Dynamics and optimal Harvesting strategy for biological models with Beverton – Holt growth

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Abstract
In this work, the dynamic behavior of discrete models is analyzed with Beverton- Holt function growth. All equilibria are found. The existence and local stability are investigated of all its equilibria. The optimal harvest strategy is done for the system by using Pontryagin’s maximum principle to solve the optimality problem. Finally numerical simulations are used to solve the optimality problem and to enhance the results of mathematical analysis.

Keywords: Beverton- Holt; Discrete System; Local Stability; Optimal Harvesting;

1- Introduction
For many organisms births occur in regular times each month or year or each circle. Discrete time function is well used to describe the life of them. Many researchers have analyzed models that described by system of difference equations[1-4]. For one dimension model, the general form is governed by first order difference equation \( x_{t+1} = f(x_t) \), where \( x_t \) denotes the size of population at year or period \( t \). For two or more dimensions model, the well known model is Lotka – Volterra, that was first introduced by Lotka and Voltera [5]. After their work, more realistic models were introduced and modified by many authors, we refer to reader for more details see [6-10].

It is well known that harvesting plays an important role in managing the renew resources, so one should consider useful strategies in order to decrease the risks of extinction as well as to increase the net gains. Scientists and researchers used different harvesting strategies in their models [11-13]. For example, Sanchez and Braner analyzed and investigated the effect of periodic harvesting in periodic environments[14,15]. A great deal of attention was given to the discrete as well as continuous logistic models in [16]. Other models are considered in the literatures, for example Ricker model, with constant depletion rate [17]. It was shown numerically that the population exhibits chaotic

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oscillations, which are not necessarily lead to extinction [18,19]. In this paper we will investigate the dynamic and an optimal harvesting of biological models with Beverton-Holt model in one and two dimensions models.

This work is organized as follows: In section 2 we study the dynamics behaviour of a single species with Beverton-Holt growth function with and without harvesting. In section 3, Stability analysis of the prey-predator model is analyzed. In section 4 the model is extended to an optimal harvesting problem. We used the extension of Pontryagin’s maximum principle to find the optimal solutions. In section 5, numerical simulations are used to solve the optimality problem and to enhance the results of mathematical analysis. Finally conclusion is provided.

2) Single species

In this section we will study and investigate the dynamic behaviour of the classical Beverton Holt models with and without constant rate harvesting for single population.

The model without harvesting is given by

$$x_{t+1} = \frac{rx_t}{1+ax_t}$$

(1)

Where $r$ is the inherent growth rate, and $a$ is the population carrying capacity. The dynamics of (1) are well known. For any point $x_0 > 0$, $r < 1$ the extinction equilibrium point is globally asymptotically stable, while if $r > 1$ then $x = 0$ is unstable and the survival equilibrium $x = \frac{r-1}{a}$ is sink point. The last case when $r = 1$ the equilibrium point is nonhyperbolic point. Now if one puts a constant rate harvesting then the model will be as the following:

$$x_{t+1} = \frac{rx_t}{1+ax_t} - qx_t$$

(2)

Where $q$ is a constant representing the intensity of harvesting due to fishing or hunting, here $q$ depends on the density of the population, so that we cannot harvest more than the population density. Thus our mathematical analysis is concerned with $0 < q < q_{max} \leq 1$. The model (2) has also two equilibria, the extinction equilibrium $x_0$ always exists, and the unique positive equilibrium $x_q = \frac{r-1+q}{a(1+q)}$ exists only when $r > 1 + q$. Next lemma describes the stability of $x_0$ and $x_q$.

Lemma 1: For the equilibrium points $x_0$ and $x_q$ of the model 2 we have the following:

1- $x_0$ is sink point if $r < 1 + q$.
2- $x_0$ is source point if $r > 1 + q$.
3- $x_0$ is nonhyperbolic point if $r = 1 + q$.
4- $x_q = \frac{r-1+q}{a(1+q)}$ is always sink point if $r > 1 + q$.

Proof: The proof of 1, 2, and 3 is easy so it is omitted. It is clear that $|f'(x_q)| < 1$ if and only if $-1 < \frac{(1+q)^2}{r} - q < 1$ and only if $r > 1 + q$. So that $x_q$ is always sink point.

