



## Nonclassical Symmetry of Differential Equations

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### Abstract

In this paper, we discuss the difference between classical and nonclassical symmetries. In addition, we found the non-classical symmetry of the Benjamin Bona Mahony Equation (BBM). Finally, we found a new exact solution to a Benjamin Bona Mahony Equation (BBM) using nonclassical symmetry.

**Keywords:** Classical symmetry, Nonclassical Symmetry, Exact Solution, Heat Equation, BBM Equation.

### التماثل غير الكلاسيكي للمعادلات التفاضلية

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### الخلاصه

في هذا البحث، ناقش الفرق بين التماثلات الكلاسيكية وغير الكلاسيكية ، وكذلك وجدنا التماثل غير الكلاسيكي لمعادلة بنيامين بونا ماهوني (BBM) . أخيراً ، نجد حلاً مضبوطاً جديداً لمعادلة بنيامين بونا ماهوني (BBM) باستخدام التناظر غير الكلاسيكي.

### 1. Introduction

It is well known that the field of partial differential equations (PDEs) is an active area of Mathematics, due to the important applications of PDEs in many real problems in physics, chemistry, and engineering. Therefore, many authors have interested in studying PDEs with different types of initial-boundary conditions, see for instance [1-10].

Over the years, there have been several generalizations of Lie's classical method for finding group invariant solutions of partial differential equations (PDEs). In fact, there exist two ways for extending Lie's symmetries:

1. Through the weakening of the invariance criterion by calculating the symmetries of the basic equation supplemented by certain differential constraints (side conditions) in order to provide us with larger Lie-point symmetry groups for the augmented system, e.g. non-classical symmetry and weak symmetry (conditional symmetry) [1-8]. Olver [6] divided Lie group symmetry into :
  - i) The strong symmetry group of the system of differential equations  $E^\sigma = 0$ , which is a group of transformations  $G$  on the space of independent and dependent variables, which has the

following two properties:

a) The elements of  $G$  transform solutions of the system to other solutions of the system.  
 b) The  $G$  invariant solutions of the system are found from a reduced system of differential equations which involve a fewer number of independent variables than the original system  $E^\sigma = 0$ .

ii) The weak symmetry group of the system  $E^\sigma = 0$  is a group of transformations that satisfies the reduction property (b), but no longer transforms solutions to solutions.

2. Extending the space of symmetries to include some derivatives and/or integrals of the dependent variables (e.g. contact, generalized or Lie Backlund symmetry, nonlocal and extend symmetry) [5, 11-13].

One can mix (1) and (2) to have both weak and strong generalized symmetry groups of the system  $E^\sigma = 0$  [6].

In this paper, we discuss the first way- that is- weakening the invariance criterion. We find the nonclassical symmetries of the Benjamin Bona Mahony Equation (BBM) and make a comparison among the classical and nonclassical symmetries.

## 2. Basic Definitions

**Definition 2.1** [13]. A  $k$ -th order ( $k \geq 1$ ) system  $E$  of  $s$  differential equations is defined by

$$E^\sigma(x, u, u_{(1)}, \dots, u_{(k)}) = 0; \quad \sigma = 1, 2, \dots, s \quad (2.1)$$

where  $x = (x^1, x^2, \dots, x^n)$ ,  $u = (u^1, u^2, \dots, u^m)$  and  $u_{(1)}, u_{(2)}, \dots, u_{(k)}$  are respectively the collection of all first, second, up to  $k$ -th order partial derivatives.

**Definition 2.2** [13]. The first order linear differential operator

$$X = \sum_{i=1}^n \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^m \eta^\alpha(x, u) \frac{\partial}{\partial u^\alpha} \quad (2.2)$$

is called symmetry operator, Lie symmetry, Lie operator, infinite operator or admitted group.

**Definition 2.3** [13]. The  $k$ -prolongation of  $X$  which is denoted by  $X^{[k]}$  is defined by:

$$X^{[k]} = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta^\alpha(x, u) \frac{\partial}{\partial u^\alpha} + \dots + \zeta_{i_1 i_2 \dots i_k}^\alpha(x, u, \dots, u_{(k)}) \frac{\partial}{\partial u_{i_1 \dots i_k}^\alpha} \quad (2.3)$$

where the Einstein summation convention is adopted.

**Definition 2.4** [13]. The generator  $X$  is admitted by the system of differential equation (2.1) with maximal rank if and only if

$$X^{[k]} E^\sigma|_{E^\sigma=0} = 0 \quad \sigma = 1, \dots, s \quad (2.4)$$

where  $X^{[k]}$  is the  $k$ -th prolongation of  $X$  and  $|_{E^\sigma=0}$  means that it is evaluated on the frame of the system of differential equations (2.1).

