ISSN: 0067-2904

Influence of heat transfer on Magneto hydrodynamics oscillatory flow for Williamson fluid through a porous medium

Wissam Sadiq Khudair*, Dheia G. Salih Al-Khafajy
Department of Mathematics, Faculty of Computer Science and Information Technology, University of Al-Qadisiyah, Diwaneyah, Iraq.

Abstract
In this paper, we have examined the influence of heat- transfer on the magnetohydrodynamics oscillatory flow of Williamson fluid during porous medium for two types of geometries "Poiseuille flow and Couette flow". We use perturbation technique in terms of the Weissenberg number to obtain explicit forms for velocity profiles. The results that obtained are illustrated by graphs.

Keywords: Williamson fluid, Magneto-hydrodynamics (MHD), Oscillatory flow**.**

تأثير انتقال الحرارة على الهايدروديناميكا الممغنطة لتدفق متذبذب لمائع ويليامسون خلال قناة مسامية

وسام صادق خضير* ، ضياء غازي صالح الخفاجي
قسم الرياضيات، كلية علوم الحاسوب وتكنولوجيا المعلومات ، جامعة القادسية ، الديوانية ، العراق.

الخلاصة
في هذا البحث، قمنا بدراسة تأثير انتقال الحرارة على الهايدروديناميكا الممغنطة لتدفق متذبذب لمائع ويليامسون خلال قناة مسامية لنوعين هندسيين "Poiseuille flow and Couette flow" . استخدمنا طريقة سلسلة الاضطراب باعتماد عدد وزنبرك للحصول على صيغة معتمده لحقل السرعة. ناقشنا نتائج المشكلة باستخدام الرسوم التوضيحية .

1. Introduction

The flow of electrically oriented fluid has a lot of applications, and this science deal with many branches. In astronomy, it helps to understand what happens in the sun, such as rotating solar spots, what happens inside other stars during their life cycle, and geology. The resulting magnetic and mechanical properties, and this science is also looking at generating electricity directly from hot gases evaporated ionizing generators that rely on this magnetic movement. It is also looking at tracking what happens in nuclear fusion by putting high electromagnetic energy on a mixture of deuterium and tritium in the laboratory to imitate what is happening inside the sun and in nuclear reactors using molten sodium molten metal. To reduce it in an area far from the walls of the container by magnetic fields, so that the temperature and pressure can be increased to values close to the corresponding values within the stars and so on.

Nigam and Singh [1], have studied the effect of heat-transfer on laminar flow among parallel flakes under the impact of transverse magnetic field. Attia and Kotb [2], have studied the heat-transfer with MHD flow of viscous fluid among two parallel flakes. The hydro-magnetic free convection flow during a porous medium among two parallel plates was discussed by Massias et al. [3]. Mustafa [4],

*Email: wissamhsse12@gmail.com

have analyzed the thermal radiation effect on unsteady magneto-hydrodynamics free convection flow past a vertical plate with temperature relied on viscosity. Hamza et al. [5], have studied the un-steady heat-transfer to magneto-hydrodynamics oscillatory flow during porous medium under slip condition. Moreover the Newtonian fluids are less appropriate than non-Newtonian fluids in many feasible applications. Examples of such fluids include ketchup, shampoo, cosmetic products, lubricants, polymers, mud, blood at low shear rate and many others. All the non-Newtonian fluids (in terms of their various characteristics), unlike the viscous fluids, cannot be portrayed by a single constitutive relationship. Hence, many models of Non-Newtonian fluids are suggested in the literature.

The development of Poiseuille flow of the yield - stress fluid was discussed by Al-Khatib and Wilson [6]. Frigaard and Ryan [7], have analyzed the flow of a viscous-plastic fluid in a canal of slowly varying width. Kavita et al. [8], have studied the effect of heat-transfer on magneto-hydrodynamics oscillatory flow of Jeffrey fluid in a canal. The effect of heat-transfer on the MHD oscillatory flow of a Jeffrey fluid with variable viscosity model during porous medium studied by Al-Khafajy [9].

We consider a mathematical model to study the influence of heat- transfer on magneto-hydrodynamics oscillatory inflow of Williamson fluid during porous medium. The numerical solutions "perturbation technique" for the two kinds of flow "Poiseuille flow and Couette flow" are addressed. We discussed the pertinent parameters that appear in the problem during the graphs.

2. Mathematical Formulation

Let us consider the flow of a Williamson fluid in the canal of breadth l qualify the effects of magnetic field and radioactive heat transference as described in Figure-1. We supposed that the fluid has very small electromagnetic force produced and the electrical conductivity is small. We are considering Cartesian coordniate system such that, $(v(y), 0,0)$ is the velocity vector in which v is the x-component of velocity and y is orthogonal to x-axis.

