DOI: 10.24996/ijs.2020.SI.1.19

ISSN: 0067-2904

Solvability of Impulsive Nonlinear Partial Differential Equations with Nonlocal Conditions

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Received: 8/12/ 2019 Accepted: 15/ 3/2020

Abstract
The aim of this paper is to investigate the theoretical approach for solvability of impulsive abstract Cauchy problem for impulsive nonlinear fractional order partial differential equations with nonlocal conditions, where the nonlinear extensible beam equation is a particular application case of this problem.

Keywords: fractional calculus, semigroup theorems, fixed point theorem, nonlocal conditions and mild solution.

قابلية الحل للمعادلات التفاضليه الجزئيه غير الخطيه من النوع الدافع مع شروط غير محليه

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الخلاصه
الهدف في هذا البحث هو التحقق من قابلية الحل بأسلوباً نظرياً لمسألة كوشي المختصرة والتي تمثل تعميماً للمعادلات التفاضلية الجزئية النبضية الغير خطية ذات الرتبة الكسرية مع شروط ابتدائية غير محلية. حيث ان معادلة الشعاع الموسع الغير خطية تعد حالة تطبيقية خاصة من هذه المسألة.

1. Introduction

The fractional differential equations are models of many applications, such as medicine, engineering, physics and other sciences. They are becoming more important tools among researchers and have been attracting many authors in recent years [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. Furthermore, fractional impulsive differential equations played a main role in the modeling phenomena, for example in describing population dynamics which are subject to abrupt changes, as well as other phenomena such as diseases, harvesting, and so forth. From this reason, there are many researchers who discussed the existence of a mild solution of impulsive fractional differential equations [11, 12, 13, 14, 15, 16, 17]. There are some authors who also studied impulsive fractional differential equations with delay [18, 19, 20]. However, there are few authors who discussed the existence of mild solutions of the impulsive fractional integro-differential equations of order $1 < \alpha \leq 2$.

In this work, we will study the existence of mild solutions for considered an abstract Cauchy problem, which is entitled the impulsive nonlinear fractional order partial differential equations with nonlocal conditions, as follows:

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$$\begin{cases} {}^{C}D_t^{\alpha}\left[z(t,x)-h\left(t,z_t(t,x)\right)\right]+{}^{C}D_x^{2\alpha}z(t,x)-\left(\beta_1+\beta_2\int_0^L\left|\frac{\partial z(y,s)}{\partial y}\right|^2dy\right){}^{C}D_x^{\alpha}z(t,x) \\ \qquad =\int_0^t f\left(s,z_s(s,x),I_s^{\gamma}z\right)ds, \\ \text{for } t\in J=[0,T],\quad t\neq t_k,\quad k=1,2,\dots.m.,\quad 1<\alpha,\gamma\leq 2, \\ \Delta z(t,x)|_{t=t_k}=I_k\left(z(t_k^-)\right), \\ \Delta z'(t,x)\Big|_{t=t_k}=\bar{I}_k\left(z(t_k^-)\right), \\ z_0(t,x)+g(t,z)=\varphi(t)\in E_v, \qquad t\in(-\infty,0], \\ z'(0,x)=z_1\in\bar{E}_v . \end{cases} \tag{1.1}$$

Now, consider the following $A^{\alpha}z(t,x)={}^{C}D_x^{\alpha}z(t,x)$. We have

$$\begin{cases} {}^{C}D_t^{\alpha}\left[z(t,x)-h\left(t,z_t(t,x)\right)\right]=\hat{A}^{\alpha}z(t,x)+\int_0^t f\left(s,z_s(s,x),I_s^{\gamma}z\right)ds, \\ \text{for } t\in J=[0,T],\quad t\neq t_k,\quad k=1.2.\dots.m,\quad 1<\alpha,\gamma\leq 2, \\ \Delta z(t,x)|_{t=t_k}=I_k\left(z(t_k^-)\right), \\ \Delta z'(t,x)\Big|_{t=t_k}=\bar{I}_k\left(z(t_k^-)\right), \\ z(t,x)+g(t,z)=\varphi(t)\in E_v, \qquad \text{for } t\in(-\infty,0], \\ z'(0,x)=z_1,\quad h'\left(0,z_t(0,x)\right)=z_2. \end{cases} \tag{1.2}$$

where $\hat{A}^{\alpha}=\left(\beta_1+\beta_2 M\left[\left\|A^{\alpha/2}z\right\|^2\right]\right)A^{\alpha}-A^{2\alpha}$ is a linear operator in the Banach space X, ${}^{C}D^{\alpha}$, I^{γ} are the Caputo fractional derivative and fractional integration of $1<\alpha,\gamma\leq 2$, respectively, $z(t,x)$ belongs to Banach space X, and $\Delta z(t,x)|_{t=t_k}$ is defined by $\Delta z(t,x)|_{t=t_k}=z(t_k^+)-z(t_k^-)$. Also, $\Delta z'(t,x)|_{t=t_k}$ is defined by $\Delta z'(t,x)|_{t=t_k}=z'(t_k^+)-z'(t_k^-)$ for all $k=1,2.\dots,m$, $0<t_0<t_1<t_2<\cdots<t_m<t_{m+1}=T$, with $z(t_k^+),z(t_k^-)$ representing the right and left limits of z at t_k respectively, $z'(t_k^+),z'(t_k^-)$ representing the right and left limits of z' at t_k respectively, and $g: E_v\to X$ is a given function. Let $z_t(.)$ denoted by $z_t(\theta)=z(t+\theta)$, $\theta\in(-\infty,0]$.

Many scientific problem of the impulsive nonlinear fractional order partial differential equations with nonlocal conditions can be expressed as in the above systems (1.1) and (1.2). Some of them do not have easy solvability. They are even difficult to study sometimes. Also their behaviors for their solutions is not appearing in general and, thus, we need more effort and practices. Therefore, these problems have taken our interest.

The aim of this article is to study and present all the results of the solvability for the proposal class impulsive nonlinear fractional order partial differential equations with nonlocal conditions. Some of special types of cases of the proposed problem have been approached, using nonlinear functional analysis theorems and abstract Cauchy problem involving semigroup operators with Krasnoselskii's fixed point theorem, to show the solvability with some recent sufficient and necessary conditions.

2. Preliminaries

In this section, we present some assumptions, notation and results needed in our proofs later.

Assumptions (2.1)

The following Assumptions of E_v are needed in the description of piecewise continuous space $PC\left((-\infty,0],X\right)$.

1. The continuous function $v:(-\infty,0]\to(0,+\infty)$ is satisfying $l=\int_{-\infty}^{0}v(t)dt<+\infty$:, and $\left(E_v, \|.\|_{E_v}\right)$ by a Banach space induced by the function v is defined as follows:

$E_v=\{\omega:(-\infty,0]\to X$: for any $c>0$, $\omega(\theta)$ is a bounded and measurable function on $[-c,0]$, and $\int_{-\infty}^{0}v(s)\sup_{s\leq\theta\leq 0}\|\omega(\theta)\|\,ds<+\infty\}$

with the norm $\|\omega\|_{E_v}=\int_{-\infty}^{0}v(s)\sup_{s\leq\theta\leq 0}\|\omega(\theta)\|\,ds$.

