamics of two –dimension model , prey-predator system, with Beverton-Holt growth in prey species . The system is given by

$$\begin{aligned} x_{t+1} &= \frac{rx_t}{1+ax_t} - bx_ty_t \\ y_{t+1} &= cy_t + dx_ty_t \end{aligned} \tag{3}$$

The parameter $a,$ r , c are the population carrying capacity , the growth rate of the prey and the predator respectively ,while the positive parameters b and d represent the maximum per capita killing rate and conversion rate of predator respectively. All equilibrium points of the system (3) can be determined by solving the following algebraic equation :

$$\begin{aligned} \frac{rx}{1+ax} - bxy &= x \\ cy + dxy &= y \end{aligned} \tag{4}$$

After simple calculation, we have the following lemma:

Lemma 2:

The system (3) has the following equilibira for all parameters values

1 - $E_0 = (0,0)$, the trivial equilibrium always exists without any restriction .

2 - $E_1 = (0, k_1)$exists only if $c = 1$, where $k_1 > 0$.
3 - $E_2 = \left(\frac{r-1}{a}, 0\right)$ exists only if $r > 1$.
4 - $E_3 = (x_1, y_1) = \left(\frac{1-C}{d}, \frac{r-(1+ax)}{b(1+ax)}\right)$, the unique positive equilibrium which exists if
$r > 1 + \frac{a(1-c)}{d}$ and $c < 1$

In order to discuss the local stability analysis of system(3) around the equilibrium points , we have to compute the general Jacobian matrix of the system (3) at point (x, y).
This is given by :

$$J = (x, y) = \begin{bmatrix} J_{11}(x, y) & J_{12}(x, y) \\ J_{21}(x, y) & J_{22}(x, y) \end{bmatrix} \quad (5)$$

where

$J_{11} = \dfrac{r}{(1 + ax)^2} - by$, $J_{12} = -bx$, $J_{21} = dy$ and $J_{22} = c + dx$.

The characteristic polynomial of(5) is
$F(x) = \lambda^2 + p\lambda + D$
Where $p = -\text{trac}(J)$ and $D = \det(J)$
The following theorems give the local stability of E_0, E_1 *and* E_2 .

Theorem 1 : For the system(3)the equilibrium E_0 has:
1-E_0 is sink if and only if $r < 1$ *and* $c < 1$.
2- E_0 is source if and only if $r > 1$ *and* $c > 1$.
3- E_0 is saddle point either $r > 1$ *and* $c < 1$ *or* $r < 1$ *and* $c > 1$.
4-E_0 is nonhyperbolic point either $r = 1$ *or* $c = 1$.
Proof:
It is clear that the eigenvalues of Jacobian matrix at E_0 *are* $\lambda_1 = r$ *and* $\lambda_2 = c$. Thus the all results can be easily obtaind.
For the equilibrium point E_1=$(0, k_1)$,one can see that E_1 is always nonhyperbolic point because of the eigenvalues of the Jacobian matrix at E_1 are $\lambda_1 = r - bk_1$ and $\lambda_2 = 1$.

Theorem 2: For system (3) the equilibrium point E_2has the following:
$1 - E_2$ is sink if $r \in (1, k_2)$and $c < 1$ where $k_2 = \frac{a}{b}(1 - c) + 1$.
$2 - E_2$ is never be source point .
$3 - E_2$ is saddle point if $r \in (k_2, \infty)$
$4 - E_2$ is nonhyperbolic point if $r = k_2$with $c < 1$.
Proof: It is clear that the Jacobian matrix at E_2 is:

$$J_{E_2} = \begin{bmatrix} \dfrac{1}{r} & \dfrac{-b(r-1)}{a} \\ 0 & c + \dfrac{d(r-1)}{a} \end{bmatrix}$$

So that the eigenvalues are $\lambda_1 = \frac{1}{r}$, $\lambda_2 = c + \frac{d(r-1)}{a}$ since $r > 1$ *therefore* $|\lambda_1| < 1$ for all values of r.
Now $|\lambda_2| < 1 \leftrightarrow \left|c + \frac{d(r-1)}{a}\right| < 1 \leftrightarrow \frac{a}{d}(-1 - c) + 1 < r < \frac{a}{d}(1 - c) + 1$.
Thus all results can be obtaind.
We need the following lemma which is found in [20] in order to study the dynamic of the positive equilibrium E_3.