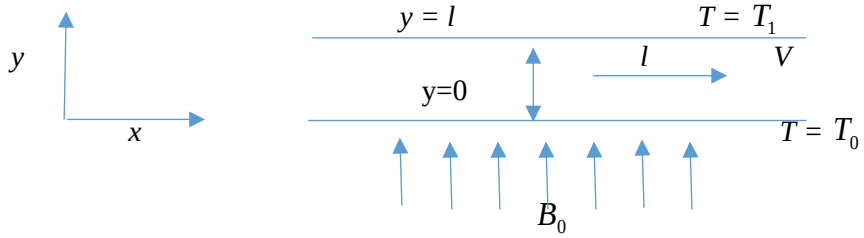

Figure 1-Graph of the problem**.**

The fundamental equation for Williamson fluid is [10] :

$$\boldsymbol{S} = -\bar{p}\boldsymbol{I} + \tau \quad (1)$$

$$\bar{\tau} = [\mu_\infty + (\mu_0 - \mu_\infty)(1 + \Gamma\dot{\bar{\gamma}})^{-1}]\dot{\bar{\gamma}} \quad (2)$$

Where $\bar{p}$ is the pressure, $\boldsymbol{I}$ is the unit tensor, $\bar{\tau}$ is the extra stress tensor, Γ is the time constant, μ_∞ and μ_0 are the infinite shear rate viscosity and zero shear rate viscosity, then $\dot{\gamma}$ is defined as :

$$\dot{\gamma} = \sqrt{\frac{1}{2}\sum_i \sum_j \dot{\gamma}_{ij}\dot{\gamma}_{ji}} = \sqrt{\frac{1}{2}\prod} \quad (3)$$

Here $\prod$ is the second invariant strain tensor. We consider the fundamental Eq. (2), the case for which $\Gamma\dot{\gamma} < 1$, and $\mu_\infty = 0$. We can write the component of extra stress tensor according to follows as :

$$\bar{\tau} = \mu_0[(1 + \Gamma\dot{\bar{\gamma}})]\dot{\bar{\gamma}} \quad (4)$$

The equations of momentum and energy governing such a flow, subjugate to the Boussinesq approximation, are :

$$\rho\frac{\partial\bar{v}}{\partial\bar{t}} = -\frac{\partial\bar{p}}{\partial\bar{x}} + \frac{\partial\bar{\tau}_{\bar{x}\bar{x}}}{\partial\bar{x}} + \frac{\partial\bar{\tau}_{\bar{x}\bar{y}}}{\partial\bar{y}} + \frac{\partial\bar{\tau}_{\bar{x}\bar{z}}}{\partial\bar{z}} + \rho g\beta(T - T_0) - \sigma B_0^2\bar{v} - \frac{\mu_0}{k}\bar{v} \quad (5)$$

$$\rho\frac{\partial T}{\partial\bar{t}} = \frac{K}{C_p}\frac{\partial^2 T}{\partial\bar{y}^2} - \frac{1}{C_p}\frac{\partial q}{\partial y} \quad (6)$$

The temperatures at the walls of the canal are given as:

$$T = T_0 \text{ at } \bar{y} = 0, \text{ and } T = T_1 \text{ at } \bar{y} = l. \quad (7)$$

In which $\bar{v}$ is the axial velocity, T is a fluid temperature, B_0 is a magnetic field strength, ρ is a fluid density, σ is a conductivity of the fluid, β is a coefficient of volume amplification due to

temperature, g is an hastening due to gravity, k is a permeability, c_p is a specific heat at constant pressure, K is a thermal conductivity and q is a radioactive heat flux.

Following Vinvent et al. [11], it is supposed that the fluid is visually thin with a relatively low density and the radioactive heat flux is given by:

$$\frac{\partial q}{\partial y} = 4\alpha^2(T_0 - T) \tag{8}$$

Here α is the mean radiation absorption coefficient.

Non-dimensional parameters are :

$$\left.\begin{array}{l} v = \frac{\bar{v}}{V}, x = \frac{\bar{x}}{l}\ , y = \frac{\bar{y}}{l}\ , \theta = \frac{T-T_0}{T_1-T_0}\ , t = \frac{\bar{t}V}{l}\ , p = \frac{\bar{p}h}{\mu V}\ , M^2 = \frac{\sigma B_0^2 h^2}{\mu}\ , Da = \frac{k}{l^2}, Gr = \frac{\rho g \beta l^2 (T-T_0)}{\mu V}\ , \\ Re = \frac{\rho l V}{\mu}\ , Pe = \frac{\rho l V c_p}{K}\ , N^2 = \frac{4\alpha^2 l^2}{K}\ , \tau_{xx} = \frac{l}{\mu_0 V}\bar{\tau}_{\bar{x}\bar{x}}, \tau_{xy} = \frac{l}{\mu_0 V}\bar{\tau}_{\bar{x}\bar{y}}, \tau_{xz} = \frac{l}{\mu_0 V}\bar{\tau}_{\bar{x}\bar{z}}, \dot{\gamma} = \frac{l}{V}\bar{\dot{\gamma}} \end{array}\right\} \tag{9}$$

Where V is the mean flow velocity, Darcy number (Da), Reynolds number (Re), Peclet number (Pe), magnetic parameter (M), Grashof number (Gr) and radiation parameter (N) .

