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Types of Fixed Points of Set-Valued Contraction Mappings for Comparable Elements

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Abstract
This paper is concerned with the study of the fixed points of set-valued contractions on ordered g −metric spaces. The first part of the paper deals with the existence of fixed points for these mappings where the contraction condition is assumed for comparable variables. A coupled fixed point theorem is also established in the second part.

Keywords: Partially Ordered General Metric Spaces, Fixed Point, Coupled Point,

أنواع من النقاط الصامدة للتطبيقات الانكماشية المتعددة القيم العاملة على العناصر القابلة للمقارنة

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الخلاصه
هذه الورقة البحثية معنية بدراسة نقطة صامدة لتطبيقات انكماشية متعددة القيم معرفة على فضاءات g −المترية. يتناول الجزء الأول من الورقة وجود نقاط الصامدة لهذه التطبيقات حيث يُفترض أن حالة الانكماش بالنسبة العناصر القابلة للمقارنة. تم اثبات مبرهنة النقطة الصامدة المقترنة أيضًا في الجزء الثاني.

1. Introduction

Recently, the fixed point theory was developed rapidly in a partially ordered metric space. Some generalization of the usual metric space is provided here. In 1993, Czerwik [1] introduced the b −metric spaces, followed by several results which dealt with the fixed point theory in such space [2,3,4]. In 2000, Brancecir [5] defined a generalized metric space as a metric space in which the triangle inequality is replaced by the rectangular one. Since then, many authors proved results in the field of metric fixed point theory [6,7]. In 2006, Mustafa and Sims [8] presented another modification of a usual metric which is known as G-metric space. Saadati *et al*. [9] proved some fixed point results for contractive mappings in partially ordered G-metric space. Lakshmikantham *et al*.[9],[10] demonstrated the notion of coupled coincidence point for a mapping T and studied coupled fixed point theorems in partially ordered metric spaces. Therefore, Mustafa and Sims together with other researchers extended some pervious results and provided new findings [1,11-14]. In 2014, Aghajani *et al*. [11] introduced a new generalization of b-metric and G-metric spaces. Recently, Mustafa *et al*. [15] obtained some coupled coincidence point theorems for G_b −metric space. Abbas and Rhoades studied common fixed point theory in generalizes metric space, while many authors obtained fixed and common fixed points in G-metric spaces [16-23]. In 2006, Bhaskar and Lakshmikantham[23] introduced the concept of mixed monotone property. In 2009, Lakshmikanthem and Ciric [10]

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generalized the concept of mixed monotone mapping and proved a common coupled fixed point theorem.

2. Preliminaries

We begin with the following definition.

Definition (2.1): [2]

Let $\mathcal{M}$ be a nonempty set and ω: $\mathcal{M}^3 \rightarrow [0,\infty)$ be satisfying the following conditions:

1-$\omega(p,q,e)$=0 if and only if $p = q = e$.

2-0< $\omega(p,p,q), \forall p,q \in \mathcal{M}$ with $p \neq q$

3- $\omega(p,p,q) \leq \omega(p,q,e)$ for all $p,q,e \in \mathcal{M}$ with q $\neq e$

4- $\omega(p,q,e) = \omega(p,e,q) = \cdots$,(symmetry in all three variables),

5- $\omega(p,q,e) \leq \omega(p,a,a) + \omega(a,q,e)$ for all $q,e,a \in \mathcal{M}$.

then the function ω is called generalized metric on $\mathcal{M}$ and the pair $(\mathcal{M},\omega)$ is called a g −metric space.

Example(22)**: [23].** Consider $\mathcal{M}$=R^+ with usual distance $d(p,q) =| p - q|$, for all p,q in $\mathcal{M}$. Define ω:$\mathcal{M}^3 \rightarrow R^+$

$\omega\ (p,q,e) = |p-q| + |q-e| + |e-p|$ for all $p,q,e \in \mathcal{M}$.

Then ω is a g −metric on $\mathcal{M}$.

Definition (2.3): [8]

Let $(\mathcal{M},\omega$) be a g-metric space, then the g-metric is called symmetric if $\omega(p,q,q) = \omega(p,p,q)$ for all $p,q, \in\mathcal{M}$.