Equation (2.4) is known as the determining equation (Invariance criteria), which decomposes into several equations, thus becoming an over-determining system of differential equation for  $\xi$  and  $\eta$ . After solving this system, one finds all generators of point transformations admitted by the differential equations.

## 3 The Nonclassical Symmetries (Q-conditional Symmetry or standard conditional Symmetry)

Recall the system (2.1), that is,

$$E^\sigma(x, u, u_{(1)}, \dots, u_{(k)}) = 0; \quad \sigma = 1, 2, \dots, s. \quad (3.1)$$

The nonclassical symmetries were suggested by a previous article [14]. In this method, the problem is to find a linear operator  $X$  defined by (2.2), which is not admitted by the differential equation (3.1), but admitted by the differential equation (3.1) and the invariant surface condition, that is

$$Xu^\alpha = Q_\alpha(x, u, u_{(1)}) = \sum_{i=1}^n \xi^i(x, u) \frac{\partial u^\alpha}{\partial x^i} - \eta^\alpha(x, u) = 0 \quad (3.2)$$

$$\alpha = 1, 2, \dots, m$$

**Definition 3.1** [15]. We say that the linear operator  $X$ , defined by (2.2). is nonclassical

symmetry of the equations (3.1), if (2.2) is Lie symmetry of the combined systems (3.1) and (3.2), that is,

$$E^\sigma(\bar{x}, \bar{u}, \bar{u}_{(1)}, \dots, \bar{u}_{(k)}) = 0; \quad \sigma = 1, 2, \dots, s. \quad (3.3)$$

and

$$Q_\alpha(\bar{x}, \bar{u}, \bar{u}_{(1)}, \dots, \bar{u}_{(k)}) = 0; \quad \alpha = 1, 2, \dots, m. \quad (3.4)$$

when (3.1) and (3.2) are satisfied.

**Remark 3.1.**

1. If the systems (3.1) and (3.2) are compatible, we can use the following equation to find nonclassical symmetries for (3.1), that is

$$\begin{aligned} X^{[k]}E^\sigma|_N &= 0; \quad \sigma = 1, 2, \dots, s \\ X^{[1]}Q_\alpha|_N &= 0; \quad \alpha = 1, 2, \dots, m \end{aligned} \quad (3.5)$$

where N is the set solution of (3.1) and (3.2), we note that the invariance criterion (3.5) may be replaced by  $X^{[k]}E^\sigma|_N = 0$  where the second equation of (3.5) is automatically satisfied [16].

2. For the systems (3.1) and (3.2) to be compatible [17], the k-th prolongation  $X^{[k]}$  of the vector field X must be tangent to M, where

$$\begin{aligned} M = \{ (x, u, u_{(1)}, \dots, u_{(n)}): E^\sigma = 0, Q_\alpha = 0, D_i Q_\alpha = 0, \dots, \\ D_i^{(k-1)} Q_\alpha = 0 \quad \sigma = 1, 2, \dots, s \quad \alpha = 1, 2, \dots, m \} \end{aligned} \quad (3.6)$$

that is

$$X^{[k]}E^\sigma|_M = 0 \quad \sigma = 1, 2, \dots, s. \quad (3.7)$$

### 3.1 Comparison Between Classical and Nonclassical Symmetries

#### 1. Invariance:

Any Lie symmetry locally maps the solution set (whole) of the corresponding system of differential equations onto itself. This is, the main characteristic of any type of symmetry (e.g. contact, generalized, nonlocally symmetry) gives rise to the possibility of generating new solutions from known ones (maybe trivial). But the basic prerequisite of the definition of nonclassical symmetry is the consideration of only the set of solutions invariant (subset of whole solutions) under the associated transformation. That is, symmetries of special solutions are not at the same time symmetries of the equations (2.1). Therefore, it is impossible to use nonclassical symmetries in order to generate new solutions from known ones.

2. Determining equations: - The determining equations (2.4) for Lie symmetry X are linear and homogeneous PDEs with respect to the coordinates  $\xi^i$  and  $\eta^\alpha$  of the operator X. Therefore, the set of solutions generates a linear vector space, which implies getting the Lie algebra [13]. While the determining equations (3.7) are nonlinear PDEs, since  $\xi^i, \eta^\alpha$  play a double role: it appears not only as the symmetry vector field, but also in the equation of (3.2). Therefore, the solution (3.7) in general does not constitute a vector space (i.e. a Lie algebra). Therefore, we need a new notion (involutive) to define module (similar to Lie algebra in case of the classical symmetry) [16].