Substituting (8) and (9) into equations (5) - (7), we obtain

$$\rho \frac{V\partial v}{\frac{l}{V}\partial t} = -\frac{\frac{\mu_0 V}{l}\partial p}{l\partial x} + \frac{\frac{\mu_0 V}{l}\partial \tau_{xx}}{l\partial x} + \frac{\frac{\mu_0 V}{l}\partial \tau_{xy}}{l\partial y} + \frac{\frac{\mu_0 V}{l}\partial \tau_{xz}}{l\partial z} + \rho g\beta(T_1 - T_0)\theta - \sigma B_0^2 V v - \frac{\mu_0 V}{k} v \tag{10}$$

$$\rho\ \frac{\partial(\theta(T_1-T_0)+T_0))}{\frac{l}{V}\partial t} = \frac{k}{C_P}\left[\frac{\partial^2(\theta(T_1-T_0)+T_0))}{l^2\partial y^2} - \frac{1}{k}4\alpha^2(T_0 - T)\right] \tag{11}$$

where $\tau_{xx} = 0$, $\tau_{xy} = \mu_0\left[\left(1 + \Gamma\frac{\partial v}{\partial y}\right)\right]\frac{\partial v}{\partial y}$, $\tau_{xz} = 0$.

The following are the non-dimensional boundary conditions corresponding to the temperature equation :

$$\theta(0) = 0\ ,\ \theta(1) = 1 \tag{12}$$

Finally, we get the following non-dimensional equations:

$$Re\frac{\partial v}{\partial t} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left[\frac{\partial v}{\partial y} + We(\frac{\partial v}{\partial y})^2\right] + Gr\theta_0 - \left(M^2 + \frac{1}{Da}\right)v \tag{13}$$

$$\rho\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial y^2} + N^2\theta \tag{14}$$

To solve the temperature equation (14) with boundary conditions (12), let

$$\theta(y,t) = \theta_0(y,t)e^{i\omega t} \tag{15}$$

where ω is the frequency of the oscillation.

Substituting the equation (15) into the equation (14), we have

$$\frac{\partial^2 \theta}{\partial y^2} + (N^2 - i\omega Pe)\theta_0 = 0 \tag{16}$$

The solution of equation (16) with boundary conditions (12) is $\theta_0(y) = \csc(\varphi)\sin(\varphi)$, where $\varphi = \sqrt{N^2 - i\omega Pe}$. Therefore

$$\theta(y,t) = \csc(\varphi)\sin(\varphi)e^{i\omega t} \tag{17}$$

The calculated of equation (13) have been solved in the next parts for two kinds of boundary conditions "Poiseuille flow and Couette flow".

3. Solution of the Problem

(i) Poiseuille flow

We suppose that the rigid flakes at $y = 0$ and $y = l$ are at rest. Therefore

$\bar{v} = 0$ at $\bar{y} = 0$, and $\bar{v} = 0$ at $\bar{y} = l$.

The non-dimensional boundary conditions are:

$$v(0) = 0\ ,\ v(1) = 0\ . \tag{18}$$

To solve the momentum equation (13), let

$$-\frac{\partial p}{\partial x} = \lambda e^{i\omega t} \tag{19}$$

$$v(y,t) = v_0(y,t)e^{i\omega t} \tag{20}$$

Where λ is a real constant.

Substituting the equations (19) and (20) into the equations (13), we have

$$Re\frac{\partial}{\partial t}(v_0(y,t)e^{i\omega t}) =$$

$$\lambda e^{i\omega t} + \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y} + We(\frac{\partial}{\partial y})^2\right](v_0(y,t)e^{i\omega t}) + Gr\theta_0 - \left(M^2 + \frac{1}{Da}\right)(v_0(y,t)e^{i\omega t}) \tag{21}$$

Equation (21) is non-linear and difficult to get an exact solution. So for waning We, the boundary value problem is agreeing to an easy analytical solution. In this case the equation can be solved. Nevertheless, we suggest a small Γ and used the perturbation technique to solve the problem. Accordingly, we write :

$$v_0 = v_{00} + Wev_{01} + We^2 v_{02} + \mathrm{O}(We^3) \quad (22)$$

Substituting Eq. (22) in Eq. (21) with boundary conditions (18), then we equality the powers of We, we obtain :

A - Zeros-order system (We^0)

$$\frac{\partial v_{00}}{\partial y^2} - \left(M^2 + Rei\omega + \frac{1}{Da}\right) v_{00} = -(\lambda + Gr\theta_0) \quad (23)$$

The associated boundary conditions are:

$$v_{00}(0) = v_{00}(1) = 0 \quad (24)$$

B - First-order system (We^1)

$$\frac{\partial v_{01}}{\partial y^2} - \left(M^2 + Rei\omega + \frac{1}{Da}\right) v_{01} = -2(\frac{\partial v_{00}}{\partial y}\frac{\partial^2 v_{00}}{\partial y^2})e^{i\omega t} \quad (25)$$

The associated boundary conditions are:

$$v_{01}(0) = v_{01}(1) = 0 \quad (26)$$

C - Second-order system (We^2)

$$\frac{\partial v_{02}}{\partial y^2} - \left(M^2 + Rei\omega + \frac{1}{Da}\right) v_{02} = -2(\frac{\partial v_{00}}{\partial y}\frac{\partial^2 v_{01}}{\partial y^2} + \frac{\partial v_{01}}{\partial y}\frac{\partial^2 v_{00}}{\partial y^2})e^{i\omega t} \quad (27)$$