Lemma 3 :Let $F(\lambda) = \lambda^2 + p\lambda + q$ suppose that F(1)>0 , and λ_1, λ_2 are the roots of $F(\lambda) = 0$ then:
$1 - |\lambda_1| < 1$ and $|\lambda_2| < 1$if and only if F(-1) > 0 and $q < 1$
$2 - |\lambda_1| > 1$and $|\lambda_2| > 1$if and only if F(-1)> 0 and $q > 1$.
$3 - |\lambda_1| > 1$and $|\lambda_2| < 1$ (or $|\lambda_1| < 1$and $|\lambda_2| > 1$) if and only if F(-1) < 0 .
$4 - \lambda_1 = -1$ and $|\lambda_2| \neq 1$if and only if F(-1)=0 and $q \neq 0,2$.

Remark:

1-The characteristic polynomial of the Jacobian matrix at $E_3$is given by:
$F(ג) = ג^2 + b - dy_1 ג - \frac{r}{k^2} ג + \frac{r}{k^2} - by_1 - ג + bdx_1y_1$,so that the
$P = by_1 - \frac{r}{k^2} - 1$,and $q = \frac{r}{k^2} - by_1 + bdx_1y_1$. Where $k = 1 + ax_1$
2- It is clear that d<a+d and if a>rd then d<ac+rd, thus we have d<a+ac.
This holds for the positive equilibruim point E_3.
The next theorem gives the dynamic of the unique positive equilibrium E_3.
Theorem 3 : For the unique positive equilibrium E_3we have :
1-$E_3$is sink if one of the following conditions satisfies:

i) $r < min\left\{\frac{a}{d}, \frac{M}{N}, \frac{M1}{N1}\right\}$ and $d > ac$

ii) $r \in \left(0, \min\left\{\frac{a}{d}, \frac{M}{N}\right\}\right)$and $ac > d$

iii) $r \in \left(0, \frac{M1}{N1}\right)$ and $d > ac + a$.

where $M = (1 + c)k^2$, $N = (ax_1 + acx_1 - dx_1), N_1 = (d - ac)$.
And $M_1 = ((1 + ax_1)^2 d)$
2-E_3 is source if one of the following coditions satisfies:

i) $r > \frac{M1}{N1}$ and $d > ac + a$.

ii) $r > \frac{M1}{N1}$ and $d > ac\ as\ well\ as\ r < min\left\{\frac{a}{d}, \frac{M}{N}\right\}$.

3- E_3 is saddle point if and only if $r \in \left(\frac{M}{N}, \frac{a}{b}\right)$.
4- E_3 is non yperbolic if $r = \frac{M}{N}$, $r \neq \frac{2(1+ax_1)^2}{ax_1}$ and $r \neq \frac{4(1+ax_1)^2}{ax_1}$.
Proof: We will apply lemma (3) ,the Jacobian matrix at E_3 is given by :

$$J_{E_3} = \begin{bmatrix} \frac{r}{k^2} - by & -bx_1 \\ dy & 1 \end{bmatrix}$$

So that the caracteristic polynomail will be as follows :
$F(ג) = ג^2 + (by_1 - \frac{r}{k^2} - 1)ג + \frac{r}{k^2} - by_1 + bdx_1y_1$
$p = by_1 - \frac{r}{k^2} - 1$ and $q = \frac{r}{k^2} - by_1 + bdx_1y_1$ .It is clear that F(-1)>0 becouse so that
$F(1) = 1 + by_1 - \frac{r}{k^2} - 1 + \frac{r}{k^2} - by_1 + bdx_1y_1 = bdx_1y_1$. Thus F(-1)> 0 ,$\forall x_1, y_1 > 0$.
To prove (1)(i-iii), we have to show that F(-1)>0 and q<1 .Consider (1) i holds then

$$F(-1) > 0 \Leftrightarrow F(-1) = 1 - by_1 + \frac{r}{k^2} + 1 + \frac{r}{k^2} - by_1 + bdx_1y_1 > 0$$

$$\Leftrightarrow 2 - 2by_1 + \frac{2r}{k^2} + by_1(1 - c) > 0$$

$$\Leftrightarrow (3 + c)k^2 > -r + cr + rax + acrx \Leftrightarrow (3 + c)k^2 > r(ax + cax - dx)$$

$$\Leftrightarrow M > rN .$$

Now if $r < \frac{a}{d}$ then N > 0 by the remark (2)
Therefore F(-1) > 0 if and only if $r < \frac{M}{N}$.