3. In the classical symmetry method, if X is a symmetry, then  $\lambda X$  is also a symmetry, where  $\lambda$  is a constant, since the over-determining equations are linear and homogenous partial differential equations. If X is nonclassical symmetry, then  $\lambda(x, u)X$  is also nonclassical symmetry [16]. This fact is useful, since it leads to two simplifying cases for  $\tau \equiv 1; \tau \equiv 0, \xi = 1$  when  $X = \xi(x, t, u)\partial_x + \tau(x, t, u)\partial_t + \eta(x, t, u)\partial_u$ .

4. Reduction: - Each classical (Lie) symmetry and nonclassical symmetry leads to an anstaz\* reducing the initial system to a system with a smaller number of independent variables, and in principle, the reduced system is more easily solvable than the initial one, that is this property is preserved in both cases.

5. Structure of the equation:- In both cases, classical or nonclassical symmetries, the structure of the equation is preserved, which means that the linear equation is converted to a linear one and the nonlinear to a nonlinear one.

\*Anstaz: It is a hypothesis used to simplify a differential equation into a differential equation

that is simpler to solve.

### 3.2 Nonclassical Symmetry (NCS) Algorithm

For the construction of nonclassical symmetry, the following algorithm is presented [5, 18]:

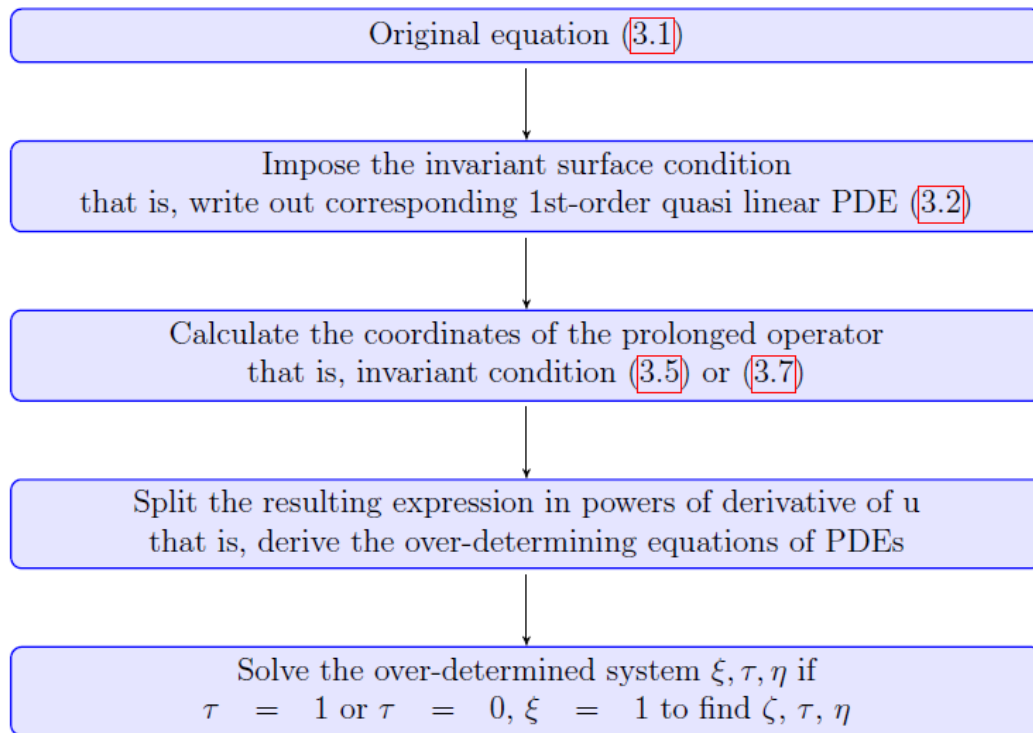


Figure 3.1: Nonclassical Algorithm

The following examples illustrate nonclassical symmetry algorithm.