2. Let the space $\bar{E}_v=\{\omega,\omega':(-\infty,T]\to X;\ \omega_k,\omega'_k\in C(J_k,X), k=0,1,2,\dots,m$ and there exist $\omega(t_k^+)$, $\omega(t_k^-)$ and $\omega'(t_k^+)$, $\omega'(t_k^-)$ with $\omega(t_k)=\omega(t_k^-)$ and $\omega'(t_k)=\omega'(t_k^-)$, $\omega_0=\omega(0)+g(\omega)=\omega\in E_v\}$ where $\omega_k,\omega_k{}'$ are the restrictions of ω,ω' to J_k, where $J_0=[0,t_1]$, $J_k=(t_k,t_{k+1}]$, $k=0,1,2,\dots,m$. Define the seminorm in the space $\bar{E}_v$ by $\|\omega\|_{\bar{E}_v}=\|\omega\|_{E_v}+sup\{\|\omega(s)\|: s\in[0,T]\}$ for $\omega\in\bar{E}_v$ and $\|\omega'\|_{\bar{E}_v}=sup\{\|\omega'(s)\|: s\in[0,T]\}$ for $\omega'\in\bar{E}_v$.

3. By the space $\bar{E}_v$ we can define the space $\left(\bar{\bar{E}}_v , \|.\|_{\bar{E}_v}\right)$ as:
$\bar{\bar{E}}_v = \{\omega \in \bar{E}_v : 0 = \omega_0 \in E_v\}$ with norm $\|\omega\|_{\bar{E}_v} = sup\{\|\omega(s)\| : s \in [0,T]\}$.
4. Let the space $E_r = \left\{\omega \in \bar{\bar{E}}_v : \|\varphi\|_{E_{v'}} \leq r\right\}$ for $r > 0$, then E_r for each r, is a closed convex, bounded subset in X.

Definition (2.1), [21]

The fractional integral of order $\alpha > 0$ with the lower limit 0 for a function h is defined as: $I^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{h(s)}{(t-s)^{1-\alpha}} ds, \ t > 0, \ \alpha > 0$
where $\Gamma(u) = \int_0^\infty e^{-s} s^{u-1} ds$, $u > 0$ (gamma function).

Definition (2.2), [21]

The Caputo fractional derivative of order $\alpha > 0$ with lower limit 0 for a function h can be written as:
$^{C}D^\alpha h(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{h^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds \, , t > 0, \ n-1 < \alpha \leq n$.

Definition (2.3), [21]

The Laplace transform, of the Caputo fractional derivative of order $\alpha > 0$ is given as:
$$\mathcal{L}\left\{ {}^{C}D_t^\alpha h(t) \right\}(\lambda) = \lambda^\alpha \mathcal{L}(h)(\lambda) - \sum_{k=0}^{n-1} \lambda^{\alpha-k-1} [D^k h(t)]|_{t=0}, \ n-1 < \alpha \leq n.$$

Lemma (2.1), [22]

Let $\in E_{v'}$, for $t \in J$, then $z_t \in E_v$ and $l\|z(t)\| \leq \|z_t\|_{E_v} \leq \|\varphi\|_{E_v} + l \sup_{0 \leq s \leq t} \|z(s)\|$.

Definition (2.4), [10]

Let $A: D \subseteq X \to X$ be a closed linear operator. Then A is the sectorial operator of type (M, θ, α, μ) if there exist $0 < \theta < \frac{\pi}{2}$, $M > 0$ and $\mu \in \mathbb{R}$ such that the α-resolvent of A exists outside the secto
$\mu + S_\theta = \{\mu + \lambda^\alpha : \lambda \in \mathbb{C}, |Arg(-\lambda^\alpha)| < \theta\}$
and $\|(\lambda^\alpha I - A)^{-1}\| \leq \frac{M}{|\lambda^\alpha - \mu|}, \ \lambda^\alpha \notin \mu + S_\theta$.

Lemma (2.2), [10]

Let $M \neq \emptyset$ be a closed convex subset of a Banach space X. Let Υ and Ψ be two operators which satisfy
1. $\Upsilon u + \Psi v \in M$ whenever $u, v \in M$;
2. Ψ is continuous and compact;
3. Υ is a contraction.

Then there exists $z \in M$ such that $z = \Upsilon z + \Psi z$.

Lemma (2.3), [23]

Let A be a densely defined operator in X satisfying the following conditions ;
(i) For some $0 < \theta < \frac{\pi}{2}$, $\mu + S_\theta = \{\mu + \lambda : \lambda \in \mathbb{C}, |Arg(-\lambda)| < \theta\}$.
(ii) There exists a constant M such that
$\|(\lambda I - A)^{-1}\| \leq \frac{M}{|\lambda - \mu|}$, $\lambda \notin \mu + S_\theta$.

Then, A is the infinitesimal generator of a semigroup $T(t)$ satisfying $\|T(t)\| \leq C$. Moreover, $T(t) = \frac{1}{2\pi i} \int_c e^{\lambda t} R(\lambda, A) d\lambda$ with c being a suitable path $\lambda \notin \mu + S_\theta$ for $\lambda \in c$.

Lemma (2.4)

If $\hat{A}^\alpha$ is a densely defined sectorial operator of type (M, θ, α, μ) , then $\hat{A}^\alpha$ is the infinitesimal generator of a α-resolvent family $\{T_\alpha(t)\}_{t \geq 0}$ in Banach space X, where
$T_\alpha(t) = \frac{1}{2\pi i} \int_c e^{\lambda t} R(\lambda^\alpha, \hat{A}^\alpha) d\lambda$.

Proof

Since $\hat{A}^\alpha$ is sectorial operator then, form definition (2.4), we get $\hat{A}^\alpha$ is closed linear operator and satisfies
(i) For some $0 < \theta < \frac{\pi}{2}$, $\mu + S_\theta = \{\mu + \lambda^\alpha : \lambda \in \mathbb{C}, |Arg(-\lambda^\alpha)| < \theta\}$.
(ii) There exists a constant M such that
$\|(\lambda^\alpha I - A)^{-1}\| \leq \frac{M}{|\lambda^\alpha - \mu|}$, $\lambda^\alpha \notin \mu + S_\theta$.

Also, $\hat{A}^{\alpha}$ is densely defined operator. By lemma (2.3), then $\hat{A}^{\alpha}$ is the infinitesimal generator of a α-resolvent family $\{T_{\alpha}(t)\}_{t\geq 0}$ in Banach space, where

$$T_{\alpha}(t)=\frac{1}{2\pi i}\int_{c} e^{\lambda t}R(\lambda^{\alpha},\hat{A}^{\alpha})d\lambda .$$

Lemma (2.5), [10]

If the sectorial operator $\hat{A}^{\alpha}$ is of type (M,θ,α,μ) in definition (2.4), then we have the following:

(i) If $\mu\geq 0$ and $\phi\in(0,\pi)$, then we have that

$$\|S_{\alpha}(t)\|\leq\frac{K_1(\theta,\phi)Me^{[K_1(\theta,\phi)(1+\mu t^{\alpha})]^{\frac{1}{\alpha}}}\left[\left(1+\frac{\sin\phi}{\sin(\phi-\theta)}\right)^{\frac{1}{\alpha}}-1\right]}{\pi\sin^{1+\frac{1}{\alpha}}\theta}(1+\mu t^{\alpha})$$
$$+\frac{\Gamma(\alpha)M}{\pi(1+\mu t^{\alpha})\left|\cos\frac{\pi-\phi}{\alpha}\right|^{\alpha}\sin\theta\sin\phi}$$

$$\|T_{\alpha}(t)\|\leq\frac{M\left[\left(1+\frac{\sin\phi}{\sin(\phi-\theta)}\right)^{\frac{1}{\alpha}}-1\right]}{\pi\sin\theta}(1+\mu t^{\alpha})^{\frac{1}{\alpha}}t^{\alpha-1}e^{[K_1(\theta,\phi)(1+\mu t^{\alpha})]^{\frac{1}{\alpha}}}$$
$$+\frac{Mt^{\alpha-1}}{\pi(1+\mu t^{\alpha})\left|\cos\frac{\pi-\phi}{\alpha}\right|^{\alpha}\sin\theta\sin\phi}$$

$$\|K_{\alpha}(t)\|\leq\frac{M\left[\left(1+\frac{\sin\phi}{\sin(\phi-\theta)}\right)^{\frac{1}{\alpha}}-1\right]K_1(\theta,\phi)}{\pi\sin\theta^{\frac{\alpha+2}{\alpha}}}(1+\mu t^{\alpha})^{\frac{\alpha-1}{\alpha}}t^{\alpha-1}e^{[K_1(\theta,\phi)(1+\mu t^{\alpha})]^{\frac{1}{\alpha}}}$$
$$+\frac{M\alpha\Gamma(\alpha)}{\pi(1+\mu t^{\alpha})\left|\cos\frac{\pi-\phi}{\alpha}\right|^{\alpha}\sin\theta\sin\phi}$$

for $t>0$, where $K_1(\theta,\phi)=\max\left\{1,\frac{\sin\phi}{\sin(\phi-\theta)}\right\}$.

(ii) If $\mu<0$ and $\phi\in(0,\pi)$, then we have that

$$\|S_{\alpha}(t)\|\leq\left(\frac{eM\left[(1+\sin\phi)^{\frac{1}{\alpha}}-1\right]}{\pi|\cos\phi|^{1+\frac{1}{\alpha}}}+\frac{\Gamma(\alpha)M}{\pi|\cos\phi|\left|\cos\frac{\pi-\phi}{\alpha}\right|^{\alpha}}\right)\frac{1}{1+|\mu|t^{\alpha}}$$

$$\|T_{\alpha}(t)\|\leq\left(\frac{eM\left[(1+\sin\phi)^{\frac{1}{\alpha}}-1\right]}{\pi|\cos\phi|}+\frac{M}{\pi|\cos\phi|\left|\cos\frac{\pi-\phi}{\alpha}\right|}\right)\frac{t^{\alpha-1}}{1+|\mu|t^{\alpha}}$$

$$\|K_{\alpha}(t)\|\leq\left(\frac{eM\left[(1+\sin\phi)^{\frac{1}{\alpha}}-1\right]t}{\pi|\cos\phi|^{1+\frac{2}{\alpha}}}+\frac{\alpha\Gamma(\alpha)M}{\pi|\cos\phi|\left|\cos\frac{\pi-\phi}{\alpha}\right|}\right)\frac{1}{1+|\mu|t^{\alpha}}\ ,\qquad\text{for } t>0.$$

3. Main results

To investigate the existence of the mild solution of the impulsive abstract Cauchy problem (1.2), we assume the following conditions:

(A_1) The semigroups $S_{\alpha}(t)$, $K_{\alpha}(t)$, $P_{\alpha}(t)$ and $T_{\alpha}(t)$ generated by the operator $\hat{A}^{\alpha}$ are compact in $\overline{D(\hat{A}^{\alpha})}$ when $t\geq 0$ and

$\sup_{t\in J}\|S_{\alpha}(t)\|\leq\breve{M}$, $\sup_{t\in J}\|K_{\alpha}(t)\|\leq\breve{M}$, $\sup_{t\in J}\|P_{\alpha}(t)\|\leq\breve{M}$, $\sup_{t\in J}\|T_{\alpha}(t)\|\leq\breve{M}$.

(A_2) The $h:J\times E_v\to X$ and there exist constants $L_1, L_2>0$ satisfying

$\|h(t_1,\varphi_1)-h(t_2,\varphi_2)\|\leq L_1(\|\varphi_1-\varphi_2\|_{E_v}+\|t_1-t_2\|)$

$\sup_{t\in J}\|h(t,0)\|=L_2$, $h'(0,z_t(0,x))=z_2$.

(A_3) The $f:J\times E_v\times X\to X$ and there exist constants $K_1, K_2>0$ satisfying

$\|f(t_1,\varphi_1,y_1)-f(t_2,\varphi_2,y_2)\|\leq K_1(\|\varphi_1-\varphi_2\|_{E_v}+\|y_1-y_2\|+\|t_1-t_2\|)$

$\sup_{t\in J}\|f(t,0,0)\|=K_2$.

(A_4) The $I_k:X\to X$, $\bar{I}_k:X\to X$ and there exist constants $a_k, b_k>0$ and $c_k, d_k>0$, $k=1,2,\dots,m$ satisfying

$\|I_k(z_1)-I_k(z_2)\|\leq a_k\|z_1-z_2\|$, $\|I_k(0)\|\leq b_k$,

$\left\|\left(\hat{A}^{\alpha}\right)^{-1}\bar{I}_k(z_1)-\left(\hat{A}^{\alpha}\right)^{-1}\bar{I}_k(z_2)\right\|\leq c_k\|z_1-z_2\|$, $\left\|\left(\hat{A}^{\alpha}\right)^{-1}\bar{I}_k(0)\right\|\leq d_k$.

(A_5) The $g:J\times E_v\to X$ and there exists a constant $N_1>0$ satisfying

$$\|g(t_1,\varphi_1)-g(t_2,\varphi_2)\| \le N_1(\|\varphi_1-\varphi_2\|_{E_v}+\|t_1-t_2\|)$$

(A_6) Let

$$\rho_1 = \breve{M}[\|\varphi(0)\|+\|g(0,u+\emptyset)\|+\left\|h\big(0,\varphi(0)-g(0,u+\emptyset)\big)\right\|]+\breve{M}\|z_1-z_2\|$$
$$+\breve{M}t_1(L_1r'+L_2)+\breve{M}t_1T\left(K_1\left(r'+\frac{t^\gamma}{\Gamma(\gamma+1)}r'\right)+K_2\right),$$

$$\rho_2=\sum_{i=1}^{k}2\breve{M}\left(\breve{M}\left\|\varphi(0)-g(0,u+\emptyset)-h\big(0,\varphi(0)-g(0,u+\emptyset)\big)\right\|+\breve{M}\|z_1-z_2\|\right)$$
$$+\sum_{i=1}^{k}\breve{M}(a_i(r'')+b_i-\|z_1-z_2\|)$$
$$+\sum_{i=1}^{k}\breve{M}\left(c_i(r'')+d_i+\left\|\varphi(0)-g(0,u+\emptyset)-h\big(0,\varphi(0)-g(0,u+\emptyset)\big)\right\|\right)$$
$$+\sum_{i=1}^{k}2\breve{M}\Big((t_i-t_{i-1})\breve{M}(L_1(r')+L_2)+(t_i-t_{i-1})\breve{M}T(K_1(r'+\frac{t^\gamma}{\Gamma(\gamma+1)}(r''))+K_2\Big)$$
$$+\breve{M}T(L_1(r')+L_2)+\breve{M}T^2(K_1(r'+\frac{t^\gamma}{\Gamma(\gamma+1)}(r''))+K_2\Big) \quad \text{for } t\in(t_k,t_{k+1}].$$

where $r' = \|\varphi(0)\|_{E_v}+lr+\breve{M}\|[\varphi(0)-g(0,u+\emptyset)]\|$ and $r''=r+\breve{M}\|[\varphi(0)-g(0,u+\emptyset)]\|$ then we can get ρ_1 , $\rho_2 > 0$, such that

$$\mathrm{N} \le \begin{cases}\rho_1, t\in[0,t_1]\\ \rho_2, t\in[t_k,t_{k+1}], k=1,2,\dots,m\end{cases} \le r.$$