$$\Leftrightarrow \frac{r}{k^2} - \frac{cr}{k} + c < 1 \quad q < 1 \Leftrightarrow \frac{r}{k^2} - by_1 + bdx_1y_1 < 1$$

$$\Leftrightarrow r(d - ca) < k^2 d$$

by assumption we have d > ac then q < 1 $\Leftrightarrow r < \frac{M1}{N1}$ and the prove is finished .
Consider (1) (ii) holds then from (1)i,we have F(-1) > 0 if $r < min\left\{\frac{a}{d}, \frac{M}{N}\right\}$and if ac > d then $N1 < 0$, $q < 1$ hold for all values of r .
1- Consider (1)(iii) holds, from (1)(i), we can see if d > ac+a then N will be less than zero ,and F(-1) > 0 if and only if $r < \frac{M1}{N1}$, therefore the proof is finished.
2-For (2)(i) From proof (1)i,we have that if d > ac +a then F(-1) > 0 for all values of r and q >1 if and only if $r > \frac{M1}{N1}$.

For (2)(ii) , if $r > \frac{M1}{N1}$ and d > ac we can see that q > 1 and from (1)(i) , one can get that F(-1) > 0 hance according to lemma (2) we have E_3 is source point .

3- We have to show that F(-1) < 0 . If $r < \frac{a}{d}$ then by remark N is greater than zero and from (1)(i),F(-1) < 0 if and only if $r < \frac{M}{N}$, so that E_3 is saddle point .

4- one can easily note that F(-1) = 0 if and only if $r = \frac{M}{N}$. Now $p = 0 \Leftrightarrow$

by $-\frac{r}{k^2} - 1 = 0 \Leftrightarrow \frac{r}{1+ax_1} - 1 - \frac{r}{k^2} - 1 = 0 \Leftrightarrow rax_1 = 2k^2 \Leftrightarrow r = \frac{2k^2}{ax_1}$.

And if p = 2 then by the same way one get $r = \frac{4k^2}{ax_1}$, So that $E_3$is nonhyperbolic point when

$$r = \frac{M}{N}, \quad r \neq \frac{2k^2}{ax_1} \text{ and } r \neq \frac{4k^2}{ax_1} .$$

4) Optimal harvesting strategy

We will study an optimal control strategy by using discrete version of PMP Pontryagin maximum principle to solve the optimality problem [21, 22,23,24]. In this optimal control problem the state equations are :

$$x_{t+1} = \frac{rx_t}{1+ax_t} - bx_ty_t - h_tx_t \tag{6}$$

$$y_{t+1} = cy_t + dx_ty_t$$

The x_t and y_t are the prey population density and the predator density at period time t respectivety . The parameters r ,b and d are defined as before while the parameter h_t refers to the cotrol variable ,which represents the harvesting amount at period time t, with $0 \leq h_t \leq A$, where A is the maximum harvesting one can get .

The objective functional that we have to maximize is :

$$J(h_t) = \sum_{t=0}^{T-1} c_1h_tx_t - c_2h_t^2 \tag{7}$$

The term $c_1h_ty_t$ is the amount of money that one can to earn and $c_2h_t^2$ is the associated with the cost of catching and supporting the animals . To solve the problem one has to form the Hamilitonian function for t = 0,1,2,………,T-1 ,this is given by :

$$H(t, x_t, y_t, h_t) = c_1h_tx_t - c_2h_t^2 + \lambda_{t+1}\left(\frac{rx_t}{1+ax_t} - bx_ty_t - h_tx_t\right) + m_{t+1}(cy_t + dx_ty_t),$$

where m_t and λ_t are called the adjoint function or as they are known in the literature by the shadow price [17] . The existence and uniqueness of the optimal control are guarantee due to the finite dimensional structure of the problem . Now according to the maximum principle of Pontrygin [21,23]. The necessary conditions of the above problem are :

$$\lambda_t = ch_t + \lambda_{t+1}\left(\frac{r}{(1+ax)^2} - by - h_t\right) + m_{t+1}(dy_t)$$

$$m_t = \lambda_{t+1}(-bx_t) + m_{t+1}(c + dx_t) \tag{8}$$

and the characterization of the optimal control solution is :

$$h_t^* = \begin{cases} o & if \quad \frac{c_1x_t - \lambda_{t+1}x_t}{2c_2} \leq 0 \\ \frac{c_1x_t - \lambda_{t+1}x_t}{2c_2} & if\ 0 < \frac{c_1x_t - \lambda_{t+1}x_t}{2c_2} < A \\ A & if\ A \leq \frac{c_1x_t - \lambda_{t+1}x_t}{2c_2} \end{cases}$$

Now finding the optimal harvesting with corrosponding optimal state solution at time t will be obtained numerically .