3- Stability analysis of the prey-predator model.

In this section we will study the dynamics of two –dimension model, prey-predator system, with Beverton-Holt growth in prey species. The system is given by

$$x_{t+1} = \frac{rx_t}{1+ax_t} - bx_t y_t$$

$$y_{t+1} = cy_t + dx_t y_t$$

(3)

The parameter $a$, $r$, $c$ are the population carrying capacity, the growth rate of the prey and the predator respectively, while the positive parameters $b$ and $d$ represent the maximum per capita killing rate and conversion rate of predator respectively. All equilibrium points of the system (3) can be determined by solving the following algebraic equation:

$$\frac{rx}{1+ax} - bxy = x$$

$$cy + dxy = y$$

(4)

After simple calculation, we have the following lemma:

Lemma 2: The system (3) has the following equilibria for all parameters values

1 - $E_0 = (0,0)$, the trivial equilibrium always exists without any restriction.
2 - $E_1 = (0, k_1)$ exists only if $c = 1$, where $k_1 > 0$.
3 - $E_2 = \left( \frac{r-1}{a}, 0 \right)$ exists only if $r > 1$.
4 - $E_3 = (x_1, y_1) = \left( \frac{1-c}{d}, \frac{r-(1+ax)}{b(1+ax)} \right)$, the unique positive equilibrium which exists if $r > 1 + \frac{a(1-c)}{d}$ and $c < 1$

In order to discuss the local stability analysis of system (3) around the equilibrium points, we have to compute the general Jacobian matrix of the system (3) at point $(x, y)$.

This is given by:

$$J = (x, y) = \begin{bmatrix} J_{11}(x, y) & J_{12}(x, y) \\ J_{21}(x, y) & J_{22}(x, y) \end{bmatrix} \quad \text{(5)}$$

where

$$J_{11} = \frac{r}{(1+ax)^2} - by, \quad J_{12} = -bx, \quad J_{21} = dy \quad \text{and} \quad J_{22} = c + dx.$$ 

The characteristic polynomial of (5) is

$$F(x) = x^2 + px + D$$

Where $p = -\text{tr}(J)$ and $D = \text{det}(J)$

The following theorems give the local stability of $E_0, E_1$ and $E_2$.

**Theorem 1**: For the system (3) the equilibrium $E_0$ has:
1 - $E_0$ is a sink if and only if $r < 1$ and $c < 1$.
2 - $E_0$ is a source point if and only if $r > 1$ and $c > 1$.
3 - $E_0$ is saddle point either $r > 1$ and $c < 1$ or $r < 1$ and $c > 1$.
4 - $E_0$ is a nonhyperbolic point either $r = 1$ or $c = 1$.

**Proof**: It is clear that the eigenvalues of Jacobian matrix at $E_0$ are $\lambda_1 = r$ and $\lambda_2 = c$. Thus all results can be easily obtained.

For the equilibrium point $E_1 = (0, k_1)$, one can see that $E_1$ is always nonhyperbolic point because of the eigenvalues of the Jacobian matrix at $E_1$ are $\lambda_1 = r - bk_1$ and $\lambda_2 = 1$.

**Theorem 2**: For system (3) the equilibrium point $E_2$ has the following:
1 - $E_2$ is a sink if $r \in (1, k_2)$ and $c < 1$ where $k_2 = \frac{a}{b}(1-c) + 1$.
2 - $E_2$ is never be a source point.
3 - $E_2$ is saddle point if $r \in (k_2, \infty)$
4 - $E_2$ is nonhyperbolic point if $r = k_2$ with $c < 1$.

**Proof**: It is clear that the Jacobian matrix at $E_2$ is:

$$J_{E_2} = \begin{bmatrix} \frac{1}{r} & \frac{-b(r-1)}{a} \\ \frac{a}{r} & \frac{d(r-1)}{a} \end{bmatrix}$$

So that the eigenvalues are $\lambda_1 = \frac{1}{r}$, $\lambda_2 = c + \frac{d(r-1)}{a}$ since $r > 1$ therefore $|\lambda_1| < 1$ for all values of $r$.