(A_7) 1) $\breve{M}(1+L_1)N_1<1$,

2) $\left[\sum_{i=1}^{k}2\breve{M}\,N_1\big(\breve{M}(1+L_1)\big)+\sum_{i=1}^{k}\breve{M}\,d_i+\sum_{i=1}^{k}\breve{M}\left(\bar{d}_i+N_1+L_1(N_1)\right)\right]<1.$

Lemma (3.6)

If the sectorial operator $\hat{A}^\alpha$ of type (M,θ,α,μ) and $f: J\times E_v\times X\to X$ is a map that satisfies (A_3), then the impulsive abstract Cauchy problem (1.2) is equivalent to the integral equation given by the following:

$$z(t,x)$$
$$=\begin{cases}\varphi(t), & \text{if } t\in(-\infty,0].\\ S_\alpha(t)\big[\varphi(0)-g(0,z)-h\big(0,\varphi(0)-g(0,z)\big)\big]+K_\alpha(t)[z_1-z_2] \\ \quad +\int_0^t P_\alpha(t-s)\,h\big(s,z_s(s,x)\big)ds \\ \quad +\int_0^t\int_0^s T_\alpha(t-s)f\big(c,z_c(c,x),I_c^\gamma z\big)\,dcds, & \text{if } t\in(0,t_1]\\ \sum_{i=1}^{k}[S_\alpha(t-t_i)+K_\alpha(t-t_i)]\big(S_\alpha(t_i)\big[\varphi(0)-g(0,z)-h\big(0,\varphi(0)-g(0,z)\big)\big] \\ \quad +K_\alpha(t_i)[z_1-z_2]+\int_{t_{i-1}}^{t_i}P_\alpha(t_i-s)\,h\big(s,z_s(s,x)\big)ds \\ \quad +\int_{t_{i-1}}^{t_i}\int_0^s T_\alpha(t_i-s)f\big(c,z_c(c,x),I_c^\gamma z\big)\,dcds\big)+\sum_{i=1}^{k}S_\alpha(t-t_i)\left(I_i\left(z(t_i^-)\right)-[z_1-z_2]\right) \\ \quad +\sum_{i=1}^{k}K_\alpha(t-t_i)\left(\left(\hat{A}^\alpha\right)^{-1}\bar{I}_i\big(z(t_i^-)\big)-\big[\varphi(0)-g(0,z)-h\big(0,\varphi(0)-g(0,z)\big)\big]\right) \\ \quad +\int_{t_k}^{t}P_\alpha(t-s)\,h\big(s,z_s(s,x)\big)ds+\int_{t_k}^{t}\int_0^s T_\alpha(t-s)f\big(c,z_c(c,x),I_c^\gamma z\big)\,dcds, \\ & \text{if } t\in(t_k,t_{k+1}],\ k=1,2,\dots,m\end{cases}$$

where, $S_\alpha(t)=\frac{1}{2\pi i}\int_0^\infty e^{\lambda t}\lambda^{\alpha-1}\left(\lambda^\alpha I-\hat{A}^\alpha\right)^{-1}d\lambda$

$K_\alpha(t)=\frac{1}{2\pi i}\int_0^\infty e^{\lambda t}\lambda^{\alpha-2}\left(\lambda^\alpha I-\hat{A}^\alpha\right)^{-1}d\lambda$

$P_\alpha(t)=\frac{1}{2\pi i}\int_0^\infty e^{\lambda t}\lambda^{\alpha}\left(\lambda^\alpha I-\hat{A}^\alpha\right)^{-1}d\lambda$

$T_\alpha(t)=\frac{1}{2\pi i}\int_0^\infty e^{\lambda t}\left(\lambda^\alpha I-\hat{A}^\alpha\right)^{-1}d\lambda$

Such that $\lambda^\alpha\notin\mu+S_\theta$ for $\lambda\in\mathbb{C}$.

Proof

By integrating the both sides of impulsive abstract Cauchy problem (1.2) of order $1<\alpha\le 2$, we get the equivalent equation

$$z(t,x)=[\varphi(0)-g(0,z)-h(0,\varphi(0)-g(0,z))]+[z_1-z_2]t+h\big(t,z_t(t,x)\big)$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\hat{A}^\alpha z(s,x)ds+\frac{1}{\Gamma(\alpha)}\int_0^t\int_0^s(t-s)^{\alpha-1}f\big(c,z_c(c,x),I_c^\gamma z\big)\,dcds \tag{3.3}$$

Now, by taking a Laplace transform of both sides of (3.3), we have

$$\mathcal{L}\{z(t,x)\}(\lambda)=\frac{1}{\lambda}\big[\varphi(0)-g(0,z)-h\big(0,\varphi(0)-g(0,z)\big)\big]+\frac{1}{\lambda^2}[z_1-z_2]+\mathcal{L}\big\{h\big(t,z_t(t,x)\big)\big\}(\lambda)$$

$$+\frac{1}{\lambda^{\alpha}}\hat{A}^{\alpha}\mathcal{L}\{z(t,x)\}(\lambda)+\frac{1}{\lambda^{\alpha}}\mathcal{L}\{\int_0^s f\big(c,z_c(c,x),I_c^{\gamma}z\big)dc\}(\lambda)$$

$$\mathcal{L}\{z(t,x)\}(\lambda)=\lambda^{\alpha-1}\big(\lambda^{\alpha}I-\hat{A}^{\alpha}\big)^{-1}\big[\varphi(0)-g(z)-h\big(0,\varphi(0)-g(z)\big)\big]$$
$$+\lambda^{\alpha-2}\big(\lambda^{\alpha}I-\hat{A}^{\alpha}\big)^{-1}[z_1-z_2]$$
$$+\lambda^{\alpha}\big(\lambda^{\alpha}I-\hat{A}^{\alpha}\big)^{-1}\mathcal{L}\big\{h\big(t,z_t(t,x)\big)\big\}(\lambda)+\big(\lambda^{\alpha}I-\hat{A}^{\alpha}\big)^{-1}\mathcal{L}\{\int_0^s f\big(c,z_c(c,x),I_c^{\gamma}z\big)dc\}(\lambda) \quad (3.2)$$

Now, by taking inverse transform of Laplace, we get

$$z(t,x)=\frac{1}{2\pi i}\int_0^{\infty}e^{\lambda t}\lambda^{\alpha-1}\big(\lambda^{\alpha}I-\hat{A}^{\alpha}\big)^{-1}\big[\varphi(0)-g(0,z)-h\big(0,\varphi(0)-g(0,z)\big)\big]d\lambda$$

$$+\frac{1}{2\pi i}\int_0^{\infty}e^{\lambda t}\lambda^{\alpha-2}\big(\lambda^{\alpha}I-\hat{A}^{\alpha}\big)^{-1}[z_1-z_2]d\lambda+\frac{1}{2\pi i}\int_0^{\infty}e^{\lambda t}\lambda^{\alpha}\big(\lambda^{\alpha}I-\hat{A}^{\alpha}\big)^{-1}\mathcal{L}\big\{h\big(t,z_t(t,x)\big)\big\}d\lambda$$
$$+\frac{1}{2\pi i}\int_0^{\infty}e^{\lambda t}\big(\lambda^{\alpha}I-\hat{A}^{\alpha}\big)^{-1}\mathcal{L}\{\int_0^s f\big(c,z_c(c,x),I_c^{\gamma}z\big)dc\}d\lambda \quad (3.4)$$