5-Numerical results:

we will use different sets of the values to show the local stability of E_0, E_2 and E_3 for the system (30 , we also choose other values for the optimal harvesting solution. We choose the values of the parameters for E_0 , as follows r =0.9 ,a=1,b = 2,c =0.3 and d = 1 with initial condition (1.1 ,0.15) . Therefore the (1) condition in theorem(1) is satisfied.

Figure-1 shows that the point E_0 is locally stable . For the point E_2 we choose the

values of parameters as following , r =1.2 ,a $= 1$, b $= 2$, c =0.5 and d =1.3 , so that $k_2 = 1.25$, with initial condition (0.5 ,0.1). Figure- 2 illustrates the local stability of E_2 according to (1) in theorem 2 .
To show the local stability of the unique positive equilibrium point E_3, these values of parameter are chosen $r = 1.3$, $a = 1$, $b = 2$, $c = 0.3$ and d =1.3 ,with initial condition (1.3,0.12), so that according to theorem (3)(1) the equilibrium point E_3 is sink point .The local stability of E_3 is shown in Figure-3 .
For solving the optimal problem we use an iterative method to compute the optimality [21,12,24] . Starting by an initial solution of the control with initial of state variable ,then we solve the state system (6)forward while the adjoint system (8) is solve backward and combine the new control with previous one to update the control . This procedure continuous until getting the optimal solutions with corresponding state variables. We choose the set of values of parameter $r = 2$, $a = 1$, $b = 2$, $c = .2$, $d = 1.3$, $c_1 = 0.03$, $c_2 = 0.01$ and $T = 80$, the total optimal harvesting is $J_{opt} = 0.3006$. Figures- 5and 6 show the effect of optimal harvesting on the prey and optimal control variable as function of time respectively. Table-1 compares the total optimal harvesting and other total harvesting starategies using the same values of the parameters.

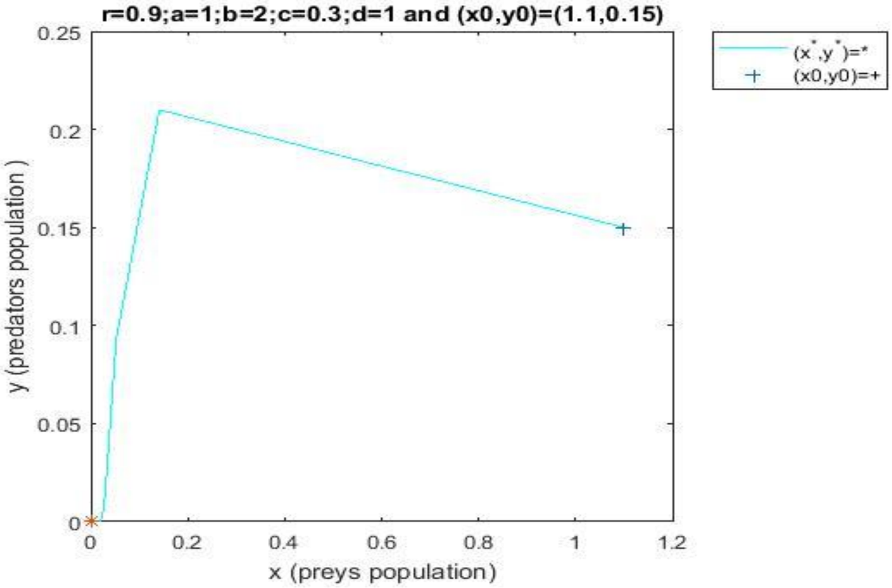

Figure 1-The local stabilty of the point E_0 with its correspndant values of parameters

