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Deriving The Upper Blow-up Rate Estimate for a Parabolic Problem

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Abstract
In this paper, the blow-up solutions for a parabolic problem, defined in a bounded domain, are studied. Namely, we consider the upper blow-up rate estimate for heat equation with a nonlinear Neumann boundary condition defined on a ball in R_n.

Keywords: Blow-up solutions; Parabolic problem; Heat equation ; Neumann boundary condition; outward normal vector.

اشتقاق صيغة لتقدير النسبة العليا للانفجار لمسألة قطع مكافئ

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الخلاصه
في هذا البحث ، تم دراسه الحلول المنفجرة لمسأله قطع مكافئ معرفة على فضاء مقيد. وبصورة ادق، نهتم بالنسبه العليا لتقدير الانفجار لمعادلة الحرارة مع شرط نيومان الحدودي غير الخطي معرفة على كرة في الفضاء R_n .

1. Introduction

In this paper, we study the follwoing parabolic problem:

$$\left.\begin{array}{ll} u_t = \Delta u, & (x,t) \in B_R \times (0,T), \\ \frac{\partial u}{\partial \eta} = \lambda u^p e^{qu} \quad , & (x,t) \in \partial B_R \times (0,T), \\ u(x,0) = u_0(x), & x \in B_R, \end{array}\right\} \quad (1)$$

where $p \geq 1$; $q, \lambda > 0$; B_R is a ball in R^n; η is the outward normal and u_0 is smooth, nonzero, nonnegative, radially symmetric, satisfiing the following condition:

$$\frac{\partial u_0}{\partial \eta} = \lambda u_0^p e^{qu_0} \quad , \quad x \in \partial B_R,$$

$$\Delta u_0 \geq 0, \quad x \in \overline{B}_R.$$

Many real problems in the fields of fluid dynamics, population, heat propagation, and others are modeled using the forms of parabolic partial differntional equations assosiated with different types of initial-boudary conditions [1-3].

In some cases, the solutions of parabolic problems cannot be continued globally in time. This is called the blow-up phenomenon which, in time-dependent problems, has been studied over the past years by many authoes [3-7]. One of these problems is the problem of the heat equation defined in a ball B_R with a nonlinear Neumann boundary condition, $\frac{\partial u}{\partial \eta} = f(u)$ on $\partial B_R \times (0,T)$, which has been introduced in previous articles [7-11].

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In an earlier study [10], it was proved that if $1/f$ is integrable at infinity for positive values, and f is nondecreasing, then with any positive initial function u_0, the blow-up occurs in a finite time. In addition, if $f \in C^2(0,\infty)$, is a convex and increasing function in $(0,\infty)$, then the blow-up can only occur on the boundary.

In another study [8], a speial case was considerd, where $f(u) = u^p$, showing that for any u_0, the blow-up occurs for $p > 1$, and it can only occur on the boundary. Futhermore, in other investigations [9,11], it was proved that the upper (lower) blow-up rate estimates are as follows:

$$C_1(T-t)^{\frac{-1}{2(p-1)}} \le \max_{x\in\overline{B}_R} u(x,t) \le C_2(T-t)^{\frac{-1}{2(p-1)}}, \quad t \in (0,T).$$

Another special case, where $f(u) = e^u$, was considered in another work [7], showing that all positive solutions blow up in finite times, and the blow-up can only occur on the boundary. Moreover, the upper (lower) blow-up rate estimates are as follows:

$$C_1(T-t)^{-1/2} \le e^{u(1,t)} \le C_2(T-t)^{-1/2}, \quad 0 < t < T.$$

A similer result has been obtained [3] for another case: $f(u) = e^{u^p}, p > 1$.
In this paper, we derive that upper blow-up rate estimate for problem (1), showing that:

$$\max_{\overline{B}_R} u(x,t) \le \log C - \frac{\lambda}{2q}\log(T-t), \quad 0 < t < T$$

The rest of this paper is organized as follows: In section two, we discuss the local existance, blow-up, and blow-up set with stating the main properties of classical solutions for problem (1). In section three, we derive the upper blow-up rate estimate. Section four is devoted to state some conclusions.

