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On The Intersection of Semi-Pure Subgroups of Abelian Group

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Abstract

The following question was raised by L.Fuchs: "what are the subgroups of an abelian group G that can be represented as intersections of pure subgroups of G ? . Fuchs also added that “One of my main aims is to give the answers to the above question". In this paper, we shall define new subgroups which are a family of the pure subgroups. Then we shall answer problem 2 of L.Fuchs by these semi-pure subgroups which can be represented as the intersections of pure subgroups.

Keywords: Abelian group; Subgroups; Semi-pure subgroups; Minimal pure subgroup.

حول تقاطعات الزمر الجزئية شبه التامة في الزمر الابيلية

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الخلاصه

طرح L.Fuchs السؤال التالي: "ما هي الزمر الجزئية لزمرة ابيلية G التي يمكن تمثيلها على أنها تقاطعات زمر جزئية نقية من G؟ أضاف Fuchs أيضًا أن" أحد أهدافي الرئيسية هو إعطاء الإجابات لما سبق سؤال". في هذا البحث، سنحدد الزمر الجزئية التي هي عائلة من الزمر الجزئية النقية. ثم سنجيب على المشكلة 2 ل L.Fuchs من خلال هذه الزمر الجزئية شبه النقية والتي يمكن تمثيلها على أنها تقاطعات الزمر الجزئية النقية.

1. Introduction

We shall give the definition of semi-pure subgroups and prove that, if L is a subgroup of an abelian group G, with for all $x \in L$, for some $n \in z^+$, $x \notin \in nL$, then $n \mid x$ in G , $L = \cap_{i \in I} G_{\alpha i}$, where $G_{\alpha i}$ are a family of all pure subgroups $G_{\alpha i}$ containing L, for all $i \in l$. We shall also study the answer to the following question:

Under which conditions the intersections of pure subgroups are pure ?

We need the following some important definitions, which are used in this paper. Moreover, we need some important properties of subgroups without the proofs:

Definition (A): A group G is said to be a primary group of p-group, if each of its elements g is a power $p^{(ng)}$ of a given prime [1].

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Definition (B): A subgroup G of A is called pure, if the equation $n x = g \in G$ is solvable in G whenever it is solvable in the whole group A. This amounts to saying that G is pure in A if $n \mid g$ in A implies $n \mid g$ in G [2].
We enumerate some of the most important properties of pure subgroups .
1- Every divisible of an abelian group is pure.
2- Purity is a transitive.
3- The union of an ascending chain of pure subgroups is pure .
4- The sum of pure subgroups in general is not pure [3].
5- Let H be a subgroup of G, and B a pure subgroup K $\ni (G / B) = (H / B) + (K / B)$. Then K is pure [4].
6- If G is a p-group and K is a pure subgroup of G containing G_n [p], then K is a direct summand [5].
7- If G is torsion-free group and contains elements $g_1, g_2, \ldots g_m$ $G = \sum_{i=1}^{n} < g_i >$, then every pure subgroup A of G , $A = \sum_{i=1}^{n} < a_i >$ where $a_i \in A$ [6].

2. Main Results

We, in similar manner to Fuchs , raised the following question:
Which are those subgroups of an abelian group G that can be represented as intersections of pure subgroups of G ?
We give some definitions and conditions to answer this question.
We start with the following definitions:
Definition (C): A pure subgroup N of G is said to be minimal-pure containing a subgroup N of G, which is said to be minimal-pure containing a subgroup L, if and only if, for all subgroup K of N, if $K \cap L = [0]$, then K={0} .
Definition (D) : A subgroup L of an abelian G is said to be semi-pure in G⇆ for all $x \notin L$,there exists a pure subgroup D of G, such that $D \supseteq L$ and $x \notin D$.
It is clear that every semi-divisible subgroup is semi-pure and every pure subgroup of G is semi-pure.
We know that the intersection of pure subgroups needs not to be pure, as in the following example .
Example (1) : Take $G = z_4 \otimes z_2$, $s_1 = \{(a, 0) | a \in z_4\}$ $s_2 = \{(0,0), (1,1), (2,0), (3,1)\}$. Consider that (a,0)∈ s_1, for all positive integers n, suppose that n|(a,0) in G, which means that n (a,0) = (a,0) for $x \in Z_4$ and $y \in Z_4$.
Then we have
n^x =a (mod 4)
n^y =0 (mod 2)
We obtain
Nx(x,0)=(nx,0)=(a,0) for (x,0) ∈ s_1
Therefore, n|(a,0) in s_1. So, s_1 is pure in G .
Next, to prove that s_2 is pure in G:
Examine (1,1) and (3,1).
Let $\in z^+$, to prove that for all $x \in s_2$, if n|x in G then n|xin s_1.
Case 1: If n is an even positive number, then $n \nmid (1,1)\big((1,1) \nmid nG\big)$. .
Case 2 : If n is an odd positive number, then n=4k+3 or n=4k+1, for k={0,1,2,3,…}, we have (4k+1)(1,1)=(1,1) and (4k+3) (3,1)=(1,1). which means that n|(1,1) in s_2 , By the same way, one can show that, if n|(3,1) in G, then n|(3,1) in s_2 .
Examine (2,0)
Let n be an even positive integer number, then k= 4k+2 or n=4k for k={0,1,2,…}. In the first case, we have n(1,1)=(4k+2)(1,1)=(2,0).
In the second case, for any (x,y), we get n(x,y)=4k(x,y)=(0,0), hence n|(2,0) in G .
If n is an add positive integer number, then it is clear that n|(2,0) in G.
Consequently, s_2 is pure in G.
Now, $s_1 \cap s_2 = \{(0,0), (2,0)\}$.
Consider the equation 2(x,y)=(2,0) , which has the following solutions in G {(1,1),(1,0),(3,0),(3,1)}. But each of these elements does not belong to $s_1 \cap s_2$. Hence, $s_1 \cap s_2$ is not pure .
Now, we need the following lemma for the next theorem.