#### Example 3.1 [1, 14] Heat equation

The first PDE considered through the nonclassical method is the homogeneous heat equation:

$$u_t = u_{xx} \quad (3.8)$$

By property (3) in 3.1, two cases will be considered

**Case 1:**  $\tau = 1$ , therefore  $X$  becomes

$$X = \xi(x, t, u)\partial_x + \partial_t + \eta(x, t, u)\partial_u$$

If  $u$  satisfies the augmented PDE system consisting of (3.8) and the invariant surface condition

$$u_t = \eta(x, t, u) - \xi(x, t, u)u_x \quad (3.9)$$

then the invariance criterion is

$$X^{[2]}[u_{xx} - u_t]|_{(3.8)(3.9)} = 0 \quad (3.10)$$

By solving the equation (3.10), we get the over-determining equations.

$$\eta_t - \eta_{xx} + 2\eta\xi_x = 0 \quad (3.11)$$

$$\xi_{xx} - \xi_t - 2\eta_{xu} - 2\xi\xi_x + 2\eta\xi_u = 0 \quad (3.12)$$

$$2\xi_{xu} - \eta_{uu} - 2\xi\xi_u = 0 \quad (3.13)$$

$$\xi_{uu} = 0 \quad (3.14)$$

and finally we obtain

$$\xi = \xi(x, t)$$

$$\eta = C(x, t)u + D(x, t) \tag{3.15}$$

$$\tau = 1$$

where  $\{\xi(x, t), C(x, t), D(x, t)\}$  is any solution of the nonlinear system

$$\begin{aligned} \xi_x - \xi_{xx} - 2\xi\xi_x + 2C_x &= 0 \\ C_t - C_{xx} + 2\xi_x C &= 0 \\ D_t - D_{xx} + 2\xi_x D &= 0 \end{aligned} \tag{3.16}$$

**Case 2:**  $\tau = 0, \xi = 1$ , therefore  $X = \partial_x + \eta(x, u, t)\partial_y$  and (3.9) become

$$\begin{aligned} u_x &= \eta(x, t, u) \\ u_{xx} &= \eta_x + \eta_u u_x = \eta_x + \eta_u \eta \end{aligned} \tag{3.17}$$

The invariance criteria is

$$X^{[2]}[u_{xx} - u_t]|_{(3.8)(3.17)} = 0$$

which leads to the infinitesimals

$$\begin{aligned} \xi &= 0 \\ \tau &= 0 \end{aligned}$$

and,

$$\eta = \eta(x, t, u)$$

where

$$\eta^2 \eta_{uu} + 2\eta \eta_{xu} + \eta_{xx} - \eta_t = 0 \tag{3.18}$$

**Theorem 3.1.** An arbitrary operator  $X$  nonclassical symmetry of the BBM equation,

$$u_{xxt} = u_x + u_x u + u_t \tag{3.19}$$

is equivalent to either the operator

$$X = \partial_t + \partial_x$$

or the operator

$$X = \partial_x + \eta(t, x, u)\partial_u$$

where  $\eta$  satisfies

$$\eta_{xx}\eta_u - 2\eta\eta_u + \eta_t - \eta_{xtu}\eta - 2\eta^2 - \eta_{xxt} - \eta_x - u(\eta_x - 2\eta\eta_x) = 0$$

Especially when  $\eta = \eta(t, x)$  then  $X = \partial_x + \frac{1}{t+c}\partial_u$

**Proof:**

The first case is taking the following

$$X = \partial_t + \xi(x, t, u)\partial_x + \eta(x, t, u)\partial_u \tag{3.20}$$

The corresponding constraint invariant surface condition is

$$u_t = \eta(x, t, u) - \xi(x, t, u)u_x \tag{3.21}$$

By differentiating (3.21), we get

$$u_{tx} = \eta_x + \eta_u u_x - \xi\eta_{xx} - (\xi_x - \xi_u u_x)u_x \tag{3.22}$$

By invariance criteria, we obtain

$$X^{[3]}[u_{xxt} - u_x - u_x u - u_t]|_{(3.30)(3.32)(3.33)} = 0 \tag{3.23}$$

which implies to

$$\eta^{xxt} - \eta^x - \eta^x u - \eta u_x - \eta^t|_{(3.30)(3.32)(3.33)} = 0 \tag{3.24}$$

By solving the equation (3.24), we get the over-determining equations

$$\begin{aligned} \eta_{xxt} - \eta_{xx}\eta + (2\eta_{xu} - \xi_{xx})\eta_x + (\eta_u - 2\xi_x)\eta - \eta_t - \eta_u\eta - \eta_x &= 0 \\ -\eta_x &= 0 \end{aligned}$$

$$\begin{aligned} \eta_{xxu}\xi + (2\eta_{xu} - \xi_{xx})(\eta_u - \xi_x) + 2\eta_{xtu} + 2\eta_{xuu}\eta - \xi_{xxt} - \xi_{xxu}\eta + 2(\eta_{uu} - 2\xi_{xu})\eta_x \\ - (\eta_u - 2\xi_x)\xi - 3\xi_u\eta - \eta + \eta_u\xi + \xi_t + \xi_u\eta - \eta_u + \xi_x + \eta_u - 2\xi_x &= 0 \end{aligned}$$