The associated boundary conditions are:

$$v_{02}(0) = v_{02}(1) = 0 \quad (28)$$

D - Zeros-order solution

The solution of equation (23) subset to the associate boundary conditions (24) is:

$$v_{00} = (\frac{B}{A} - \frac{Be^{\sqrt{A}y}}{A}(1 + e^{\sqrt{A}})^{-1} - \frac{Be^{\sqrt{A}}}{A}(1 + e^{\sqrt{A}})^{-1}e^{-\sqrt{A}y}) \quad (29)$$

E - First-order solution

The solution of equation (25) subset to the associate boundary conditions (26) is:

$$v_{01} = -\frac{2B^2e^{itw-2\sqrt{A}y}(-e^{2\sqrt{A}}+e^{4\sqrt{A}y})}{3A^{3/2}(1+e^{\sqrt{A}})^2} + \frac{2B^2e^{itw}}{3A^{3/2}(1+e^{\sqrt{A}})}e^{\sqrt{A}y} - \frac{2B^2e^{\sqrt{A}+itw}}{3A^{3/2}(1+e^{\sqrt{A}})}e^{-\sqrt{A}y} \quad (30)$$

Where $A = \left(M^2 + \mathrm{Re}i\omega + \frac{1}{\mathrm{Da}}\right) and\ B = (\lambda + \mathrm{Gr}\theta_0)$

The formula of v_{02} is a long.

Finally, the perturbation solutions up to second order for v_0 is given by

$$v_0 = v_{00} + Wev_{01} + We^2 v_{02} + \mathrm{O}(We^3)$$

Therefore, the fluid velocity is given as:

$$v(y,t) = v_0(y,t)e^{i\omega t} \quad (31)$$

(ii) Couette flow

The upper flake is locomotion and the lower flake is fixed with the velocity V_h. The boundary conditions for the Couette flow problem as defined:

$$v(0) = 0\ ,\ v(1) = V_0 \quad (32)$$

We have same defined as the governing equations in Poiseuille flow (Eq. 21). The solution in this case has been calculated by the perturbation technique and the results have been discussed during graphs.

4. Results and Discussion

We discuss the Influence of heat-transfer on magnetohydrodynamics oscillatory flow of Williamson fluid during porous medium for Poiseuille flow and Couette flow in some results during the graphical illustrations. Numerical assessments of analytical results and some of the graphically significant results are presented in Figures (2-14).We used the MATHEMATICA program to find the numerical results and illustrations. The momentum equation is resolved by using " perturbation technique " and all the results are discussed graphically.

The velocity profile of Poiseuille flow is shown during Figures-(2-6). Figure-2 illustrates the influence Da and M on the velocity profiles function v vs. y. It is found by the increasing Da the velocity profiles function v increases, while v decreases with increasing M. Figure-3 show that velocity profile v rising up by the increasing influence of both the parameters Gr and λ. Figure-4 we observed that v increases by the increasing influence of both the parameters Re and Pe. Figure-5 show

the velocity profile v increases by the increasing N, while v decreases by the increasing ω. The fluid velocity starts to be constant at the walls and increasing, as fixed by the boundary conditions. Figure-6 show that velocity profiles increases with the increasing of the parameters We when $0.45 < y < 1$, while v decreases by the increasing of We when $0 < y < 0.45$. The velocity profile of Couette flow is shown during Figures-(7–11). It is noted that by the increasing Each of parameters Re, Pe, Gr, Da, N and λ the velocity profile v increases, while v decreases by the increasing We, M and ω . Based on equation (17), Figure-12 show that influence of N on the temperature function θ. The temperature increases by the increase in N. Figure-13 we observed that the influence Pe in temperature θ by the increasing Pe then θ increases. Figure-14 show as that by the increasing of ω the temperature θ decreases .

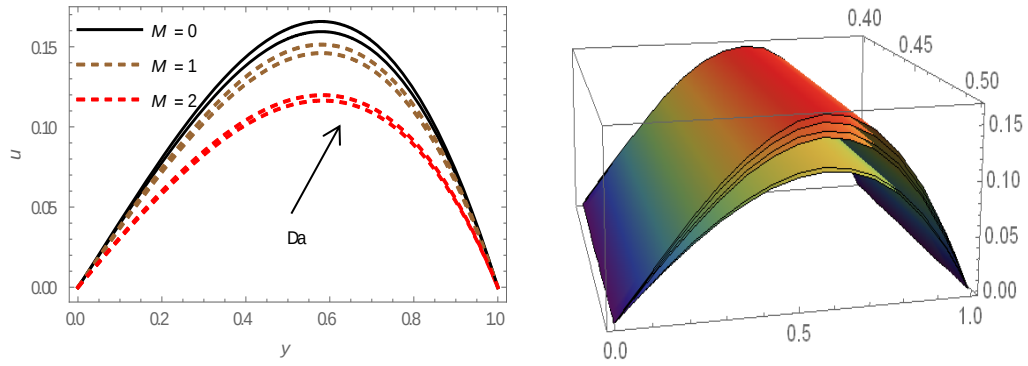

Figure 2-Velocity profile for Da and M with $\omega = 1, N = 1, \mathrm{Gr} = 1, \mathrm{Re} = 1, Pe = 1, \lambda = 1, \mathrm{We} = 0.05, t = 0.5$ in Poiseuille flow.