Now $|\lambda_2| < 1 \iff \left| c + \frac{d(r-1)}{a} \right| < 1 \iff r < \frac{a}{d} (1-c) + 1$. Thus all results can be obtained.

**Remark**: We need the following lemma which is found in [20] in order to study the dynamic of the positive equilibrium $E_3$.

**Lemma 3**: Let $F(x) = x^2 + px + q$ suppose that $F(1) > 0$, and $\lambda_1, \lambda_2$ are the roots of $F(x) = 0$ then:
1 - $|\lambda_1| < 1$ and $|\lambda_2| < 1$ if and only if $F(1) > 0$ and $q < 1$
2 - $|\lambda_1| > 1$ and $|\lambda_2| > 1$ if and only if $F(-1) > 0$ and $q > 1$.
3 - $|\lambda_1| > 1$ and $|\lambda_2| < 1$ (or $|\lambda_1| < 1$ and $|\lambda_2| > 1$) if and only if $F(-1) < 0$.
4 - $\lambda_1 = -1$ and $\lambda_2 \neq 1$ if and only if $F(-1) = 0$ and $q \neq 0, 2$. 

**Remark**: 225
1- The characteristic polynomial of the Jacobian matrix at $E_3$ is given by:

$$F(z) = z^2 + b - dy_1 y_2 + \frac{r}{k^2} z + \frac{r}{k^2} - by_1 - z + bd{x_1}y_1,$$

so that the

$$P = by_1 - \frac{r}{k^2} - 1,$$

and $q = \frac{r}{k^2} - by_1 + bd{x_1}y_1$. Where $k = 1 + a_1$

2- It is clear that $d < a + d$ and if $a > rd$ then $d < ac + rd$, thus we have $d < a + ac$. This holds for the positive equilibrium point $E_3$.

The next theorem gives the dynamic of the unique positive equilibrium $E_3$.

**Theorem 3:** For the unique positive equilibrium $E_3$ we have:

1- $E_3$ is sink if one of the following conditions satisfies:

i) $r < \min \left\{ \frac{a}{d}, \frac{M}{N}, \frac{M_1}{N_1} \right\}$ and $d > ac$

ii) $r \in \left( 0, \min \left\{ \frac{a}{d}, \frac{M}{N}, \frac{M_1}{N_1} \right\} \right)$ and $ac > d$

iii) $r \in \left( 0, \frac{M_1}{N_1} \right)$ and $d > ac + a$.

where $M = (1 + c)k^2$, $N = (ax_1 + acx_1 - dx_1), N_1 = (d - ac)$.

And $M_1 = ((1 + ax_1)^2 d)$

2- $E_3$ is source if one of the following conditions satisfies:

i) $r > \frac{M_1}{N_1}$ and $d > ac + a$.

ii) $r > \frac{M_1}{N_1}$ and $d > ac$ as well as $r < \min \left\{ \frac{a}{d}, \frac{M}{N}, \frac{M_1}{N_1} \right\}$.

3- $E_3$ is saddle point if and only if $r \in \left( \frac{M}{N}, \frac{M}{N}, \frac{M_1}{N_1} \right)$.

4- $E_3$ is non hyperbolic if $r = \frac{M}{N}, r \neq \frac{2(1+ax_1)^2}{ax_1}$ and $r \neq \frac{4(1+ax_1)^2}{ax_1}$.

**Proof:** We will apply lemma (3), the Jacobian matrix at $E_3$ is given by:

$$J_{E_3} = \begin{bmatrix} \frac{r}{k^2} & -by_1 \\ dy & 1 \end{bmatrix}$$

So that the characteristic polynomial will be as follows:

$$F(z) = z^2 + \left( \frac{by_1 - \frac{r}{k^2} - 1}{k^2} \right)z + \frac{r}{k^2} - by_1 + bd{x_1}y_1$$

By assumption we have $d > ac$ then $q < 1$ and the proof is finished.
For (2)(ii), if \( r > \frac{M}{N} \) and \( d > \alpha c \) we can see that \( q > 1 \) and from (1)(i), one can get that \( F(-1) > 0 \) hence according to lemma (2) we have \( E_3 \) is source point.