Lemma (A): Let G be an abelian group, L be a subgroup of G, and M be a minimal pure subgroup of G containing L, then

a) For all positive integer n, $M_{[n]} = L_{[n]}$.

b) For $x \in M$. ∃ positive integer n ∋ $nx \in nL$.

c) For all positive integer n, for all $\ell \in L$, $(L \notin nL)$, and for all $r \in z^+$ if (r,n)=1, then $r\ell \notin nL$.

Proof: a) Claim that $M_{[n]} \subseteq L_{[n]}$.If not, then ∃ positive integer n ∋ $x \in M_{[n]}$ and $x \notin L$.

Now, we shall show that $< x > \cap \{0\}$. If not, thus ∃ $y \in < x > \cap L \neq \{0\}$, and hence y=rx, where (r,n)=1 .

Thus, λr+7n=1 (λ,7∈ z^+) .

Therefore, λrx+7nx=x . But we have 7px=0, implies that λrx=x∈ L. This is a contradiction.

Consequently,

$< x > \cap L = \{0\}$. But M is a minimal pure subgroup, so this is impossible due to the fact that $x \neq 0$.

We get $x \in L$ and $\in L[n]$. Therefore, we obtain the result.

b) If not, then for all $n \in z^+$, $nx \notin L$, so we have $< x > \cap L = \{0\}$. But M is a minimal pure subgroup containing L, hence x=0 , thus we consequently have ∃ $n \in z^+$ ∃$nx \in L$.

c) Suppose that $n\ell \in nL$, so ∃ ℓ_0 ∋ $n = n\ell_0$, and we have mr+λn=1 for m , λ∈ z^+.

Thus, ℓmr+7λn=$\ell \in L$, then nmℓ+λnℓ=ℓ, so n(m20+ℓ)=ℓ .

Since mℓ+λℓ=$L_1 \in L$, then $n\ell_1 = \ell \in nL$. But this is a contradiction $\ell \notin nL$.

Then we get the result.

Now, we are ready to give an answer to the earlier question at the beginning of the main results:

Theorem (1). Let G be an abelian group, L be a subgroup of G, and $(G_{\alpha i}\}$ be a family of all pure subgroups $G_{\alpha i}$ and containing L . Then $L = \cap_{i=l} G_{\alpha i}$ ⇆ L is semi-pure in G.

Proof: Suppose that L is a semi-pure in G, to show that$\cap_{i=l} G_{\alpha i} \subseteq L$.

Consider any element $x \in \cap_{i=l} G_{\alpha i}$, hence $x \in G_{\alpha i}$ for all $i \in l$.

Consequently,

$$L = \cap_{i=l} G_{\alpha i}$$

Conversely, let $L = \cap_{i=l} G_{\alpha i}$ to prove that L is semi-pure.

If not, then ∃ $x \in G$ and $x \notin L$, for any pure subgroup (say D) $D \supseteq L$, $x \in D$ .Therefore $x \in G_{\alpha i}$ for all $i \in l$.

This implies that $x \in \cap_{i=l} G_{\alpha i}$ =L .Thus, we get a contradiction.

Consequently, L is a semi-pure in G.

Now, we are ready to give another answer of our question:

Theorem (2): Let G be an abelian group and L be a subgroup of G.

If, for all $x \in L$, $x \notin nL$ ⇆ $n|x$ in G, then the following conditions are equivalent:

1- $L = \cap_{i=l} G_{\alpha i}$, for $G_{\alpha i}$ are the family of pure subgroups containing L .

2- For all positive integer numbers n, $[n] \subseteq L$ ⇆ $nL = L$.

Proof: (1)→(2)

Assume that $L = \cap_{i=l} G_{\alpha i}$ for all $i \in l$ and for all $\in z^+$, $G[n] \subseteq L$.