Example (2.4):[8] Let $\mathcal{M} = \{p,q\}$and$\omega(p,p,p) = \omega(q,q,q) = 0, \omega(p,p,q) = 1$, $\omega(p,q,q) = 2$ and extend ω to all of $\mathcal{M}\ x\ \mathcal{M}\ x\ \mathcal{M}$ by symmetry in the variables. Then it is easy to verify that ω is a g −metric, but $\omega(p,q,q) \neq \omega(p,p,q)$.

Proposition (2.5): [2]

Let $(\mathcal{M},\omega$) be a g −metric space, then the following are equivalent:

1- ($\mathcal{M}$,ω) is symmetric.

2-$\omega(p,q,q) \leq \omega(p,q,a), \forall p,q,a \in \mathcal{M}$.

3- $\omega(p,q,e) \leq \omega(p,q,a) + \omega(e,p,b), \forall p,q,e,a,b \in \mathcal{M}$.

Definition(2.6): [2]

Let $(\mathcal{M},\omega)$ be a g –metric space and $\{r_j\}$ be a sequence of points of $\mathcal{M}$, if there exist $L \in \mathbb{N} \in>$ 0 for $j,i,l \geq L$ then the sequence $\{r_j\}$ is said to be

1- ω - convergent to r if $\omega(r,r_j,r_i) <\in$ for all $i\ ,j\ \geq L$.

That is, $\lim_{i,j\to\infty} \omega\ (r,r_j,r_i)$ =0 as $i\ ,j \rightarrow \infty$.

2- ω – Cauchy if $\omega(r_j,r_i,r_l) <\in$ for all $i\ ,j\ , l \geq L$.

That is, $\omega(r_j,r_i,r_l) \rightarrow 0$ as $i\ ,j\ , l \rightarrow \infty$.

Proposition (2.7): [2]. Let $(\mathcal{M},\omega)$ be a g-metric space, then the following statements are equivalent:

1- $\{r_j\}$ is ω-convergent to r, if and only if $\omega(r_j,r_j,r) \rightarrow 0, asj \rightarrow \infty$.

2- $\omega(r_j,r,r) \rightarrow 0, as\ \ j \rightarrow \infty$. if and only if $\omega(r_j,r_i,r) \rightarrow 0, as\ j,i \rightarrow \infty$.

Remark (2.8): [8]

Every g –metric $(\mathcal{M},\omega)$ on $\mathcal{M}$ defines a metric d_ω on $\mathcal{M}$ given by

$d_\omega\ (p,q\ ) = \ \omega(p,q,q) + \omega(q,p,p)$ for all $p,q \in \mathcal{M}$ and

$\omega(p,q,e) = max\{\ |\ p-q\ \ |, |\ q-e\ \ |, |\ e-p\ \ |\}$.

Definition (2.9): [2]

A g–metric space ($\mathcal{M},\omega$) is complete if every ω-Cauchy sequence is ω-convergent in $(\mathcal{M},\omega)$.

Proposition(2.10): [8]. Let ($\mathcal{M}$,ω) be a g-metric space, then, for any p,q,e, and a$\in \mathcal{M}$, it follows that

1. If $\omega(\ p\ ,q,e)\ = 0$ then $p = q = e$.
2. $\omega(\ p\ ,q,e) \leq\ \omega(\ p\ ,p,q\ ) +\ \omega(\ q\ ,q,e\ )$.
3. $\omega(\ p\ ,q,q) \leq 2\ \omega(\ q\ ,p,p\ )$.
4. $\omega(\ p\ ,q,e\ ) \leq\ \omega(\ p\ ,a,e\ ) + \omega(\ a\ ,q,e\ )$.
5. $\omega(\ p\ ,q,e\ ) \leq\ \ 2/3\ (\ \omega\ p\ ,q,a\ ) +\ \omega(\ p\ ,a,e\ ) +\ \omega(\ a\ ,q,e\ ))$.
6. $\omega(\ p\ ,q,e\ ) \leq\ \ (\omega(\ p\ ,a,a\ ) +\ \omega(\ q\ ,a,a\ ) +\ \omega(\ e\ ,a,a\ ))$.

Definition (2.11): [5] The point $\mathcal{M}$ in $\mathcal{M}$ is called a fixed point of the multivalued mapping : $\mathcal{M} \rightarrow 2^{\mathcal{M}}$ if $\mathcal{M} \in S\mathcal{M}$ and $\mathcal{M}$ is the fixed point of a single mapping $S\colon \mathcal{M} \rightarrow \mathcal{M}$ if $\mathcal{M} = S\mathcal{M}$.