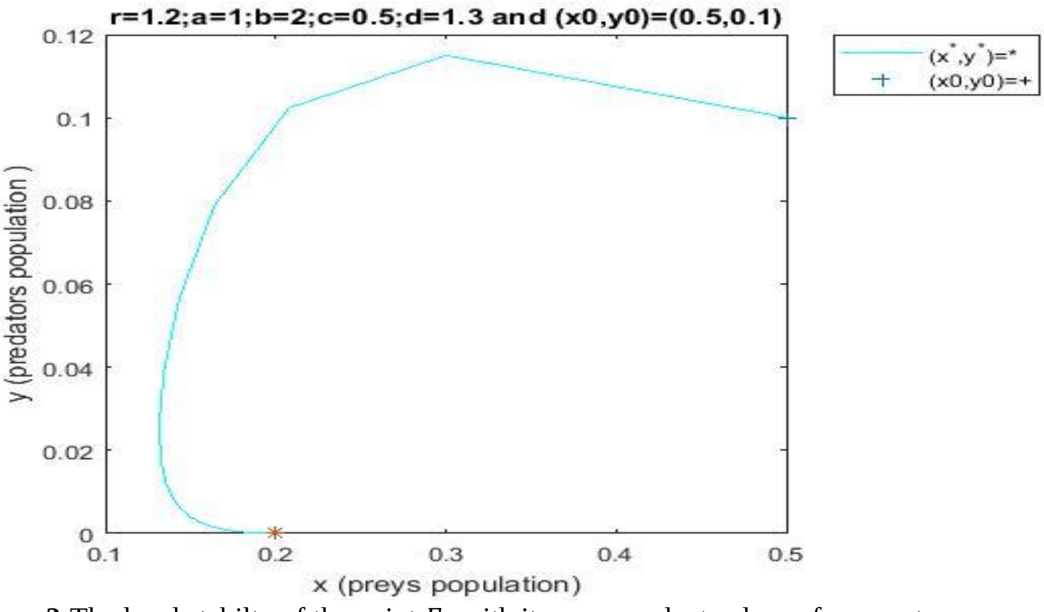

Figure 2-The local stabilty of the point E_2 with its correspndant values of parameters

