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Relationship of Essentially Small Quasi-Dedekind Modules with Scalar and Multiplication Modules

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Abstract

Let $\mathbb{R}$ be a ring with 1 and D is a left module over $\mathbb{R}$. In this paper, we study the relationship between essentially small quasi-Dedekind modules with scalar and multiplication modules. We show that if D is a scalar small quasi-prime $\mathbb{R}$-module, thus D is an essentially small quasi-Dedekind $\mathbb{R}$-module. We also show that if D is a faithful multiplication $\mathbb{R}$-module, then D is an essentially small prime $\mathbb{R}$-module iff $\mathbb{R}$ is an essentially small quasi-Dedekind ring.

Keywords: Essentially small Quasi-Dedekind modules, scalar modules, multiplication modules.

علاقة المقاسات شبه الديدكانية الجوهرية الصغيرة مع المقاسات العددية والضربية

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الخلاصة

لتكن R حلقة ذات عنصر محايد و لتكن D مقاس ايسر معرف على R. في هذا البحث درسنا العلاقة بين المقاسات الشبه ديدكانية الجوهرية الصغيرة مع المقاسات العددية ومقاسات الضرب .بينا ان كل مقاس عددي اولي صغير يكون مقاس شبه ديدكاني جوهري صغير و بينا ايضا في حالة مقاسات الضرب الصحيحة فان المقاس يكون جوهري اولي صغير اذا وفقط اذا كانت الحلقة شبه ديدكانية جوهرية صغيرة .

Introduction

A submodule S of an $\mathbb{R}$-module D is small in D($S \ll D$) if whenever a submodule H of D such that $D = S + H$ then $H = D$[1]. A submodule S of an $\mathbb{R}$- module D is essentially small($S \ll_e D$), if for every non zero small submodule G of D, $G \cap S \neq 0$. Equivalently, for each $0 \neq d \in D$, $\exists\ 0 \neq r \in \mathbb{R}$ such that $0 \neq rd \in S$ [2]. An $\mathbb{R}$-module D is essentially small quasi-Dedekind(ESQD) if $Hom(D/H, D) = 0$ for all $H \ll_e D$ [2]. A ring $\mathbb{R}$ is ESQD if $\mathbb{R}$ is an ESQD $\mathbb{R}$-module [2]. An $\mathbb{R}$-module D is a scalar