$2^{\mathcal{M}} = \{A: \emptyset \neq A \subset \mathcal{M}\}$ and $CB(\mathcal{M}) = \{A: \emptyset \neq A \subset \mathcal{M}, A$ is closed and bounded$\}$and $K(\mathcal{M}) = \{: \emptyset \neq A \subset \mathcal{M}, A$ is compact$\}$.

Definition(2.12):[21]. Let $\mathcal{M}$ be a g-metric space. The mapping $H : \mathcal{M} \rightarrow R^+$ is called the Hausdorff g −distance on CB($\mathcal{M}$), if

$H_g(A,B,C) = max\{sup_{p\in A}\ g(p,B,C), sup_{p\in B}\ g(p,C,A), sup_{p\in C}\ g(p,A,B)\}$,

where $g(p,B,C) = d_g(p,B) + d_g(B,C) + d_g(p,C)$, $d_g(p,B) = inf\{d_g(p,q), q \in B\}$, $d_g(A,B) = \inf\{d_g(a,b), a \in A, b \in$ B, and $A, B, C \in CB(\mathcal{M})\}$.

Lemma(2.13): [7]. If $A, B \in CB(\mathcal{M})$ with $\Omega(A,B,B) < \varepsilon$ then for each $a \in A$ there exists an element $b \in B$ such that $\omega(a,b,b) < \varepsilon$.

Lemma(2.14) :[7] If $A, B \in CB(\mathcal{M})$ and $a \in A$, then for each $\varepsilon > 0$, there exists $b \in B$ such that $\omega(a,b,b) \leq \Omega(A,B,B) + \varepsilon$.

Lemma(2.15):[7] If $A \in CB(\mathcal{M})$ and $b \in K(\mathcal{M})$ then for any $a \in A$, there is $b \in B$ such that:$\omega(a,b,b) \leq \Omega(A,B,B)$.

Lemma(2.16): [6]. Let $\{A_j\}$ be a sequence in $CB(\mathcal{M})$ and $\lim_{j\rightarrow\infty} \Omega(A_{j,} A, A) = 0$ for $A \in CB(\mathcal{M})$.If$p_j \in A_j$ and $\lim_{j\rightarrow\infty} \omega(p_j, p, p) = 0$, then $p \in A$.

Definition(2.17):[6]. Let $\mathcal{M}$ be a non-empty set**,** $S: \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$ be a mapping. An ordered pair $(p,q) \in \mathcal{M} \times \mathcal{M}$ is called coupled fixed point if $S(p,q) = p$ and $S(q,p) = q$.

Example[22] (2.18)

Let $\mathcal{M} = R$. Define $\omega: \mathcal{M} \times \mathcal{M} \times \mathcal{M} \rightarrow R^+$by

$\omega\ (p,q,e) = [|\ p - q\ | + |\ p - e\ | + |\ q - e\ |]$. Define a mapping $S: \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$ by

$$S(p,q) = \begin{cases} \frac{p^2 + q^2}{4}, & p \geq q \\ 0, & p < q \end{cases}$$

And : $\mathcal{M} \rightarrow \mathcal{M}, T(p) = p^2$. Then (0,0) is the only coupled fixed point of S and T.

3. Main Results

For the next part, $(\mathcal{M}, \omega, \leq)$ denotes the partially ordered complete $g -$ metric space.

Theorem(3.1): Let $S: \mathcal{M} \rightarrow CB(\mathcal{M})$ be satisfying the following

1. There exists $k \in (0,1)$ with $\Omega(Sp, Sq, Se) \leq K\omega(p,q,e)$, for all $p \leq q, q \leq e$.
2. If $\omega(p,q,e) < \in < 1$ for some $e \in Sp$ then $p \leq e$.
3. There exists $p_0 \in \mathcal{M}$, and some $p_1 \in Sp_{0,} p_2 \in SP_1$ with $p_0 \leq p_1, p_1 \leq p_2$ such that $\omega(p_0, p_1, p_2) < 1$.
4. If a non-decreasing sequence $p_j \rightarrow p$ in $\mathcal{M}$, then $p_j \leq p$, for all j.

Then S has a fixed point.