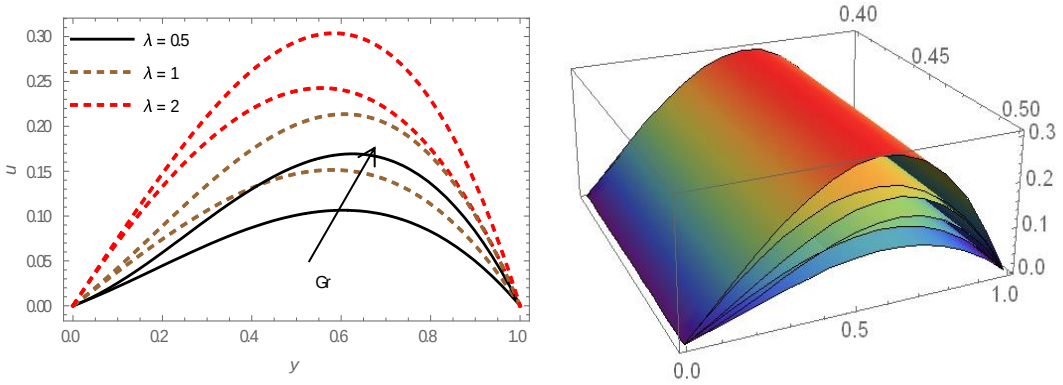

Figure 3-Velocity profile for λ and Gr with $\omega = 1, N = 1, M = 1, \mathrm{Re} = 1, Pe = 1, Da = 0.8, \mathrm{We} = 0.05, t = 0.5$ in Poiseuille flow.

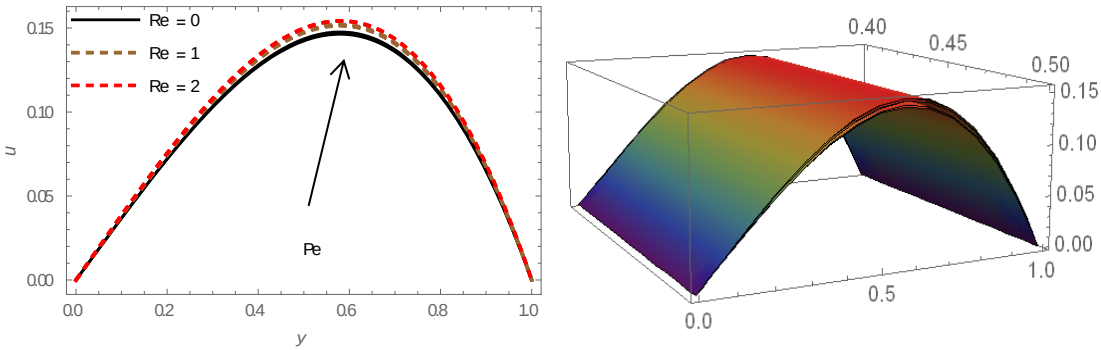

Figure 4-Velocity profile for Re and Pe with $\omega = 1, N = 1, M = 1, \lambda = 1, Gr = 1, Da = 0.8, \mathrm{We} = 0.05, t = 0.5$ in Poiseuille flow.

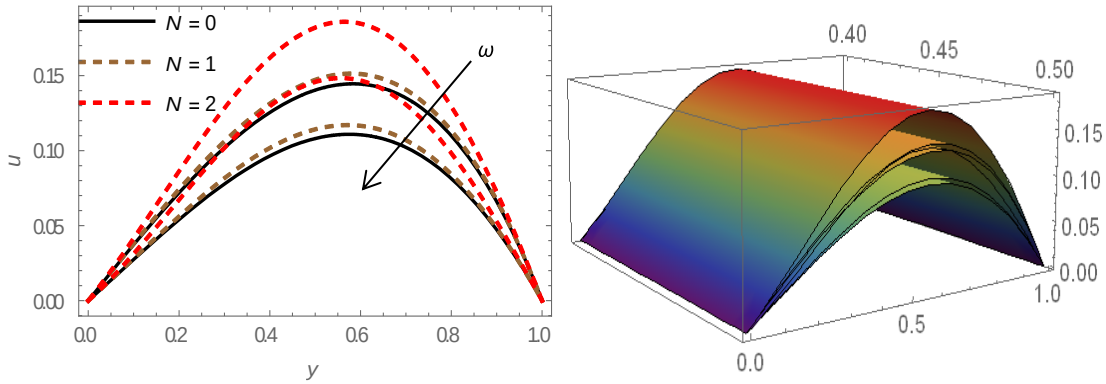

Figure 5-Velocity profile for ω and N with $Re = 1, Pe = 1, M = 1, \lambda = 1, Gr = 1, Da = 0.8, \mathrm{We} = 0.05, t = 0.5$ in Poiseuille flow.