3- We have to show that \( F(-1) < 0 \). If \( r < \frac{d}{\alpha} \) then by remark \( N \) is greater than zero and from (1)(i),\( F(-1) < 0 \) if and only if \( r < \frac{M}{N} \), so that \( E_3 \) is saddle point.

4- one can easily note that \( F(-1) = 0 \) if and only if \( r = \frac{M}{N} \). Now \( p = 0 \) \( \iff \) \( \frac{r}{k^2} - 1 = 0 \) \( \iff \) \( \frac{r}{1 + ax_t} - 1 = \frac{r}{k^2} - 1 = 0 \) \( \iff \) \( r = \frac{2k^2}{ax_1} \) \( \iff \) \( r = 2k^2 \) \( \iff \) \( r = \frac{2k^2}{ax_1} \).

And if \( p = 2 \) then by the same way one get \( r = \frac{4k^2}{ax_1} \), So that \( E_3 \) is nonhyperbolic point when \( r = \frac{M}{N} \), \( r \neq \frac{2k^2}{ax_1} \) and \( r \neq \frac{4k^2}{ax_1} \).

### 4) Optimal harvesting strategy

We will study an optimal control strategy by using discrete version of PMP Pontryagin maximum principle to solve the optimality problem [21, 22,23,24]. In this optimal control problem the state equations are:

\[
x_{t+1} = \frac{rx_t}{1 + ax_t} - bx_ty_t - h_t x_t
\]

\[
y_{t+1} = cy_t + dx_ty_t
\]

The \( x_t \) and \( y_t \) are the prey population density and the predator density at period time \( t \) respectively. The parameters \( r, b \) and \( d \) are defined as before while the parameter \( h_t \) refers to the control variable, which represents the harvesting amount at period time \( t \), with \( 0 \leq h_t \leq A \), where \( A \) is the maximum harvesting one can get.

The objective functional that we have to maximize is:

\[
f(h_t) = \sum_{t=0}^{T-1} \left( c_1 h_t x_t - c_2 h_t^2 \right)
\]

The term \( c_1 h_t y_t \) is the amount of money that one can earn and \( c_2 h_t^2 \) is the associated with the cost of catching and supporting the animals. To solve the problem one has to form the Hamiltonian function for \( t = 0,1,2, \ldots ,T-1 \), this is given by:

\[
H(t, x_t, y_t, h_t) = c_1 h_t x_t - c_2 h_t^2 + \lambda_{t+1} \left( \frac{rx_t}{1 + ax_t} - bx_t y_t - h_t x_t \right) + m_{t+1} (cy_t + dx_t y_t),
\]

where \( m_t \) and \( \lambda_t \) are called the adjoint function or as they are known in the literature by the shadow price [17]. The existence and uniqueness of the optimal control are guaranteed due to the finite dimensional structure of the problem. Now according to the maximum principle of Pontryagin [21,23].

The necessary conditions of the above problem are:

\[
\lambda_t = ch_t + \lambda_{t+1} \left( \frac{r}{1 + ax_t^2} - by_t - h_t \right) + m_{t+1} (dy_t)
\]

\[
m_t = \lambda_{t+1} (-bx_t) + m_{t+1} (c + dx_t)
\]

and the characterization of the optimal control solution is:

\[
h_t^* = \begin{cases} 
0 & \text{ if } \frac{c_1 x_t - \lambda_{t+1} x_t}{2c_2} \leq 0 \\
\frac{c_1 x_t - \lambda_{t+1} x_t}{2c_2} & \text{ if } 0 < \frac{c_1 x_t - \lambda_{t+1} x_t}{2c_2} < A \\
A & \text{ if } A \leq \frac{c_1 x_t - \lambda_{t+1} x_t}{2c_2}
\end{cases}
\]

Now finding the optimal harvesting with corresponding optimal state solution at time \( t \) will be obtained numerically.

### 5) Numerical Results:

We will use different sets of the values to show the local stability of \( E_0, E_2 \) and \( E_3 \) for the system (30), we also choose other values for the optimal harvesting solution. We choose the values of the parameters for \( E_0 \), as follows \( r = 0.9, a = 1, b = 2, c = 0.3 \) and \( d = 1 \) with initial condition \( (1.1, 0.15) \). Therefore the (1) condition in theorem (1) is satisfied.