Proof: Let $p_0 \in \mathcal{M}$ by assumption 3, there exists $p_1 \in Sp_0$ with $p_0 \leq p_1, p_1 \leq p_2$ such that

$$\omega\ (p_0, p_1, p_2) < 1. \quad (1)$$

By using assumptions (1) and (2), we have $\Omega(Sp_0, S\,p_1, Sp_2) \leq K\omega(p_0, p_1, p_2) < K$. Using assumption (2) and Lemma (2.14), we have the existence of $p_3 \in Sp_2$ with $p_2 \leq p_3$ such that

$$\omega(p_1, p_2, p_3) < K \quad (2)$$

Again, by assumptions (1) and (2), we have Ω ز $(Sp_1, Sp_2, S\,p_3) \leq K\omega(p_1, p_2, p_3) < k^2$.

By continuing in this way, we obtain $p_j \in Sp_{j-1}$ with $p_{j-1} \leq p_j$ and $p_{j+1} \in Sp_j$ with $p_j \leq p_{j+1}$ such that

$\omega\ (p_{j-1}, p_j, p_{j+1}) < K^{j-1}$ and $\omega\ (p_j, p_j, p_{j+1}) < K^j$.

From the above inequality and by the assumption (2), we have the existence of $p_{j+1} \in Spj$ with $p_j \leq p_{j+1}$ and $p_{j+2} \in Sp_{j+1}$with $p_{j+1} \leq p_{j+2}$ such that

$$\omega(p_j, p_{j+1}, p_{j+2}) < K^j \quad (3)$$

Next, we will show that $\{p_j\}$ is a $g -$ Cauchy sequence in $\mathcal{M}$. Let $i > j$. Then

$$\omega(p_j, p_i, p_i) \leq \omega(p_j, p_{j+1}, p_{j+1}) + \omega(p_{j+1}, p_{j+2}, p_{i+2}) + \cdots \omega(p_{i-1}, p_i, p_i)$$
$$< [K^j + K^{j+1} + \cdots + K^{i-1}]$$

$= K^j[1 + K^j + \cdots + K^{i-j-1}] = K^j[1 - K^{i-j}/1 - k]$

$< K^j/1 - k$. Because $k \in (0,1), 1 - K^{i-j} < 1$.

Therefore, $\omega(p_j, p_i, p_i) \rightarrow 0$ as $j \rightarrow \infty$ implies that $\{p_j\}$ is a $g -$ Cauchy sequence and hence

converges to some point (say) p in the complete $g-$ metric space $\mathcal{M}$.
Next, we have to show that p is the fixed point of the mapping S. Since $\{p_j\}$ is a non-decreasing sequence in $\mathcal{M}$ such that $p_j \to p$, therefore we have $p_j \leq p$ for all j. From assumption1, it follows that $\Omega(Sp_j, Sp, Sp) \leq k\omega(p_j, p, p) \to 0$, because $p_{j+1} \in Sp_j$, it follows by using Lemma(2.16) that $p \in Sp$, i.e.,p is fixed under the set-valued mapping S.

Remark (3.2). If we replace assumption 2 in Theorem (3.1) by the condition: if $p, q \in \mathcal{M}$ with $p \leq q$ and if for all $u \in Sp$ there exists $v \in Sq$ such that $\omega(u, v, v) < 1$ then $u \leq v$, and assuming all other hypotheses, we hypothesize that S has a fixed point. The proof is clear.

Corollary(3.3): Let $S: \mathcal{M} \to \mathcal{M}$ be satisfying the following:
1. There exists $k \in (0,1)$ with $\Omega(Sp, Sq, Se) \leq K\omega(p, q, e)$, for all $p \leq q, q \leq e$
2 S is order-preserving, i.e., if $p, q\ e \in \mathcal{M}$, with $p \leq q$ then $Sp \leq Sq$.
3.There exists $p_0 \in \mathcal{M}$ with $p_0 \leq Sp_0 = p_1$ (Say)
4. If a non-decreasing sequence $p_j \to p$ in $\mathcal{M}$, then $p_j \leq p$, for all j.
Then S has a fixed point. The proof is clear.

Theorem(3.4) Let $S: \mathcal{M} \to CB(\mathcal{M})$ be satisfying the following:
1. There exists $k \in (0,1)$ with $\Omega(Sp, Sq, Se) \leq K\omega(p, q, e)$, for all $p \leq q, q \leq e$
2. If $\omega(p, q, e) < \epsilon < 1$ for some $e \in Sp$ then $p \leq e$
3. There exists $p_0 \in \mathcal{M}$, and some $p_1 \in S\ p_{0,} p_2 \in SP_1$ with $p_1 \leq p_0, p_2 \leq p_1$ such that $\omega(p_0, p_1, p_2) < 1$
4. A non-increasing sequence $p_j \to p$ in $\mathcal{M}$, then $p \leq p_j$, for all j.
Then S has a fixed point

Proof: It follows on the similar lines as Theorem (3.1).