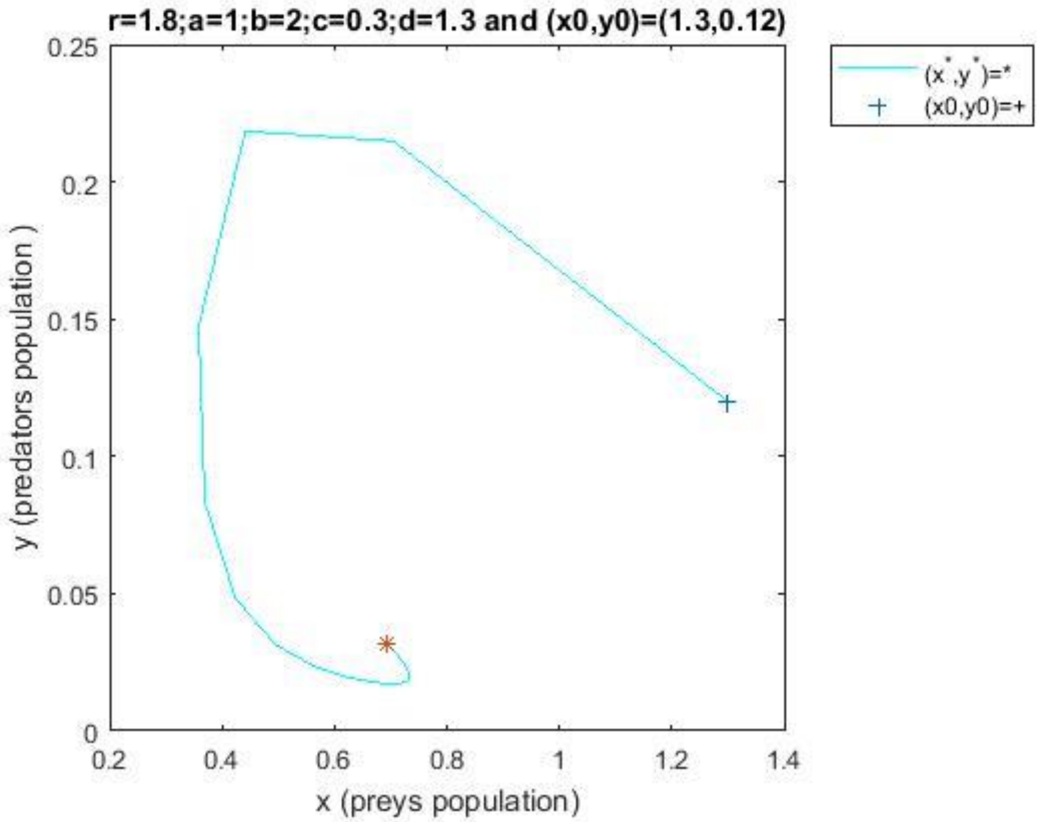

Figure 3-The plot shows the local stabilty of the positive equilibruim point E_3

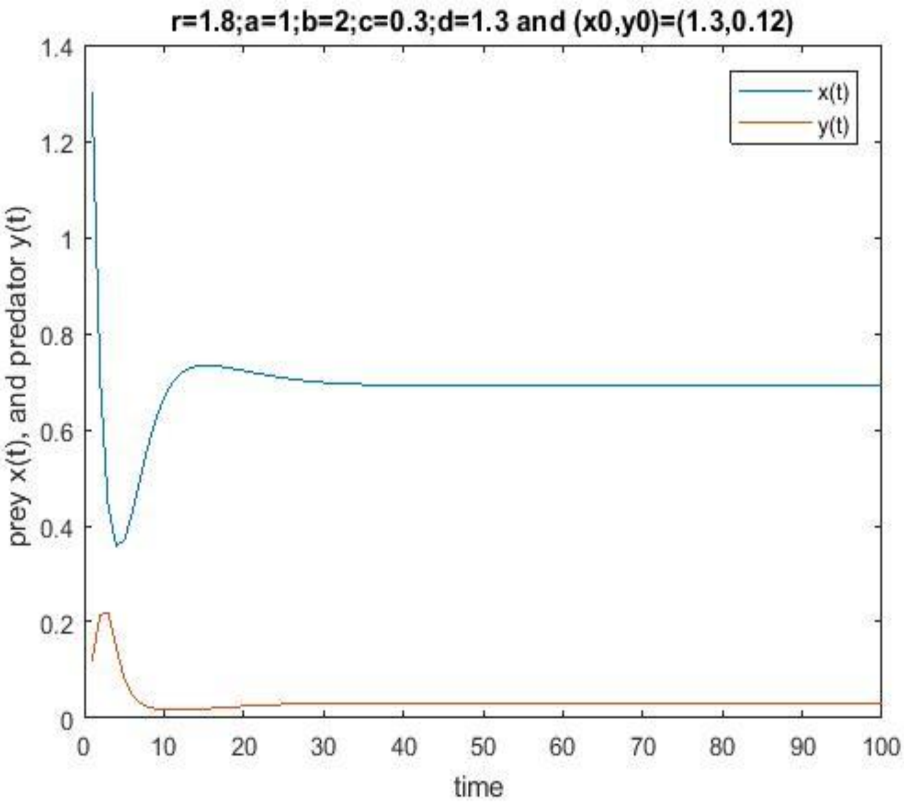

Figure 4-The trajectories of system as the function of time this plot shows the local stabilty of the positive equilibruim point E_3

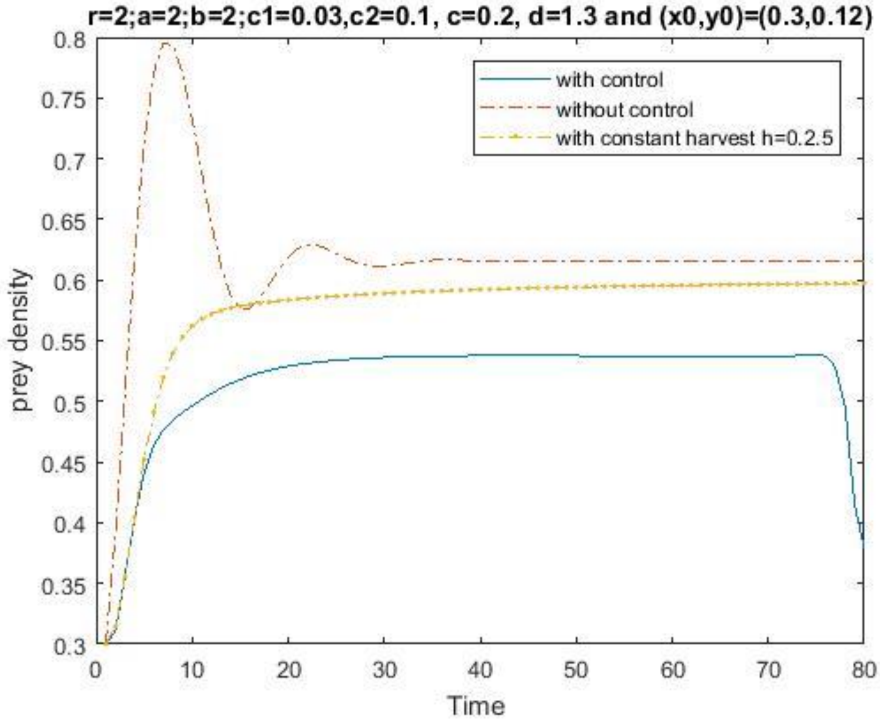

Figure 5-The effect of harvesting on the prey dnsity. All values of parameters are the same .