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ℝ-module if, $\forall\, g \in End_R(D)$, $\exists\ u \in R$ such that g(d) = ud $\forall\, d \in D$ [3]. We will ask the following: If D is ESQD ℝ-module, then $End_R(D)$ will be ESQD ring.
First, we give the following proposition.
Proposition1. Assume that D is a scalar ℝ-module with $ann_R(D)$ is a semiprime ideal of ℝ, thus $End_R(D)$ is ESQD ring.
Proof: Since D is a scalar ℝ-module, thus, as previously described [4, Lemma6.2, p.80], $End_R(D) \cong R/ann_R(D)$, since $ann_R(D)$ is semiprime ideal of ℝ, then $\overline{R} = R/ann_R(D)$ is a semiprime ring. Thus, $End_R(D)$ is a semiprime ring and hence, as in another article [2, Prop.9], $End_R(D)$ is an ESQD ring.
An ℝ-module D is essentially small prime (ESP) if $ann_R(D) = ann_R(H)$ for all $H \ll_e D$[5].
Corollary2. Let D be a scalar ℝ-module. Then (1)⇒(2)⇒(3),(3)⇏(1) and (3)⇏(2).
1) D is ESQD ℝ-module.
2) D is ESP ℝ-module.
3) $End_R(D)$ is ESQD ring.
Proof: (1) ⇒ (2): As previously described [5, Prop.18].
(2)⇒(3): Since D is an ESP, then, by a previous article [5, Coro 28], $\overline{R} = R/ann_R(D)$ is an ESQD ring. But D is a scalar ℝ-module, thus, by another article [4, Lemma 6.2, p.80], $End_R(D) \cong R/ann_R(D)$ Then $End_R(D)$ is an ESQD ring.
In the following example, we explain that (3)⇏ (1) and (3) ⇏ (2).
Example3. Z_{P^∞} as Z-module is not ESQD. But $End_Z(Z_{P^\infty})$ is an integral domain. It is clear that it is an ESQD ring. Notice that Z_{P^∞} as Z-module is not ESP, since if $H = (\frac{1}{P} + Z) \leq_e Z_{P^\infty}$, thus $ann_Z(H) = PZ \neq ann_Z(D) = (0)$, where P is prime number.
The following corollary shows that under the class of faithful scalar modules, $End_R(D)$ is ESQD ring iff ℝ is ESQD ring.
Corollary4. Assume that D is a faithful scalar ℝ-module. Then $End_R(D)$ is ESQD ring iff ℝ is ESQD ring.
Proof: Since D is a scalar ℝ-module, thus, as previously shown [4, Lemma 6.2, p.80], $End_R(D) \cong R/ann_R(D) \cong R$. Hence we get the result.
An ℝ-module D is a small quasi-prime (SQP) if $ann_R(H)$ is a prime ideal of ℝ for each non zero small submodule H of D. In addition, a proper small submodule H of D is SQP if [H:d] be small prime ideal of ℝ $\forall\, d \in D,\ d \notin H$.
Theorem5. Assume that D is an ℝ-module. Then (1) ⇒ (2) ⇒ (3) ⇒ (4)
1. D is SQP ℝ-module.
2. $ann_{\mathbb{R}}H = ann_{\mathbb{R}}rH$ for each small submodule H of D such that $rH = (0)$, $r \in \mathbb{R}$.
3. $ann_{\mathbb{R}}(d) = ann_{\mathbb{R}}(rd)$ for each $d \in D$ such that $rd \neq 0, r \in \mathbb{R}$.
4. $ann_{\mathbb{R}}(d)$ be small prime ideal of ℝ for each $d \in D$.
Proof: (1) ⇒ (2) Since $rH \subseteq H$ then $ann_{\mathbb{R}}H \subseteq ann_{\mathbb{R}}rH$. Let $a \in ann_{\mathbb{R}}rH$ so $arH = 0$ which implies that $ar \in ann_{\mathbb{R}}H$ is a prime ideal. Thus either $a \in ann_{\mathbb{R}}H$ or $r \in ann_{\mathbb{R}}H$. If $r \in ann_{\mathbb{R}}H$, then $rH = 0$, which is a contradiction. Thus, $a \in ann_{\mathbb{R}}H$.
(2) ⇒ (3) Clear.
(3) ⇒ (4) Let $ab \in ann_{\mathbb{R}}(d)$ and suppose that $b \in ann_{\mathbb{R}}(d)$. Thus $abd = 0$ and $bd = 0$, which implies that $a \in ann_{\mathbb{R}}(bd)$. But by (3), $a \in ann_{\mathbb{R}}(d)$.
Proposition6. Let H be proper submodule of an ℝ-module D. Thus, the following are equivalent:
1. H is SQP submodule of D.
2. $[H:_{\mathbb{R}} U]$ is small prime ideal of ℝ for each submodule U of D where $[H:_{\mathbb{R}} W] = \{h \in H, hW \subset H\}$.
3. $[H:_{\mathbb{R}} (d)] = [H:_{\mathbb{R}} W]$ for each $d \in D, r \in \mathbb{R}, [H:_{\mathbb{R}} (d)]$

Proof: (1) ⇒ (2) Let H be a SQP submodule of D. Thus $[H:_{\mathbb{R}}(d)]$ is small prime ideal of ℝ, for each d ∈ D. Then $[H:_{\mathbb{R}}(d)]$ is a small prime ideal for each d ∈ W and $[H:_{\mathbb{R}} W]$ is a small prime ideal of ℝ.
(2) ⇒ (3) It is clear that $[H:_{\mathbb{R}}(d)] \subseteq [H:_{\mathbb{R}}(wd)]$. Let $x \in [H:_{\mathbb{R}}(wd)]$ for each $w \in [H:_{\mathbb{R}}(d)]$ and d ∈ D. Hence x(wd) ⊆ H. It follows that $xw \in [H:_{\mathbb{R}}(d)]$ which is a small prime ideal by (2). But $w \in [H:_{\mathbb{R}}(d)]$ thus $x \in [H:_{\mathbb{R}}(d)]$. Then, $[H:_{\mathbb{R}}(wd)] \subseteq [H:_{\mathbb{R}}(d)]$. Therefore, $[H:_{\mathbb{R}}(d)] = [H:_{\mathbb{R}}(wd)]$.
(3) ⇒ (1) Let d ∈ D and x, y ∈ ℝ such that $xy \in [H:_{\mathbb{R}}(d)]$. Suppose that $y \in [H:_{\mathbb{R}}(d)]$, thus by (3), $[H:_{\mathbb{R}}(yd)] = [H:_{\mathbb{R}}(d)]$. But $x \in [H:_{\mathbb{R}}(yd)]$, then $x \in [H:_{\mathbb{R}}(d)]$ and hence H is a SQP submodule.
Proposition7. An ℝ-module D is SQP iff (0) is a SQP submodule of D.
Proof: Since D is SQP ℝ-module, thus by Theorem5, $ann_{\mathbb{R}}(d)$ is small prime ideal of ℝ for every d ∈ D. But $ann_{\mathbb{R}}(d) = [0:_{\mathbb{R}}(d)]$ ∀ d ∈ D, then by prop.6, we get that (0) is a SQP submodule of D.
Proposition8. Assume that D is a scalar SQP ℝ-module. Thus D is ESQD ℝ-module, and ℝ is ESQD ring.
Proof: First: Assume that $g \in End_R(D)$, $g \neq 0$. To prove that Ker g $\ll_e$ D. But D is a scalar ℝ-module, thus ∃ $0 \neq v \in R$ such that g(w) = vw, ∀ $w \in D$. Suppose that Ker g $\ll_e$ D, thus for any $0 \neq d \in D$, ∃ $0 \neq s \in R$ such that $0 \neq sd \in Kerg$. Hence $g(sd) = 0$; that is vsd = 0, so $vs \in ann_R(d)$. But D is a SQP ℝ-module, implies $ann_R(d)$ is a prime ideal of ℝ, thus either $v \in ann_R(d)$ or $s \in ann_R(d)$; that is either vd = 0 or sd = 0. But $sd \neq 0$, therefore vd = 0 for any $d \in D$. Thus, g = 0, which is a contradiction. Thus, Ker g $\ll_e$ D and then D is an ESQD ℝ-module.
Second: Since D is a SQP ℝ-module, thus by Prop.7, (0) is a SQP submodule of D and hence (0) is a semiprime ideal of ℝ. Then ℝ is a semiprime ring. Thus, as previously shown [2, Prop. 9], ℝ is ESQD ring.