Theorem(3.5): Let $S: \mathcal{M} \to CB(\mathcal{M})$ be satisfying the following:
1. There exists $k \in (0,1)$ with $\Omega(Sp, Sq, Se) \leq K\omega(p, q, e)$for all $p \leq q, q \leq e$
2. If $\omega(p, q, e) < \varepsilon < 1$ for some $e \in Sp$ then $p \leq e$or $e \leq p$.
3. There exists $p_0 \in \mathcal{M}$, and some $p_1 \in Sp_{0,} p_2 \in SP_1$ with $p_1 \leq p_0, p_2 \leq p_1$ such that $\omega(p_0, p_1, p_2) < 1$.
4. If $p_j \to p$ is any sequence in $\mathcal{M}$ for which the terms are comparable, then $p_j \leq p$ or $p \leq pj$ for all j.
Then S has a fixed point.

Proof: It follows on a similar line by using Theorem (3.1) and Theorem (3.4).

Theorem(3.6): Let $S: \mathcal{M}\ x\ \mathcal{M} x \mathcal{M} \to CB(\mathcal{M})$ be satisfying the following:
1. There exists $K \in (0, 1)$ with $\Omega(S(p, q, e), S(u, v, w)) < K\omega(p, q, e), (u, v, w))$, for all $(u, v, w) \leq (p, q, e$).
2. If $p_1 \leq p_2$, $q_2 \leq q_1, e_2 \leq e_1 p_i, q_i, e_i \in \mathcal{M}$ $(i = 1{,}2)$, then for all $u_1 \in S(p_1, q_1, e_1)$, there exists $u_2 \in (p_2, q_2, e_2)$, there exists $u_3 \in S(p_3, q_3, e_3)$, with $u_1 \leq u_2$, $u_2 \leq u_3$, and for all $v_1 \in S(q_1, p_1, e_1)$ there exists $v_2 \in S(q_2, p_2, e_2)$ and $v_3 \in S(q_3, p_3, e_3)$ with $v_2 \leq v_1, v_3 \leq v_2$, and for all $w_1 \in S(e_1, p_1, q_1)$ there exists $w_2 \in S(e_2, p_{2,} q_2)$ and $w_3 \in S(e_3, p_3, q_3)$, such that $w_2 \leq w_1, w_3 \leq w_2$, provided that $\omega((u_1, v_1, w_1), (u_2, v_2, w_2)) < 1$.
3. There exists $p_0, q_0, e_0 \in \mathcal{M}$ and some $p_1 \in S(p_0, q_0, e_0), q_1 \in S(q_0, p_0, e_0), e_1 \in S(e_0, p_0, q_0)$ with $p_0 \leq p_1, q_1 \leq q_0, e_1 \leq e_0$ such that $\omega((p_0\ q_0, e_0), (p_1, q_1, e_1)) < 1 - K$, where $K \in (0, 1)$.
4. If a non-decreasing sequence $p_j \to p$ in $\mathcal{M}$ then $p_j \leq p$, for all j, and if a non-increasing sequence $q_j \to q$ in $\mathcal{M}$ then $q \leq q_j$, for all j, and if a non-increasing sequence $e_j \to e$ in $\mathcal{M}$ then $e \leq e_j$, for all j. Then S has a coupled fixed point.

Proof: Let $p_0, q_0, e_0 \in \mathcal{M}$ then by assumption 3 there exists $p_1 \in S(p_0, q_0, e_0)$,$q_1 \in S(q_0, p_0, e_0) e_1 \in S(e_0, p_0, q_0)$ with $p_0 \leq p_1$, $q_1 \leq q_0, e_1 \leq e_0$, such that

$$\omega((p_0, q_0, e_0), (p_1, q_1, e_1)) < 1 - K \tag{4}$$

Since $(p_0, q_0, e_0) \leq (p_1, q_1, e_1)$, then by using assumptions (1) and (4), we have

$$\Omega(S(p_0, q_0, e_0), S(p_1, q_1, e_1)) \leq K/2\ \omega((p_0, q_0, e_0), (p_1, q_1, e_1) < K/2(1 - K)$$