$$\begin{aligned} -(2\eta_{xu} - \xi_{xx})\xi_u - 2\eta_{xuu}\xi + \xi_{xxu}\xi + 2(\eta_{uu} - 2\xi_{xu})(\eta_u - \xi_x) + \eta_{uut} + \eta_{uuu}\eta - 2\xi_{xtu} \\ - 2\xi_{xuu}\eta - 3\xi_{uu}\eta_x - 3\xi_{uu}\eta_u + 3\xi_u\xi + \xi_u\xi + \xi_u - 3\xi_u &= 0 \end{aligned}$$

$$-2(\eta_{uu} - 2\xi_{xu}) - \eta_{uuu}\xi + 2\xi_{xuu}\xi + 3\xi_{uu}\xi_x - \xi_{uut} - \xi_{uuu}\eta = 0$$

$$3\xi_{uu}\xi_u + \xi_{uuu}\xi = 0$$

$$-(2\eta_{xu} - \xi_{xx})\xi + \eta_{ut} + \eta_{uu}\eta - 2\xi_{xt} - 2\xi_{xu}\eta - 3\xi_u\eta_x = 0$$

$$-2(\eta_{uu} - 2\xi_{xu})\xi - \eta_{uu}\xi - 3\xi_u(\eta_u - \xi_x) - 3\xi_{ut} - 3\xi_{uu}\eta + 2\xi_{xu}\xi = 0$$

$$(\eta_u - 2\xi_x) - \eta_u + \xi_x = \xi_x = 0$$

$$\begin{aligned}
3\xi_{uu}\xi_{uwx} + 3\xi_u\xi_u + 3\xi_{uu}\xi &= 0 \\
-3\xi_u + \xi_u &= 0 \\
3\xi_u\xi &= 0 \\
\xi_u\xi &= 0 \\
\xi_u\eta - \xi_t &= 0
\end{aligned}$$

This case leads to the generator

$$X = \partial_t + \partial_x$$

The second case is taking

$$X = \partial_x + \eta(x, t, u) \partial_u$$

The corresponding constraint invariant surface condition is:

$$u_x = \eta(x, t, u) \quad (3.25)$$

So, we can write (3.19) in the equivalent form

$$u_t = A(x, t, u) \quad (3.26)$$

where

$$A(x, t, u) = u_{xxt} - u_x - uu_x \quad (3.27)$$

By substituting (3.25) in (3.27), we get

$$A(x, t, u) = \eta_{xt} - \eta - u\eta \quad (3.28)$$

from the invariance criteria

$$X^{(1)}(u_t - A)|_{(3.25)(3.26)} = 0$$

that is,

$$\eta^{(1)}\eta A_u - A_x|_{(3.25)(3.26)} = 0$$

we obtain the equation

$$\eta_t + \eta_u A - \eta A_u - A_x = 0 \quad (3.29)$$

By substituting (3.28) and its derivatives in (3.29), we get

$$\begin{aligned}
\eta_{xx}\eta_u - 2\eta\eta_u + \eta_t - \eta_{xtu}\eta - 2\eta^2 - \eta_{xxt} - \eta_x - u(\eta_x - 2\eta\eta_x) \\
= 0
\end{aligned} \quad (3.30)$$

It is difficult to solve this equation, therefore, we take simple the case that  $\eta = \eta(x, t)$ , then the equation (3.30) becomes

$$\eta_t - 2\eta^2 - \eta_{xxt} - \eta_x - u\eta_x = 0 \quad (3.31)$$

where its solution

$$\eta = \frac{1}{t+c}$$

Therefore X becomes

$$X = \partial_x + \frac{1}{t+c} \partial_u \quad (3.32)$$

### Theorem 3.2

The function  $u(x, t) = \frac{x-t+b}{t+c}$  is a solution of BBM equation, where b and c are arbitrary constants.

#### Proof:

By using the nonclassical (3.32), we can obtain a new solution for BBM equation, that is,

$$u(x, t) = \frac{x}{t+c} + F(t) \quad (3.33)$$

By substituting (3.33) in (3.19), we get the ODE

$$F'(t) + \frac{F(t)}{t+c} = -\frac{1}{t+c}$$

The solutions takes the form

$$F(t) = \frac{-t+b}{t+c}$$

Therefore (3.33) becomes

$$u(x, t) = \frac{x-t+b}{t+c}$$

where  $b$  and  $c$  are arbitrary constants.

#### 4 Conclusion

From this work, we notice that, when we can find the non-classical symmetry of the partial differential equation, we can find a new solution to this equation.

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