2. Basic properties and Blow-up Set

It is well known that with any smooth initial function, problem (1) has a local unique classical solution [12]. Moreover, since $f(u) = \lambda u^p e^{qu}$ is positive, increasing , convex, and $1/f$ is integrable at infinity for $u > 0$, then each solution of problem (1) blows up in a finite time and the blow-up can only occur on the boundary [10] .

The next lemma, proved in an earlier study [3], presents some solution properties of problem (1).
For simplicity, we denote $u(r,t) = u(x,t)$.

Lemma 2.1. Let $u \in C^{4,2}(B_R \times (0,T))$ be a solution to problem (1). Then

1. u is positive and radial on $\overline{B}_R \times (0,T)$, and u_r is nonnegative in $[0,R] \times [0,T)$.
2. u_t is positive in $\overline{B}_R \times (0,T)$, and if $\Delta u_0 \ge a > 0$, in $\overline{B}_R$, then $u_t \ge a$, in $\overline{B}_R \times [0,T)$.

3. Upper Blow-up Rate Estimate

The next theorem is concerned with deraving the upper blow-up rate estimate for problem (1).

Theorem 3.2 Let $u \in C^{4,2}(B_R \times (0,T))$ be a blow-up solution to (1), such that $\Delta u_0 \ge a > 0$ $\forall\, x \in \overline{B}_R$. Then there is a positive constant C such that

$$\max_{\overline{B}_R} u(x,t) \le \log C - \frac{1}{2q}\log(T-t), \quad 0 < t < T. \tag{2}$$

Proof.

As in another work [5], we define the function:

$$F(x,t) = u_t(r,t) - \varepsilon u_r^2(r,t), \quad (x,t) \in B_R \times (0,T).$$

By some dirct calculations, it follows that

$$F_t - \Delta F = 2\varepsilon\left(\frac{n-1}{r^2}u_r^2 + u_{rr}^2\right) \ge 0.$$

Since $\Delta u_0 \ge a > 0$, and $u_{0r} \in C(\overline{B}_R)$, for a samll enough value ε, we have

$$F(x,0) = \Delta u_0(r) - \varepsilon u_{0r}^2(r) \ge 0, \quad x \in B_R,$$

In addition,

$$\frac{\partial F}{\partial \eta}\Big|_{x\in S_R} = u_{rt}(R,t) - 2\varepsilon u_r(R,t)u_{rr}(R,t)$$
$$= (\lambda u^p e^{qu})_t - 2\varepsilon\lambda u^p e^{qu}\left(u_t(R,t) - \frac{n-1}{r}u_r(R,t)\right)$$
$$\ge (qu + p - 2\varepsilon)\lambda u^{p-1}e^{qu}\, u_t(R,t).$$

Since $u_t > 0$, *on* $\overline{B}_R \times (0,T)$, it follows that

$$\frac{\partial F}{\partial \eta}\Big|_{x\in S_R} \ge 0, \quad t \in (0,T),$$

provided that

$$\varepsilon \leq \frac{qu_0+p}{2}.$$

From the comparison principles [13] and [14], we obtain

$$F(x,t) \geq 0, \quad (x,t) \in \overline{B}_R \times (0,T),$$

In particular, $F(x,t) \geq 0$, for $|x| = R$, which leads to

$$u_t(R,t) \geq \varepsilon u_r^2(R,t) = \varepsilon\lambda^2 u^{2p} e^{2qu} \quad , \quad t \in (0,T).$$