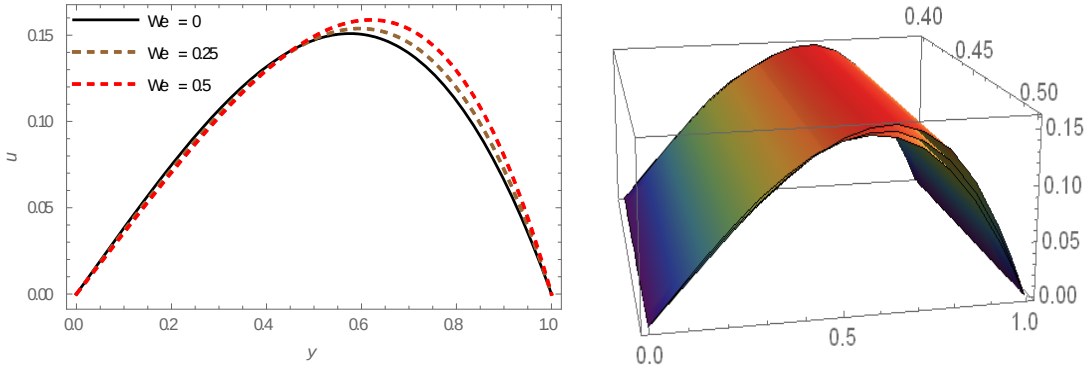

Figure 6 -Velocity profile for We with $\omega = 1, N = 1, Re = 1, Pe = 1, M = 1, \lambda = 1, Gr = 1, Da = 0.8, t = 0.5$ in Poiseuille flow.

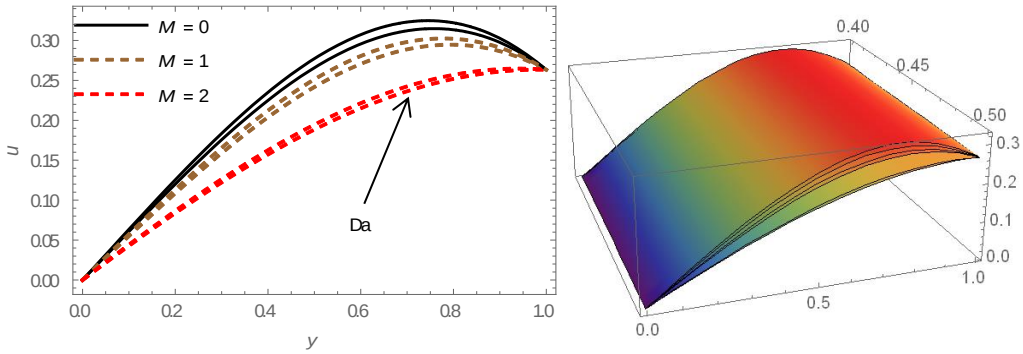

Figure 7-Velocity profile for M and Da with $\omega = 1, N = 1, \mathrm{Gr} = 1, \mathrm{Re} = 1, Pe = 1, \lambda = 1, \mathrm{We} = 0.05, V_0 = 0.3, t = 0.5$ in Couette flow.

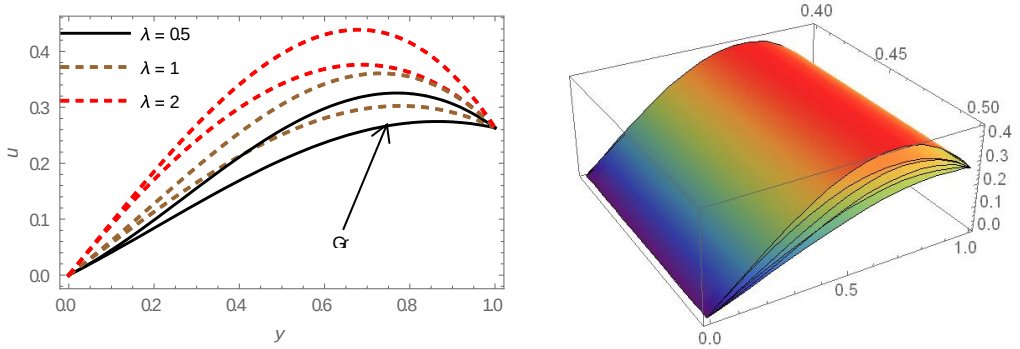

Figure 8-Velocity profile for λ and Gr with $\omega = 1, N = 1, M = 1, \mathrm{Re} = 1, Pe = 1, Da = 0.8, \mathrm{We} = 0.05, V_0 = 0.3, t = 0.5$ in Couette flow.