Similarly,

$$\Omega(S(q_0, p_0, e_0), S(q_1, p_1, e_1)) \leq K/2(1 - K), \Omega(S(e_0, p_0, q_0), S(e_1, p_1, q_1)) \leq K/2(1 - K)$$

Using assumption (2) and Lemma (2.14), we have the existence of $p_2 \in S(p_1, q_1, e_1), q_2 \in S(q_1, p_1, e_1), e_2 \in S(e_1, p_1, q_1)$ with $p_1 \leq p_2, q_2 \leq q_1, e_2 \leq e_1$, such that

$$\omega\ (p_1, p_2, p_3) \leq K/2(1-K) \quad \dots (5)$$

and

$$\omega(\ q_1, q_2, q_3)\ \leq K/2(1-K)$$

and

$$\omega\ (e_1, e_2, e_3)\ \leq K/2(1-K)$$

From (4) and (5)

$$\omega((p_1, q_1, e_1), (p_2, q_2, e_2))\ \leq K\ (1-K) \quad \dots. (6)$$

Again, by assumptions (1) and (6), we have $\Omega(S(p_1, q_1, e_1), S(p_2, q_2, e_2)\ \leq K^2/2(1-K)$ and $\Omega(S(q_1, p_1, e_1), S(q_2, p_2, e_2))\ \leq K^2/2(1-K\ )$ and $\Omega(S(e_1, p_1, q_1), S(e_2, p_2, q_2)\ \leq K^2/2(1-K)$.

From Lemma (2.14) and assumption (2), we have the existence of $p_3 \in\ S(p_2, q_2, e_2\ ), q_3 \in S(q_2, p_2, e_2), e_3 \in S(e_2, p_2, q_2)$ with $p_2 \leq p_3, q_3 \leq q_2, e_3 \leq e_2$ such that $\omega\ (p_2, p_3, p_4)\ \leq K^2/2(1-K)\omega\ (q_2, q_3, q_4)\ \leq K^2/2(1-K), \omega\ (e_2, e_3, e_4)\ \leq K^2/2(1-K)$

It follows that $\omega((p_2, q_2, e_2\ ), (p_3, q_3, e_3)) \leq K^2/2(1-K)$. By continuing in this way, we obtain: $p_{j+1}\ \in\ S(p_j, q_j, e_j), q_{j+1}\ \in S(q_j, p_j, e_j)\ , e_{j+1} \in\ S(e_j, p_j, q_j\ )$with $p_j\ \leq p_{j+1}\ , q_{j+1} \leq q_j, e_{j+1} \leq e_j$ such that $\omega\left(p_j, p_{j+1}, p_{j+2}\right) \leq \frac{K^j}{2(1-K)}$ and $\omega\left(q_j, q_{j+1}, q_{j+2}\right) \leq \frac{K^j}{2(1-K)}, \omega(e_j, e_{j+1}, e_{j+2})K^j/2(1-K)$.

$$\text{Thus} \quad \Omega(S(p_j, q_j, e_j\ ), S(p_{j+1}, q_{j+1}, e_{j+1}) \leq K^j\ (1-K). \quad \dots.. \ (7)$$

Next, we will show that $\{pj\}$ is a g −Cauchy sequence in $\mathcal{M}$. Let $i\ >\ j$. Then

$(p_j, p_i, p_i) \leq \omega(p_j, p_{j+1}, p_{j+1}) + \omega(p_{j+1}, p_{j+2}, p_{i+2}) + \cdots \omega(p_{i-1}, p_i, p_i)$

$\leq \frac{[K^j + K^{j+1} + \cdots + K^{i-1}](\ 1-k)}{2} = \frac{K^j(1-K^{i-j})}{2} < \frac{K^j}{2}$. Because $k \in (0, 1), 1 - K^{i-j} < 1$.

Therefore $\omega(p_j, p_i, p_i) \to 0$, as $j\ \to\ \infty$, implies that $\{\ p_j\ \}$ is a $g-$ Cauchy sequence and hence converges to some point (say) p in the complete $g-$ metric space $\mathcal{M}$. Similarly, we can show that $\{q_j\}$ is also a$g-$ Cauchy sequence in $\mathcal{M}$, and we can show that $\{e_j\}$ is also a$g-$ Cauchy sequence in $\mathcal{M}$. Since $\mathcal{M}$ is a complete g-metric space, there exists $p, q, e \in\ \mathcal{M}$ such that $p_j \to p$ and $q_j \to q, e_j \to e$ as $j \to \infty$. Finally, we have to show that $p \in\ S(p, q, e)$ and $q\ \in S(q, p, e), e \in S(e, p, q)$.