Since u blows up at T, and increases in time, there exists $\tau \leq T$ such that

$$u(R,t) \geq 1 \quad for \quad \tau \leq t < T,$$

which leads to

$$u_t(R,t) \geq \varepsilon\lambda^2 e^{2qu(R,t)}, \quad t \in [\tau,T).$$

By integrating the last inequality from t to T, we obtain

$$\int_t^T u_t e^{-2qu(R,t)} \geq \varepsilon\lambda^2(T-t).$$

Thus

$$-\frac{1}{2q} e^{-2qu(R,t)}|_t^T \geq \varepsilon\lambda^2(T-t). \tag{3}$$

Since

$$u(R,t) \to \infty, \quad e^{-qu(R,t)} \to 0 \quad as \quad t \to T,$$

So that, from the inequality (3), we get

$$\frac{1}{e^{qu(R,t)}} \geq (2\lambda^2 q\varepsilon(T-t))^{1/2},$$

Thus

$$(T-t)^{\frac{1}{2}} e^{qu(R,t)} \leq \frac{\lambda}{\sqrt{2q\varepsilon}} \ ,$$

So, for some postiive constant C , we have

$$\max_{\overline{B}_R} u(x,t) \leq \log C - \frac{\lambda}{2q}\log(T-t), \quad 0 < t < T.$$

4. Conclusions

This paper is devoted to derive the upper blow-up rate estimate for problem (1). The results show that the upper blow-up rate estimate formula does not depend on p, which means that the power function, appeared in the boundary condition, does not make any effect on the blow-up profile to problem (1). Therefore, the influence of the power function may only appear on the blow-up time.

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References

1. Rasheed, M.A., Balasim, A.T. and Jameel, A.F. **2019**. Some Results for the Vorticity Transport Equation by using A.D.I Scheme, AIP Conference Proceedings, 2138, 030031, doi.org/10.1063/1.5121068
2. Rasheed, M.A., Laverty, S. and Bannish, B. **2019**. Numerical solutions of a linear age-structured population model, AIP Conference Proceedings, 2096, 020002, doi.org/10.1063/1.5097799
3. Rasheed, M.A. **2012**. On Blow-up Solutions of Parabolic Problems, Ph.D. thesis, University of Sussex,UK.
4. Rasheed, M.A. , Al-Dujaly, H.A.S. and Aldhlki, T.J. **2019**. Blow-Up Rate Estimates for a System of Reaction-Diffusion Equations with Gradient Terms, *International Journal of Mathematics and Mathematical Sciences*, **2019**(1): 1-7.
5. Rasheed, M.A. and Barghooth, L.J. **2019**. Blow-up Set and Upper Rate Estimate for A Semilinear Heat Equation, *Journal of physics*: Conf. Series 129, 032013.
6. Rasheed, M.A. and Chlebik, M. **2020**. Blow-up Rate Estimates and Blow-up Set for a System of Two Heat Equations with Coupled Nonlinear Neumann Boundary Conditions, *Iraqi journal of Science*, **61**(1): 147-152.
7. Deng, K. **1994**. The blow-up behavior of the heat equation with Neumann boundary conditions, *J. Math. Anal. Appl*. **188**: 641-650.
8. Fila, M. **1989**. Boundedness of global solutions for the heat equation with nonlinear boundary conditions, *Comment. Math. Univ. Carolin*. **30**: 479-484.

9. Fila, M. and Quittner, P. **1991**. The blow-up rate for the heat equation with a nonlinear boundary condition, *Math. Methods Appl. Sci*. **14**: 197-205.
10. Gomez, J.L., Marquez, V. and Wolanski, N. **1991**. Blow up results and localization of blow up points for the heat equation with a nonlinear boundary condition, *J. Differ. Equ*. **92**: 384-401.
11. Hu, B. and Yin, H**.**M**. 1994**. H.M. 1The profile near blow-up time for solution of the heat equation with a non-linear boundary condition, *Trans. Amer. Math. Soc*. **346**: 117-135.
12. Ladyzenskaja, O.A., Solonnikov, V.A. and Uralceva, N.N. **1968**. Linear and Quasilinear Equations of Parabolic Type, Translations of Mathematical Monographs, *American Mathematical Society,* 23.
13. Pao., C.V. **1992**. *Nonlinear Parabolic and Elliptic Equations*, New York and London: Plenum Press.
14. Quittner, P. and Souplet, PH. **2007**. *Superlinear Parabolic Problems. Blow-up, Global Existence and Steady States*, Birkhuser Advanced Texts, Birkhuser, Basel.