By Laplace transform of the last term of (3.4), we have

$$z(t,x)=\frac{1}{2\pi i}\int_0^{\infty}e^{\lambda t}\lambda^{\alpha-1}\big(\lambda^{\alpha}I-\hat{A}^{\alpha}\big)^{-1}\big[\varphi(0)-g(0,z)-h\big(0,\varphi(0)-g(0,z)\big)\big]d\lambda$$
$$+\frac{1}{2\pi i}\int_0^{\infty}e^{\lambda t}\lambda^{\alpha-2}\big(\lambda^{\alpha}I-\hat{A}^{\alpha}\big)^{-1}[z_1-z_2]d\lambda$$
$$+\frac{1}{2\pi i}\int_0^{\infty}\int_0^t e^{\lambda(t-s)}\lambda^{\alpha}\big(\lambda^{\alpha}I-\hat{A}^{\alpha}\big)^{-1}h\big(s,z_s(s,x)\big)dsd\lambda$$
$$+\frac{1}{2\pi i}\int_0^{\infty}\int_0^t\int_0^s e^{\lambda(t-s)}\big(\lambda^{\alpha}I-\hat{A}^{\alpha}\big)^{-1}f\big(c,z_c(c,x),I_c^{\gamma}z\big)\,dcdsd\lambda \quad (3.5)$$

We can rewrite the equation (3.5) as follows:

$$z(t,x)=S_{\alpha}(t)\big[\varphi(0)-g(0,z)-h\big(0,\varphi(0)-g(0,z)\big)\big]+K_{\alpha}(t)[z_1-z_2]$$
$$+\int_0^t P_{\alpha}(t-s)\,h\big(s,z_s(s,x)\big)ds+\int_0^t\int_0^s T_{\alpha}(t-s)f\big(c,z_c(c,x),I_c^{\gamma}z\big)\,dcds$$

where

$$S_{\alpha}(t)=\frac{1}{2\pi i}\int_0^{\infty}e^{\lambda t}\lambda^{\alpha-1}\big(\lambda^{\alpha}I-\hat{A}^{\alpha}\big)^{-1}\,d\lambda$$
$$K_{\alpha}(t)=\frac{1}{2\pi i}\int_0^{\infty}e^{\lambda t}\lambda^{\alpha-2}\big(\lambda^{\alpha}I-\hat{A}^{\alpha}\big)^{-1}\,d\lambda$$
$$P_{\alpha}(t)=\frac{1}{2\pi i}\int_0^{\infty}e^{\lambda t}\lambda^{\alpha}\big(\lambda^{\alpha}I-\hat{A}^{\alpha}\big)^{-1}d\lambda$$
$$T_{\alpha}(t)=\frac{1}{2\pi i}\int_0^{\infty}e^{\lambda t}\big(\lambda^{\alpha}I-\hat{A}^{\alpha}\big)^{-1}\,d\lambda$$

If $t\in(t_1,t_2]$, then

$$z(t,x)=S_{\alpha}(t-t_1)\big[\varphi(0)-g(0,z)-h\big(0,\varphi(0)-g(0,z)\big)\big]+K_{\alpha}(t-t_1)[z_1-z_2]$$
$$+\int_{t_1}^t P_{\alpha}(t-s)\,h\big(s,z_s(s,x)\big)ds+\int_{t_1}^t\int_0^s T_{\alpha}(t-s)f\big(c,z_c(c,x),I_c^{\gamma}z\big)\,dcds \quad (3.6)$$

$$\Delta z(t,x)|_{t=t_1}=z(t_1^+)-z(t_1^-)=I_1\big(z(t_1^-)\big)$$
$$=\big[\varphi(0)-g(0,z)-h\big(0,\varphi(0)-g(0,z)\big)\big]+[z_1-z_2]$$
$$-\Big(S_{\alpha}(t_1)\big[\varphi(0)-g(0,z)-h\big(0,\varphi(0)-g(0,z)\big)\big]$$
$$+K_{\alpha}(t_1)[z_1-z_2]+\int_0^{t_1}P_{\alpha}(t_1-s)\,h\big(s,z_s(s,x)\big)ds$$
$$+\int_0^{t_1}\int_0^s T_{\alpha}(t_1-s)f\big(c,z_c(c,x),I_c^{\gamma}z\big)\,dcds\Big)=I_1\big(z(t_1^-)\big)$$

$$\big[\varphi(0)-g(0,z)-h\big(0,\varphi(0)-g(0,z)\big)\big]=-[z_1-z_2]$$
$$+\Big(S_{\alpha}(t_1)\big[\varphi(0)-g(0,z)-h\big(0,\varphi(0)-g(0,z)\big)\big]$$
$$+K_{\alpha}(t_1)[z_1-z_2]+\int_0^{t_1}P_{\alpha}(t_1-s)\,h\big(s,z_s(s,x)\big)ds$$
$$+\int_0^{t_1}\int_0^s T_{\alpha}(t_1-s)f\big(c,z_c(c,x),I_c^{\gamma}z\big)\,dcds\Big)+I_1\big(z(t_1^-)\big) \quad (3.7)$$

$$\Delta z'(t,x)|_{t=t_1}=z'(t_1^+)-z'(t_1^-)=\bar{I}_1\big(z(t_1^-)\big)$$
$$z'(t_1^+)-z'(t_1^-)=\hat{A}^{\alpha}\big[\varphi(0)-g(0,z)-h\big(0,\varphi(0)-g(0,z)\big)\big]+\hat{A}^{\alpha}[z_1-z_2]$$
$$-\Big(\hat{A}^{\alpha}S_{\alpha}(t_1)\big[\varphi(0)-g(0,z)-h\big(0,\varphi(0)-g(0,z)\big)\big]$$
$$+\hat{A}^{\alpha}K_{\alpha}(t_1)[z_1-z_2]+\int_0^{t_1}\hat{A}^{\alpha}P_{\alpha}(t_1-s)\,h\big(s,z_s(s,x)\big)ds$$
$$+\int_0^{t_1}\int_0^s\hat{A}^{\alpha}T_{\alpha}(t_1-s)f\big(c,z_c(c,x),I_c^{\gamma}z\big)\,dcds\Big)=\bar{I}_1\big(z(t_1^-)\big)$$

$$[z_1-z_2]=-\big[\varphi(0)-g(0,z)-h\big(0,\varphi(0)-g(0,z)\big)\big]$$
$$+\Big(S_{\alpha}(t_1)\big[\varphi(0)-g(0,z)-h\big(0,\varphi(0)-g(0,z)\big)\big]$$

$$+K_\alpha(t_1)[z_1-z_2]+\int_0^{t_1}P_\alpha(t_1-s)\,h\big(s,z_s(s,x)\big)ds$$
$$+\int_0^{t_1}\int_0^{s}T_\alpha(t_1-s)f\big(c,z_c(c,x),I_c^\gamma z\big)\,dcds\Big)+\big(\hat{A}^\alpha\big)^{-1}\bar{I}_1\big(z(t_1^-)\big) \tag{3.8}$$