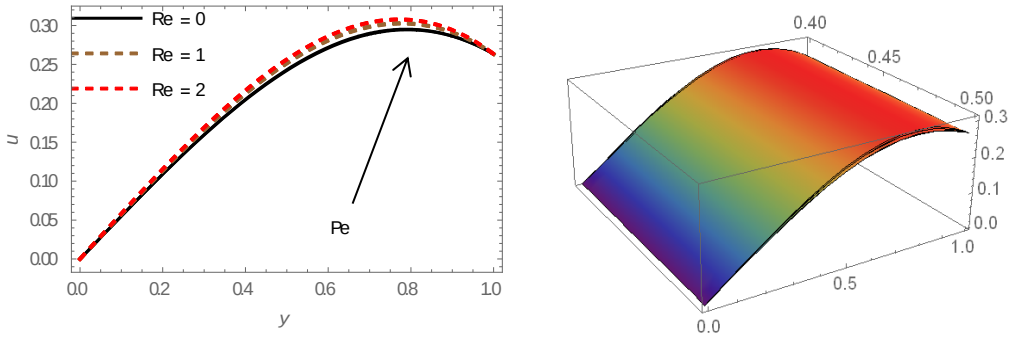

Figure 9-Velocity profile for Re and Pe with $\omega = 1, N = 1, M = 1, \lambda = 1, Gr = 1, Da = 0.8, \mathrm{We} = 0.05, V_0 = 0.3, t = 0.5$ in Couette flow.

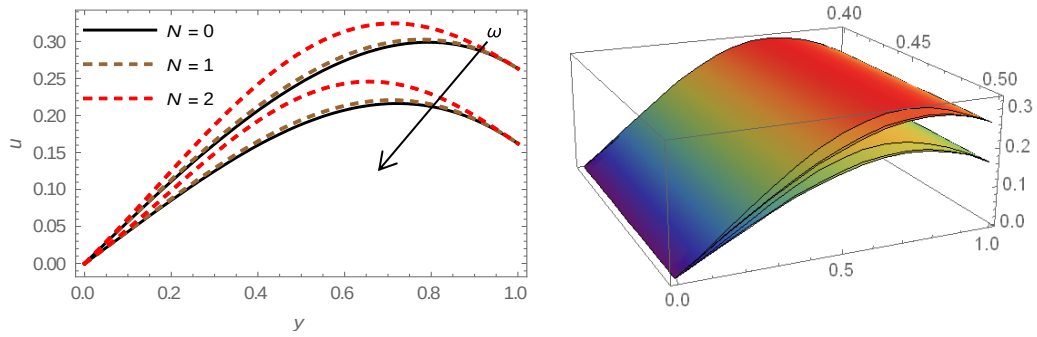

Figure 10-Velocity profile for ω and N with $Re = 1, Pe = 1, M = 1, \lambda = 1, Gr = 1, Da = 0.8, \mathrm{We} = 0.05, V_0 = 0.3, t = 0.5$ in Couette flow.

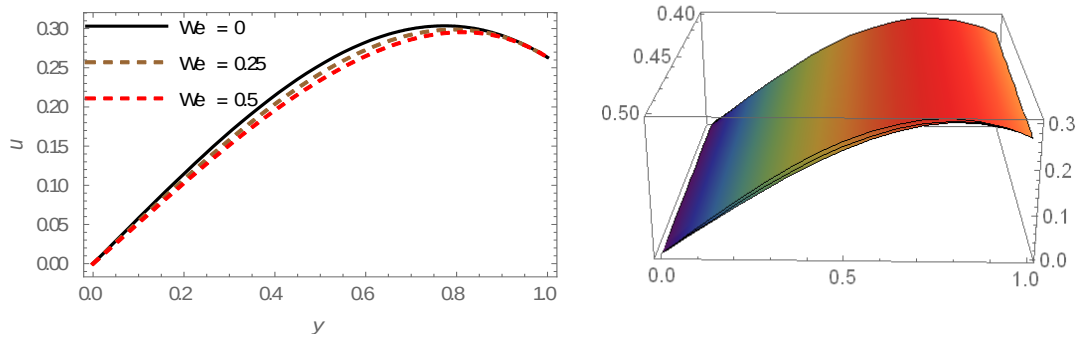

Figure 11-Velocity profile for We with
$\omega = 1, N = 1, Re = 1, Pe = 1, M = 1, \lambda = 1, Gr = 1, Da = 0.8, V_0 = 0.3, t = 0.5$ in Couette flow.

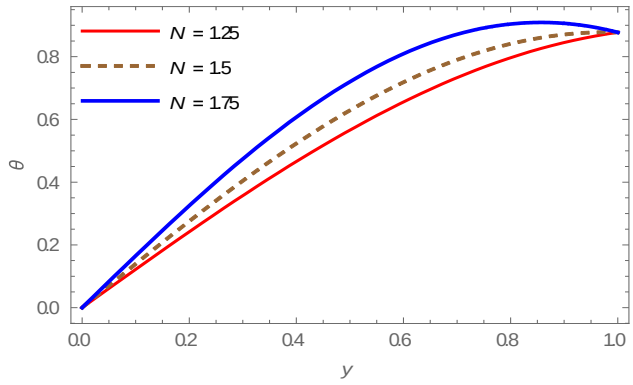

Figure 12-Influence of N on Temperature θ for $\omega = 1, Pe = 0.7, t = 0.5$

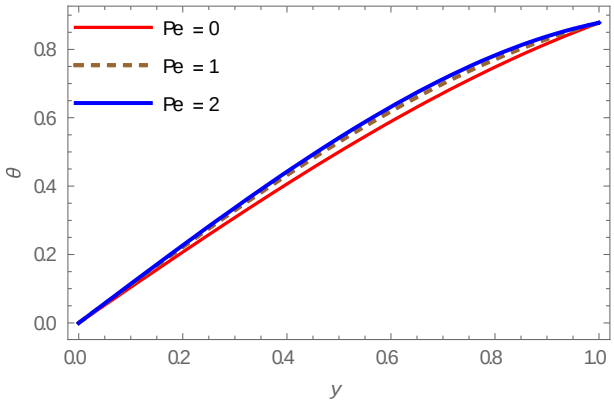

Figure 13-Influence of Pe on Temperature θ for $t = 0.5, N = 1, \omega = 1$.