Since $\{\ p_j\ \}$is a non-decreasing sequence, $\{\ q_j\ \}$is a non-increasing sequence, and $\{\ e_j\ \}$is a non-increasing sequence in $\mathcal{M}$, such that $p_j\ \to p$ and $q_j\ \to\ q, e_j \to e$, therefore we have $p_j \leq\ p$ and $q\ \leq q_j\ , e \leq e_j$ for all j. From assumption 1, it follows that $\Omega\left(S\left(p_j, q_j, e_j\right), S(p, q, e)\right) \leq k\ \omega\left(\left(p_j, q_j, e_j\right), (p, q, e)\right) \to 0$ Because $p_{j+1}\ \in S\left(p_j, q_j, e_j\right)$ and $\lim_{j\to\infty} \omega\left(p_{j+1}, p, p\right) =\ 0$, it follows, by using Lemma(2.16), that $p \in S(p, q, e)$. Again, by assumption 1, $\Omega(S(q_j, p_j, e_j), S(q, p, e))\ \leq\ K\ \omega((q_j, p_j, e_j), (q, p, e))\ \to\ 0$.

Since $q_{j+1}\ \in S(q_j, p_j, e_j)$ and $\lim_{j\to\infty} \omega\left(q_{j+1}, q, q\right) = 0$, it follows by using Lemma(2.16) that $q \in S(q, p, e)$. Again, by assumption 1, $\Omega(S(e_j, p_j, q_j), S(e, p, q) \leq K\omega(e_j, p_j, q_j), (e, p, q)\ )\ \to\ 0$.

Hence, (p, q, e) is a coupled fixed point of the set-valued mapping S.

Corollary (3.7): Let $\mathcal{M}$ be a partially ordered set and ω be a $g-$ metric on $\mathcal{M}$ such that $(\mathcal{M}, \omega)$ is a complete $g-$ metric space. Let S: $\mathcal{M}x\ \mathcal{M}x\mathcal{M}\ \to\ \mathcal{M}$ be a single-valued mapping satisfying:

1. There exists $K\ \in\ (0, 1)$ with $\Omega(S(p, q, e), S(u, v, w))\ \leq K/2[\ \ (\ \ \omega \ \ (p, u, u\ ) +\ \omega\ (q, v, v)) +\ \omega\ (e, w, w)]$, for all $(u, v, w) \leq (p, q, e)$.2. S is a mixed monotone mapping.
3. There exists $p_0, q_0, e_0 \in\ \mathcal{M}$ with $p_0 \leq\ S(p_0, q_0, e_0) = p_1,\ q_1\ =\ S(q_0, p_0, e_0)\ \leq q_{0,}$ and $e_1 = S(e_0, p_0, q_0) \leq e_0$.
4. If a non-decreasing sequence$p_j\ \to p$ in $\mathcal{M}$, then $p_j \leq p$, for all j , and if a non-increasing sequence $q_j \to q$ in $\mathcal{M}$ then $q \leq q_j$, for all j , and if a non-increasing sequence $e_j \to e$ in $\mathcal{M}$ then $e \leq e_j$, for all j .Then S has a coupled fixed point.

The proof is clear.

Remark(3.8): If in assumption (4) of theorem (3.6), p, q, e are comparable, then $p = q = e$ and $p \in S(p, p, p)$. Let $p \leq q, q \leq e\ or\ q \leq p, e \leq q$, then

$$\Omega(S(p, q, e), S(q, p, e) \leq K/2[\omega(p, q, q) + \omega(q, p, p)] = K\omega(p, q, e).$$

Because $p \in\ S(p, q, e),\ q \in S(q, p, e)$, and $e \in S(e, p, q)$, by Lemma(2.16),

$\omega\ (p, q, e) \leq \omega\ (p, q, e)$, this implies that $\omega\ (p, q, e) = 0$. Since $K \in (0{,}1\ )$, thus $p = q = e$ and $p \in S(p, p, p)$. The proof is clear.

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