By (3.6), (3.7) and (3.8), we get that

$$\begin{aligned}z(t,x)=&[S_\alpha(t-t_1)+K_\alpha(t-t_1)]\Big(S_\alpha(t_1)\big[\varphi(0)-g(0,z)-h\big(0,\varphi(0)-g(0,z)\big)\big]\\&+K_\alpha(t_1)[z_1-z_2]+\int_0^{t_1}P_\alpha(t_1-s)\,h\big(s,z_s(s,x)\big)ds\\&+\int_0^{t_1}\int_0^{s}T_\alpha(t_1-s)f\big(c,z_c(c,x),I_c^\gamma z\big)\,dcds\Big)\\&-S_\alpha(t-t_1)[z_1-z_2]+S_\alpha(t-t_1)I_1\big(z(t_1^-)\big)\\&-K_\alpha(t-t_1)\big[\varphi(0)-g(0,z)-h\big(0,\varphi(0)-g(0,z)\big)\big]+K_\alpha(t-t_1)\big(\hat{A}^\alpha\big)^{-1}\bar{I}_1\big(z(t_1^-)\big)\\&+\int_{t_1}^{t}P_\alpha(t-s)\,h\big(s,z_s(s,x)\big)ds+\int_{t_1}^{t}\int_0^{s}T_\alpha(t-s)f\big(c,z_c(c,x),I_c^\gamma z\big)\,dcds\end{aligned}$$

for $t\in(t_1,t_2]$, if $t\in(t_k,t_{k+1}]$, then we get

$$z(t,x)=\begin{cases}\varphi(t), & \text{if } t\in(-\infty,0].\\ S_\alpha(t)\big[\varphi(0)-g(0,z)-h\big(0,\varphi(0)-g(0,z)\big)\big]+K_\alpha(t)[z_1-z_2]\\ \quad+\int_0^{t}P_\alpha(t-s)\,h\big(s,z_s(s,x)\big)ds\\ \quad+\int_0^{t}\int_0^{s}T_\alpha(t-s)f\big(c,z_c(c,x),I_c^\gamma z\big)\,dcds\,, & \text{if } t\in(0,t_1]\\ \sum_{i=1}^{k}[S_\alpha(t-t_i)+K_\alpha(t-t_i)]\Big(S_\alpha(t_i)\big[\varphi(0)-g(0,z)-h\big(0,\varphi(0)-g(0,z)\big)\big]\\ \quad+K_\alpha(t_i)[z_1-z_2]+\int_{t_{i-1}}^{t_i}P_\alpha(t_i-s)\,h\big(s,z_s(s,x)\big)ds\\ \quad+\int_{t_{i-1}}^{t_i}\int_0^{s}T_\alpha(t_i-s)f\big(c,z_c(c,x),I_c^\gamma z\big)\,dcds\Big)+\sum_{i=1}^{k}S_\alpha(t-t_i)\Big(I_i\Big(z\big(t_i^-\big)\Big)-[z_1-z_2]\Big)\\ \quad+\sum_{i=1}^{k}K_\alpha(t-t_i)\Big(\big(\hat{A}^\alpha\big)^{-1}\bar{I}_i\big(z(t_i^-)\big)-\big[\varphi(0)-g(0,z)-h\big(0,\varphi(0)-g(0,z)\big)\big]\Big)\\ \quad+\int_{t_k}^{t}P_\alpha(t-s)\,h\big(s,z_s(s,x)\big)ds+\int_{t_k}^{t}\int_0^{s}T_\alpha(t-s)f\big(c,z_c(c,x),I_c^\gamma z\big)\,dcds,\\ & \text{if } t\in(t_k,t_{k+1}],\ \ k=1,2,\dots,m.\end{cases}$$

Now, we define the mild solution of impulsive abstract Cauchy problem (1.2) for every $z\in E_r$.

Definition (3.5)

A function $z:(-\infty,T]\to X$ is called a mild solution of the impulsive abstract Cauchy problem (1.2), if $z(t,x)+g(t,z)=\varphi(t)\in E_v$, the impulsive conditions $\Delta z(t,x)\big|_{t=t_k}=I_k\big(z(t_k^-)\big)$, $\Delta z'(t,x)\big|_{t=t_k}=\bar{I}_k\big(z(t_k^-)\big)$ $k=1,2.\dots,m$ are verified, the restriction of $z(\cdot)$ to the interval $J_k=[t_k,t_{k+1}]$ is continuous and the following integral equation holds for $t\in J$.

$$z(t,x)=\begin{cases}\varphi(t), & \text{if } t\in(-\infty,0].\\ S_\alpha(t)\big[\varphi(0)-g(0,z)-h\big(0,\varphi(0)-g(0,z)\big)\big]+K_\alpha(t)[z_1-z_2]\\ \quad+\int_0^{t}P_\alpha(t-s)\,h\big(s,z_s(s,x)\big)ds\\ \quad+\int_0^{t}\int_0^{s}T_\alpha(t-s)f\big(c,z_c(c,x),I_c^\gamma z\big)\,dcds\,, & \text{if } t\in(0,t_1].\\ \sum_{i=1}^{k}[S_\alpha(t-t_i)+K_\alpha(t-t_i)]\Big(S_\alpha(t_i)\big[\varphi(0)-g(0,z)-h\big(0,\varphi(0)-g(0,z)\big)\big]\\ \quad+K_\alpha(t_i)[z_1-z_2]+\int_{t_{i-1}}^{t_i}P_\alpha(t_i-s)\,h\big(s,z_s(s,x)\big)ds\\ \quad+\int_{t_{i-1}}^{t_i}\int_0^{s}T_\alpha(t_i-s)f\big(c,z_c(c,x),I_c^\gamma z\big)\,dcds\Big)+\sum_{i=1}^{k}S_\alpha(t-t_i)\Big(I_i\Big(z\big(t_i^-\big)\Big)-[z_1-z_2]\Big)\\ \quad+\sum_{i=1}^{k}K_\alpha(t-t_i)\Big(\big(\hat{A}^\alpha\big)^{-1}\bar{I}_i\big(z(t_i^-)\big)-\big[\varphi(0)-g(0,z)-h\big(0,\varphi(0)-g(0,z)\big)\big]\Big)\\ \quad+\int_{t_k}^{t}P_\alpha(t-s)\,h\big(s,z_s(s,x)\big)ds+\int_{t_k}^{t}\int_0^{s}T_\alpha(t-s)f\big(c,z_c(c,x),I_c^\gamma z\big)\,dcds,\\ & \text{if } t\in(t_k,t_{k+1}],\ \ k=1,2,\dots,m.\end{cases}$$

Definition (3.6):

Let $z(t,x) = y(t,x) + \emptyset(t,x)$ and for $\in E_v$. To introduce the following main theorem, we define the function $\emptyset$ as follows:

$$\emptyset(t,x) = \begin{cases} \varphi(t,x) & \text{for } t \in (-\infty,0], \\ S_\alpha(t)[\varphi(0) - g(z)] & \text{for } t \in J. \end{cases}$$

Lemma (3.7)

If the assumptions (A_1)-(A_3) hold, then the $\overline{\Upsilon}(E_r)$ is equicontinuous, where the operator $\overline{\Upsilon}$ on E_r defined by

$$(\overline{\Upsilon}y)(t) = \begin{cases} 0, & \text{if } t\in(-\infty,0] \\ \int_0^t P_\alpha(t-s)h(s,y_s+\emptyset_s)ds + \int_0^t \int_0^s T_\alpha(t-s)f\left(c,y_c+\emptyset_c,I_c^\gamma(y+\emptyset)\right)dcds, & \text{if } t\in(0,t_1] \\ \sum_{i=1}^k [S_\alpha(t-t_i)+K_\alpha(t-t_i)]\Big(\int_{t_{i-1}}^{t_i} P_\alpha(t_i-s)h(s,y_s+\emptyset_s)ds \\ \quad + \int_{t_{i-1}}^{t_i}\int_0^s T_\alpha(t_i-s)f\left(c,y_c+\emptyset_c,I_c^\gamma(y+\emptyset)\right)dcds\Big) + \int_{t_k}^t P_\alpha(t-s)h(s,y_s+\emptyset_s)ds \\ \quad + \int_{t_k}^t \int_0^s T_\alpha(t-s)f\left(c,y_c+\emptyset_c,I_c^\gamma(y+\emptyset)\right)dcds, & \text{if } t\in(t_k,t_{k+1}], k=1,2,\dots,m \end{cases} \tag{3.9}$$