Claim that $L \subseteq nL$. If not, then ∃ $x \in L$, $x \notin nL$.

Therefore, n|x in G which means that ∃ $g \in G$ ∋ $ng = x$ (it is clear that $g \notin L$)

But $L = \cap_{i=l} G_{\alpha i}$, so by Theorem (1), L is semi-pure in G and hence $B \supseteq L$ is a pure subgroup and $g \notin B$.

Since n|x in G, it follows that B in G. Which means that ∃ $b \in B$ ∋ $nb = x$.

Thus n(b-g)=0 ⇆ (b-g) ∈ G[n] .

But G[n] ⊆L⊆B , therefore (b-g) ∈B → g∈B, which is a contradiction. We obtain L-nL .

(1) → (2) To prove that $\cap_{i=l} G_{\alpha i} \subseteq L$.

If not, then ∃ $y \in \cap_{i=l} G_{\alpha i}$ and $\notin L$.

Suppose that M is a minimal pure subgroup of G containing L, so $y \in M$.

Now, by using Lemma (A), we have $\ell =$ for some $\ell \in L$, $n \in z^+$.

Suppose that nx_y in L, if this is not true, then $\ell_0 = ny = \ell$ $(y - \ell_0) \in M[n]$.

Now, by Lemma (A), we have $(y - \ell_0) \in M[n] = L[n] \subseteq L$ $y \in L$, which implies that $y \in L$ and this is a contradiction .

Therefore, $n \nmid y$ in L.

We get $nL \nsubseteq L$ and hence $G[n] \nsubseteq L$.
Therefore, $\exists\ z \in G[n], z \notin L$.
Suppose that $w = z + y$ and $L = L + < w >$ then $nw = nz + ny - nw = ny = L$.
Claim. $L[n] \subseteq L$.
If not, then $\exists\ \ell \in L[n]$ and $\ell \notin L$.
Thus, $\ell = \ell + rw$ where $0 \le r < n$,which implies that
$n\ell = n\ell + nrw = 0$. Hence, $n\ell = nrw = r\ell$
We obtain $n \in nL$. Now, by using Lemma (A), So, we get a contradiction .
Suppose that M is a minimal pure subgroup containing L, so $M \supseteq L \supseteq L$, and hence $y \in M$.
But we have $w \in L \subseteq M$, so again by the above lemma we get $z = w - y \in M[n] = L[n] = L[n] \subseteq L$.
But this is a contradiction. Consequently, $y \in L$ and hence $L = \cap_{i=I} G_{\alpha i}$.
As a consequence of the above results, we obtain the following result :
Theorem (3) : Let G be an abelian group and assume that for all $x \epsilon L \backslash nL$ $n|x$ in G , Lbe is a subgroup of G and $G_{\alpha i}$ is a family of all pure subgroups containing L.
Then the following conditions are equivalent:
1) L is a semi-pure in G.
2) $L = \cap_{i=I} G_{\alpha i}$.
3) For all $n \in z^+$. If $G[n] \subseteq L \rightarrow nL = L$
Proof: We need only to show (3)→(1), because (1)→(2)→(3) are obvious.
If not, then $\exists\ y \in L, y \in D$, and for any pure subgroup (say D) containing L, suppose that M is a minimal pure subgroup of G containing L, so by Lemma (A) we have $ny = \ell \in L$.
Claim. $n \nmid \ell$ in L. If not, then $\exists\ \ell_0 \in L \ni n\ell_0 = \ell$, thus $ny = n\ell = \ell \rightarrow y - \ell_o \in M[n] = L[n] \subseteq L$. Therefore, $y \in L$ and this is a contradiction .
We get $n \nmid \ell$ L, which means that $nL \nsubseteq L$, thus $G[n] \nsubseteq L$, so $\exists\ z \in G[n]$ and $z \notin L$.
Suppose that $L = L + w$, where $w = z + y$, then $nw = nz + ny\ nw = ny = L$.
Claim. $L[n] \subseteq L[n]$. If not, then $\exists\ L \in L \ni \ell \in L[n], \ell \notin L[n]$. thus $L[n] = \ell_1 + rw$ for some $r \in z^+$.
Which implies that $n\ell = n\ell + nrw = 0 \leftrightarrows -n = nrw = r\ell$. We obtain $r\ell \in nL$. But this is a contradiction (see Lemma(A))
Consequently $L[n] = L[n]$
Suppose that M is a minimal pure subgroup containing L, so $M \supseteq L \supseteq L$ and hence $y \in M$.
Since $w \in L \subseteq M$, so $z = (w - y) \in M[n] = L[n] = L[n] \subseteq L$. We get a contradiction .
Therefore for $y \notin L$ $\exists$ a pure subgroup of G containing L, (say D) and $y \notin D$.

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