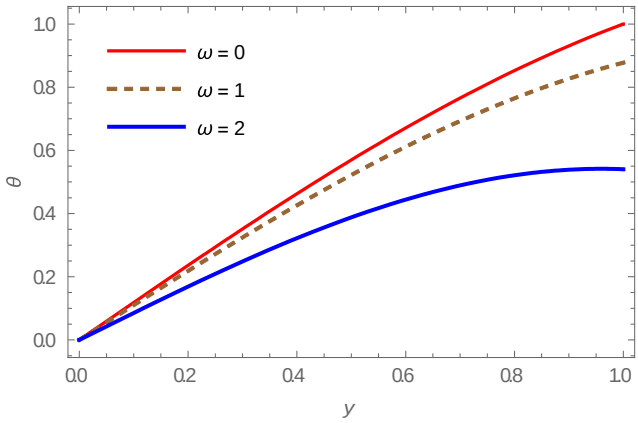

Figure 14-Influence of ω on Temperature θ for $t = 0.5, N = 1, Pe = 0.7$.

5. Conclusion and Remarks

We discussion the influence of heat-transfer on magneto-hydrodynamics oscillatory flow of Williamson fluid during porous medium. The "perturbation technique" for the two kinds of flow "Poiseuille flow and Couette flow" are addressed. We found the velocity and temperature are analytically. We used different values to finding the results of pertinent parameters namely Darcy number (Da), Reynolds number (Re), Peclet number (Pe), magnetic parameter (M), Grashof number (Gr), Weissenberg number (We), frequency of the oscillation (ω) and radiation parameter (N) for the velocity and temperature. The keys point are:

- The velocity profiles increases by the increasing , Re , N, Da ,Gr and λ for both the Poiseuille and Couette flow.
- The velocity profiles decreases by the increasing ω and M for both the Poiseuille and Couette flow.
- The velocity profiles increases by the increasing of the parameters We when $0.45 < y < 1$, while v decreases with increasing of We when $0 < y < 0.45$, for Poiseuille flow. The velocity profiles decreases with the increasing of the parameters We, for Couette flow.
- The parameter that has the most incremental effect on fluid movement is λ.
- We show that by the increases N and Pe the temperature increasing θ and the temperature θ decreases by the increasing ω**.**

References

1. Nigam S.D. and Singh S.N. **1960**. Heat-transfer by laminar flow between parallel plates under the action of transverse magnetic field. *Q. J. Mech. Appl. Math.*, **13**: 85-87.

2. Attia H.A. and Kotb N.A. **1996**. MHD flow between two parallel plates with heat transfer. *Acta Mechanic*, **117**: 215-220.

3. Raptis, A., Massias, C. and Tzivanidis,G. **1982**. Hydromagnetic free convection flow during a porous medium between two parallel plates MHD. *Phys. Lett.*, **90A**: 288-289.

4. Mostafa, A.A. **2009**.Thermal radiation effect on unsteady MHD free convection flow past a vertical plate with temperature dependent viscosity. *The Canadian journal of chemical engineering*, **87**(1): 171-181.
5. Hamza, M.M., Isah, B.Y. and Usman, H. **2011**. Unsteady heat-transfer to MHD oscillatory flow during a porous medium under slip condition. *International Journal of Computer Applications*, **33**(4): 12-17.
6. Al Khatib M.A and Wilson S.D. **2001**. The development of Poiseuille flow of yield – stress fluid. *J. Non-Newtonian Fluid Mech*., **100**: 1-8.
7. Frigaard, I.A and Ryan D.P. **2004**. Flow of a visco-plastic fluid in a canal of slowly varying width. *J. Non-Newtonian Fluid Mech*.,**123**: 67-83.
8. Kavita, K., Prasad K.R. and Kumari, B.A. **2012**. Influence of heat-transfer on MHD oscillatory flow of Jeffrey fluid in a canal. *Advances in Applied Science Research*, **3**(4): 2312-2325.
9. Al-Khafajy Dheia, G. **2016**. Effects of heat-transfer on MHD oscillatory flow of Jeffrey fluid with variable viscosity during porous medium. *Advances in Applied Science Research*,**7**(3): 179-186.
10. Nadeem, S., Youssef, M.M., Zehra, I. **2015**. Numerical solutions of Williamson fluid with pressure dependent Viscosity, Elsevier Results in Physics,pp.20-25.
11. Vinvent, W.G., Cogley, A.C.L. and Giles, E.S. **1968**. Differential approximation for radiative heat-transfer in non-linear equations-grey gas near equilibrium, American Institute of Aeronautics and Astronautics, pp. 551-553.