A submodule H of an ℝ-module D is small invertible if $H^{-1}H = D$, where $H^{-1} = \{r \in \mathbb{R}_T : rH \ll D\}$ and $\mathbb{R}_T$ is the localization of ℝ at T in the usual sence, T = {g ∈ G: gd = 0 for some d ∈ D, then d = 0}, where G is the set of all nonzero divisors of ℝ[2].

An ℝ-module D is small quasi-invertible if Hom(D/H, D) = 0, ∀ 0 ≠ H ≪ D [2].

An ℝ-module D is small quasi-Dedekind (SQD) if every non zero submodule H of D is small quasi-invertible [2].

A ring ℝ is SQD if ℝ is SQD ℝ-module [2].

Theorem 9. Assume that D is a faithful multiplication ℝ-module. Then D is ESP ℝ- module iff ℝ is ESQD ring.
Proof: ⇐) Let H $\ll_e$ D. But M is a faithful multiplication ℝ-module, thus by a previous article [6], ∃ W $\ll_e$ ℝ, such that H = WD. It is clear that $ann_R(H) = ann_R(W)$. Since ℝ is an ESQD ring, then W is a small quasi-invertible ideal of ℝ, thus $ann_R(W) = 0$. It follows that $ann_R(H) = 0 = ann_R(D)$. Then D is ESP ℝ-module.
⇒) Follows a previous work [5, Prop26].
Proposition 10. Assume that D is a multiplication ℝ-module. If $End_R(D)$ is an integral domain then D is a SQD ℝ-module.
Proof: Let $g \in End_R(D)$, $g \neq 0$. Since $End_R(D)$ is an integral domain, g is nonzero divisor. But D is a multiplication ℝ-module, so g is monomorphism, as shown previously [7, Lemma 2.2]. Then D is a SQD ℝ-module.

Proposition 11 Assume that D is an ESP ℝ-module with $ann_R(D) = ann_R(\overline{D})$, thus $\overline{D}$ is an ESP ℝ-module.
Proof: Let H $\ll_e$ $\overline{D}$. Since D $\ll_e$ $\overline{D}$ implies $0 \neq H \cap D \ll_e \overline{D}$. Let $U \leq D, U \neq 0$, so $U \leq \overline{D}$. Thus, $(H \cap D) \cap U \neq 0$, so $(H \cap D) \ll_e$ D. But D is an ESP ℝ-module, thus $ann_R(H \cap D) = ann_R(D)$.But $ann_R(H) + ann_R(D) \subseteq ann_R(H \cap D)$,hence

$ann_R(H)+ann_R(D)\subseteq ann_R(D)$, then $ann_R(H)+ann_R(\overline{D})\subseteq ann_R(\overline{D})$, so $ann_R(H)\subseteq ann_R(\overline{D})$. But $ann_R(\overline{D})\subseteq ann_R(H)$ which implies that $ann_R(H)=ann_R(\overline{D})$. Then $\overline{D}$ is an ESP ℝ-module.

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