Figure-1 shows that the point \( E_0 \) is locally stable. For the point \( E_2 \) we choose the
values of parameters as following, \( r = 1.2, a = 1, b = 2, c = 0.5 \) and \( d = 1.3 \), so that \( k_2 = 1.25 \), with initial condition \((0.5, 0.1)\). Figure-2 illustrates the local stability of \( E_2 \) according to (1) in theorem 2.

To show the local stability of the unique positive equilibrium point \( E_3 \), these values of parameter are chosen \( r = 1.3, a = 1, b = 2, c = 0.3 \) and \( d = 1.3 \), with initial condition \((1.3, 0.12)\), so that according to theorem (3)(1) the equilibrium point \( E_3 \) is sink point. The local stability of \( E_3 \) is shown in Figure-3.

For solving the optimal problem we use an iterative method to compute the optimality \([21, 12, 24]\).

Starting by an initial solution of the control with initial of state variable, then we solve the state system (6) forward while the adjoint system (8) is solve backward and combine the new control with previous one to update the control. This procedure continues until getting the optimal solutions with corresponding state variables. We choose the set of values of parameter \( r = 2, a = 1, b = 2, c = 0.2, d = 1.3, c_1 = 0.03, c_2 = 0.01 \) and \( T = 80 \), the total optimal harvesting is \( J_{opt} = 0.3006 \).

Figures-5 and 6 show the effect of optimal harvesting on the prey and optimal control variable as function of time respectively. Table-1 compares the total optimal harvesting and other total harvesting strategies using the same values of the parameters.

![Figure 1](image1)

**Figure 1**-The local stability of the point \( E_0 \) with its correspondent values of parameters

![Figure 2](image2)

**Figure 2**-The local stability of the point \( E_2 \) with its correspondent values of parameters
Figure 3-The plot shows the local stability of the positive equilibrium point $E_3$

$\begin{align*}
r &= 1.8; a = 1; b = 2; c = 0.3; d = 1.3 \quad \text{and} \quad (x_0, y_0) = (1.3, 0.12)
\end{align*}$

Figure 4-The trajectories of system as the function of time this plot shows the local stability of the positive equilibrium point $E_3$

$\begin{align*}
r &= 1.8; a = 1; b = 2; c = 0.3; d = 1.3 \quad \text{and} \quad (x_0, y_0) = (1.3, 0.12)
\end{align*}$
Figure 5 - The effect of harvesting on the prey density. All values of parameters are the same.

Figure 6 - The optimal harvesting is a function of time.
Table 1 - This table shows the results of optimal harvesting with the constant harvesting. All value of parameters are same in all strategies.

<table>
<thead>
<tr>
<th>The harvesting variable</th>
<th>Total harvesting(f)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h_t = h^*$</td>
<td>$J_{opt} = 0.3006$</td>
</tr>
<tr>
<td>$h_t = 0.2$</td>
<td>$J = 0.2531$</td>
</tr>
<tr>
<td>$h_t = 0.25$</td>
<td>$J = 0.2901$</td>
</tr>
<tr>
<td>$h_t = 0.3$</td>
<td>$J = 0.2990$</td>
</tr>
<tr>
<td>$h_t = 0.31$</td>
<td>$J = 0.2985$</td>
</tr>
<tr>
<td>$h_t = 0.32$</td>
<td>$J = 0.2975$</td>
</tr>
<tr>
<td>$h_t = 0.35$</td>
<td>$J = 0.2909$</td>
</tr>
<tr>
<td>$h_t = 0.4$</td>
<td>$J = 0.2690$</td>
</tr>
<tr>
<td>$h_t = 0.45$</td>
<td>$J = 0.2348$</td>
</tr>
</tbody>
</table>

6- Conclusions

Discrete time models with Beverton-Holt function growth has been studied and analyzed. All equilibria are found. The local stability for all equilibria is investigated, then the model has been extended to an optimal control problem. The Pontryagin’s maximum principle used to solve the optimality problem. All theoretical results confirmed by numerical simulations.

References