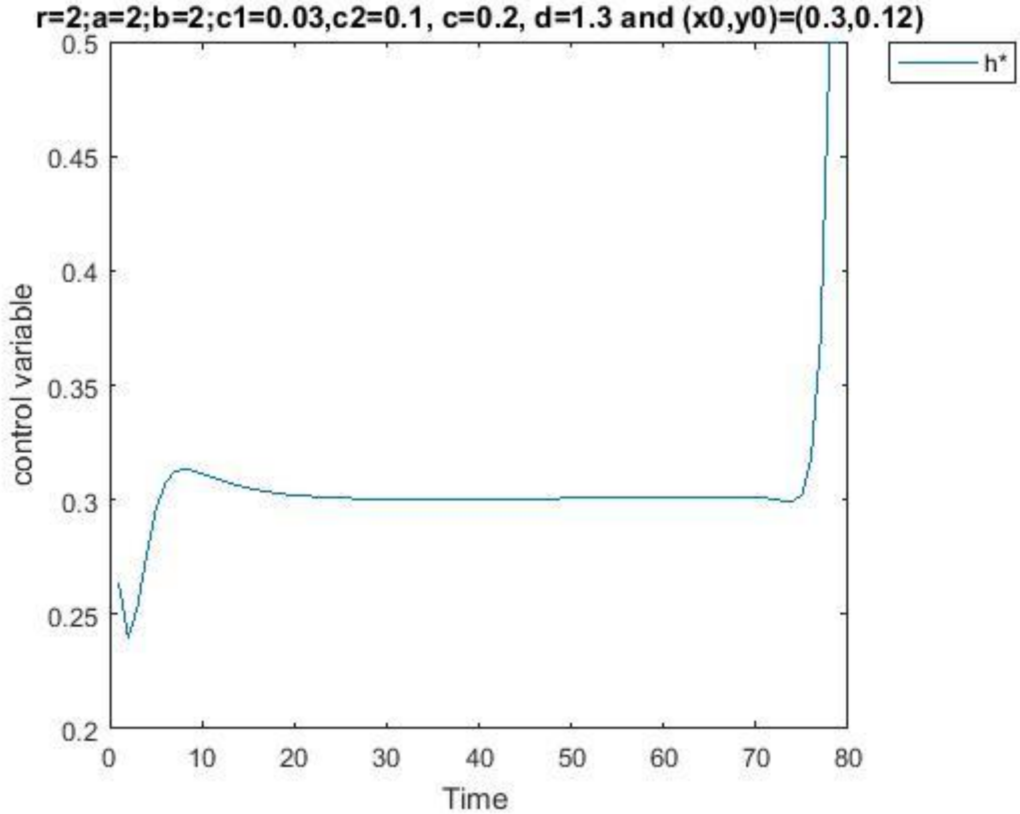

Figure 6-The optimal harevsting is function of time .

Table 1 -This table shows the results of optimal harvesting with the constant harvesting. All value of parameters are same in all strategies.

The harvesting variable	Total harvesting(J)
$h_t = h^*$	$J_{opt} = 0.3006$
$h_t = 0.2$	$J = 0.2531$
$h_t = 0.25$	$J = 0.2901$
$h_t = 0.3$	$J = 0.2990$
$h_t = 0.31$	$J = 0.2985$
$h_t = 0.32$	$J = 0.2975$
$h_t = 0.35$	$J = 0.2909$
$h_t = 0.4$	$J = 0.2690$
$h_t = 0.45$	$J = 0.2348$

6- Conclusions

Discrete time models with Beverton- Holt function growth has been studied and analyzed. All equilibria are found. The local stability for all equilibria is investigated, then the model has been extended to an optimal control problem. The Pontryagin's maximum principle used to solve the optimality problem. All theoretical results confirmed by numerical simulations.

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