Proof

For $y \in E_r$, if $s_1, s_2 \in J$ and $0 < s_1 < s_2 \le t_1$, then we have

$$\begin{aligned} &\|(\overline{\Upsilon}y)(s_2) - (\overline{\Upsilon}y)(s_1)\| \\ &\le \int_0^{s_1}\|P_\alpha(s_2-s) - P_\alpha(s_1-s)\|\,\|h(s,y_s+\emptyset_s)\|ds + \int_{s_1}^{s_2}\|P_\alpha(s_1-s)\|\,\|h(s,y_s+\emptyset_s)\|ds \\ &\quad + \int_0^{s_1}\int_0^s\|T_\alpha(s_2-s) - T_\alpha(s_1-s)\|\left\|f\left(c,y_c+\emptyset_c,I_c^\gamma(y+\emptyset)\right)\right\|dcds \\ &\quad + \int_{s_1}^{s_2}\int_0^s\|T_\alpha(s_1-s)\|\left\|f\left(c,y_c+\emptyset_c,I_c^\gamma(y+\emptyset)\right)\right\|dcds \\ &\le s_1\|P_\alpha(s_2-s) - P_\alpha(s_1-s)\|(L_1(r') + L_2) + (s_2-s_1)\breve{M}(L_1(r') + L_2) \\ &\quad + s_1T\|T_\alpha(s_2-s) - T_\alpha(s_1-s)\|\left(K_1\left(r' + \frac{t^\gamma}{\Gamma(\gamma+1)}(r'')\right) + K_2\right) + (s_2-s_1)T\breve{M}\left(K_1\left(r' + \frac{t^\gamma}{\Gamma(\gamma+1)}(r'')\right) + K_2\right) \end{aligned} \tag{3.10}$$

Also, if $t_k < s_1 < s_2 \le t_{k+1}$, then we get

$$\begin{aligned} &\|(\overline{\Upsilon}y)(s_2) - (\overline{\Upsilon}y)(s_1)\| \\ &\le \left\|\sum_{i=1}^k[S_\alpha(s_2-t_i) + K_\alpha(s_2-t_i)] - \sum_{i=1}^k[S_\alpha(s_1-t_i) + K_\alpha(s_1-t_i)]\right\| \\ &\quad \times \left\| \int_{t_{i-1}}^{t_i} P_\alpha(t_i-s)\,h(s,y_s+\emptyset_s)ds + \int_{t_{i-1}}^{t_i}\int_0^s T_\alpha(t_i-s)f\left(c,y_c+\emptyset_c,I_c^\gamma(y+\emptyset)\right)dcds\right\| \\ &\quad + \int_{t_k}^{s_1}\|P_\alpha(s_2-s) - P_\alpha(s_1-s)\|\,\|h(s,y_s+\emptyset_s)\|ds + \int_{s_1}^{s_2}\|P_\alpha(s_2-s)\|\,\|h(s,y_s+\emptyset_s)\|ds \\ &\quad + \int_{t_k}^{s_1}\int_0^s\|T_\alpha(s_2-s) - T_\alpha(s_1-s)\|\left\|f\left(c,y_c+\emptyset_c,I_c^\gamma(y+\emptyset)\right)\right\|dcds \\ &\quad + \int_{s_1}^{s_2}\int_0^s\|T_\alpha(s_2-s)\|\left\|f\left(c,y_c+\emptyset_c,I_c^\gamma(y+\emptyset)\right)\right\|dcds \\ &\le \sum_{i=1}^k\|S_\alpha(s_2-t_i) - S_\alpha(s_1-t_i)\| + \sum_{i=1}^k\|K_\alpha(s_2-t_i) - K_\alpha(s_1-t_i)\| \\ &\quad \times \left((t_i - t_{i-1})\breve{M}\,(L_1(r') + L_2) + (t_i - t_{i-1})\breve{M}T\left(K_1\left(r' + \frac{t^\gamma}{\Gamma(\gamma+1)}(r'')\right) + K_2\right)\right) \\ &\quad + (s_1 - t_k)\|P_\alpha(s_2-s) - P_\alpha(s_1-s)\|(L_1(r') + L_2) + (s_2-s_1)\breve{M}(L_1(r') + L_2) \\ &\quad + (s_1 - t_k)T\|T_\alpha(s_2-s) - T_\alpha(s_1-s)\|\left(K_1\left(r' + \frac{t^\gamma}{\Gamma(\gamma+1)}(r'')\right) + K_2\right) + (s_2-s_1)\breve{M}T\left(K_1\left(r' + \frac{t^\gamma}{\Gamma(\gamma+1)}(r'')\right) + K_2\right) \end{aligned} \tag{3.11}$$

From Assumption (A_1), the compactness of $S_\alpha(t)$, $K_\alpha(t)$, $P_\alpha(t)$ and $T_\alpha(t)$ for $t > 0$ satisfy the continuity in the uniform operator topology. Then the right-hand sides of (3.10) and (3.11) tend to zero as $s_2 \to s_1$. Therefore, the family $\{(\overline{\Upsilon}y)(t): y \in E_r\}$ satisfies the equicontinuous functions. Since the proof of equicontinuities to the case $s_1 < s_2 < 0$ and $s_1 < 0 < s_2$ is simple, then the proof is omitted.

Lemma (3.8)

If the assumptions (A_1)-(A_3) hold, then the $W(t) = \{(\overline{\Upsilon}y)(t): y \in E_r\}$ is relatively compact for any $t \in J \setminus \{t_1, \dots, t_m\}$, where the operator $\overline{\Upsilon}$ on E_r is defined in (3.9).

Proof

Let $t \in (0, t_1]$ be a fixed and ε be a real number, implies $0 < \varepsilon < t_1$. For $\in E_r$, we define $W_\varepsilon(t) = \{(\overline{\Upsilon}_\varepsilon y)(t): y \in E_r\}$ as follows:

$$(\overline{\Upsilon}_\varepsilon y)(t) = \int_0^{t-\varepsilon} P_\alpha(t-s)\, h(s, y_s + \emptyset_s)ds + \int_0^{t-\varepsilon}\int_0^s T_\alpha(t-s) f\left(c, y_c + \emptyset_c, I_c^\gamma(y+\emptyset)\right) dcds$$
$$= P_\alpha(\varepsilon)\int_0^{t-\varepsilon} P_\alpha(t-s-\varepsilon)\, h(s, y_s + \emptyset_s)ds$$
$$+ T_\alpha(\varepsilon)\int_0^{t-\varepsilon}\int_0^s T_\alpha(t-s-\varepsilon) f\left(c, y_c + \emptyset_c, I_c^\gamma(y+\emptyset)\right) dcds \quad (3.12)$$

Also, for $t \in (t_k, t_{k+1}]$ to be a fixed and ε be a real number, implies $t_k < \varepsilon < t_{k+1}$, we define the subset $W_\varepsilon(t) = \{(\overline{\Upsilon}_\varepsilon y)(t): y \in E_r\}$

$$(\overline{\Upsilon}_\varepsilon y)(t) = \sum_{i=1}^k [S_\alpha(t-t_i) + K_\alpha(t-t_i)]\Big(\int_{t_{i-1}}^{t_i} P_\alpha(t_i - s)\, h(s, y_s + \emptyset_s)ds$$
$$+ \int_{t_{i-1}}^{t_i}\int_0^s T_\alpha(t_i - s) f\left(c, y_c + \emptyset_c, I_c^\gamma(y+\emptyset)\right) dcds\Big) + P_\alpha(\varepsilon)\int_{t_k}^{t-\varepsilon} P_\alpha(t-s-\varepsilon)\, h(s, y_s + \emptyset_s)ds$$
$$+ T_\alpha(\varepsilon)\int_{t_k}^{t-\varepsilon}\int_0^s T_\alpha(t-s-\varepsilon) f\left(c, y_c + \emptyset_c, I_c^\gamma(y+\emptyset)\right) dcds \quad (3.13)$$

From assumption (A_1)-(A_3), we have $S_\alpha(t)$, $K_\alpha(t)$, $P_\alpha(t)$ and $T_\alpha(t)$ are compact and the right-hand side of (3.12) and (3.13) are bounded, then the set $W_\varepsilon(t) = \{(\overline{\Upsilon}_\varepsilon y)(t): y \in E_r\}$ is relatively compact in X, for every $\varepsilon, t_k < \varepsilon < t_{k+1}$.

Moreover, for $y \in E_r$, $t \in (0, t_1]$, we have

$$\|(\overline{\Upsilon} y)(t) - (\overline{\Upsilon}_\varepsilon y)(t)\| \le \varepsilon\|P_\alpha(t-s)\|\|h(s, y_s + \emptyset_s)\| + \varepsilon\|T_\alpha(t-s)\|\left\|f\left(c, y_c + \emptyset_c, I_c^\gamma(y+\emptyset)\right)\right\|$$
$$\le \varepsilon\breve{M}\left[(L_1 r' + L_2) + (K_1(r' + \tfrac{t^\gamma}{\Gamma(\gamma+1)}(r'')) + K_2)\right]$$

and for $t \in (t_k, t_{k+1}]$, we get

$$\|(\overline{\Upsilon} y)(t) - (\overline{\Upsilon}_\varepsilon y)(t)\| \le \left\|\sum_{i=1}^k [S_\alpha(t-t_i) + K_\alpha(t-t_i)] - \sum_{i=1}^k [S_\alpha(t-t_i-\varepsilon) + K_\alpha(t-t_i-\varepsilon)]\right\|$$
$$\times\Big[(t_i - t_{i-1})\|P_\alpha(t_i - s)\|\|h(s, y_s + \emptyset_s)\| + (t_i - t_{i-1})T\|T_\alpha(t_i - s)\|\left\|f\left(c, y_c + \emptyset_c, I_c^\gamma(y+\emptyset)\right)\right\|\Big]$$
$$+ \varepsilon\|P_\alpha(t-s)\|\|h(s, y_s + \emptyset_s)\| + \varepsilon T\|T_\alpha(t-s)\|\left\|f\left(c, y_c + \emptyset_c, I_c^\gamma(y+\emptyset)\right)\right\|$$
$$\le \left\|\sum_{i=1}^k [S_\alpha(t-t_i) + K_\alpha(t-t_i)] - \sum_{i=1}^k [S_\alpha(t-t_i-\varepsilon) + K_\alpha(t-t_i-\varepsilon)]\right\|$$
$$\times\left[(t_i - t_{i-1})\breve{M}(L_1(r') + L_2) + (t_i - t_{i-1})T\breve{M}(K_1(r' + \tfrac{t^\gamma}{\Gamma(\gamma+1)}(r'')) + K_2)\right]$$
$$+ \varepsilon\breve{M}(L_1(r') + L_2) + \varepsilon T\breve{M}(K_1(r' + \tfrac{t^\gamma}{\Gamma(\gamma+1)}(r'')) + K_2)$$

Therefore, as $\varepsilon \to 0$, there are relatively compact sets arbitrarily closed to the set $W(t) = \{(\overline{\Upsilon} y)(t): y \in E_r\}$ for each $\in J \setminus \{t_1, \dots, t_m\}$. Hence, the set $W(t) = \{(\overline{\Upsilon} y)(t): y \in E_r\}$, $t \in J \setminus \{t_1, \dots, t_m\}$ is relatively compact in X.

Theorem (3.1)

If the assumptions (A_1)-(A_7) hold, then the impulsive abstract Cauchy problem (1.2) has at least one mild solution z that belongs to E_r .

Proof

It suffices to prove that the operator Ψ defined as follows has a fixed point $z(\cdot)$

$$(\Psi z)(t,x) = \begin{cases} \varphi(t), & \text{if } t \in (-\infty, 0]. \\ S_\alpha(t)\big[\varphi(0) - g(0,z) - h\big(0, \varphi(0) - g(0,z)\big)\big] + K_\alpha(t)[z_1 - z_2] \\ \quad + \int_0^t P_\alpha(t-s)\, h\big(s, z_s(s,x)\big) ds \\ \quad + \int_0^t \int_0^s T_\alpha(t-s) f\big(c, z_c(c,x), I_c^\gamma z\big)\, dcds\,, & \text{if } t \in (0, t_1]. \\ \sum_{i=1}^k [S_\alpha(t-t_i) + K_\alpha(t-t_i)]\,\big(S_\alpha(t_i)\big[\varphi(0) - g(0,z) - h\big(0, \varphi(0) - g(0,z)\big)\big] \\ \quad + K_\alpha(t_i)[z_1 - z_2] + \int_{t_{i-1}}^{t_i} P_\alpha(t_i - s)\, h\big(s, z_s(s,x)\big) ds \\ + \int_{t_{i-1}}^{t_i} \int_0^s T_\alpha(t_i - s) f\big(c, z_c(c,x), I_c^\gamma z\big)\, dcds\Big) + \sum_{i=1}^k S_\alpha(t-t_i)\Big(I_i\Big(z\big(t_i^-\big)\Big) - [z_1 - z_2]\Big) \\ \quad + \sum_{i=1}^k K_\alpha(t-t_i)\Big(\big(\hat{A}^\alpha\big)^{-1} \bar{I}_i\big(z(t_i^-)\big) - \big[\varphi(0) - g(0,z) - h\big(0, \varphi(0) - g(0,z)\big)\big]\Big) \\ \quad + \int_{t_k}^t P_\alpha(t-s)\, h\big(s, z_s(s,x)\big) ds + \int_{t_k}^t \int_0^s T_\alpha(t-s) f\big(c, z_c(c,x), I_c^\gamma z\big)\, dcds, \\ & \text{if } t \in (t_k, t_{k+1}],\ \ k = 1,2,\dots,m. \end{cases}$$

We consider the operators Υ and $\bar{\Upsilon}$ on E_r defined as follows:

$$(\Upsilon y)(t) = \begin{cases} 0\,, \ \text{if } t \in (-\infty, 0] \\ S_\alpha(t)\big[\varphi(0) - g(0, y + \emptyset) - h\big(0, \varphi(0) - g(0, y+\emptyset)\big)\big] + K_\alpha(t)[z_1 - z_2] \quad \text{if } t \in (0, t_1] \\ \sum_{i=1}^k [S_\alpha(t-t_i) + K_\alpha(t-t_i)]\,\big(S_\alpha(t_i)\big[\varphi(0) - g(0, y+\emptyset) - h\big(0, \varphi(0) - g(y+\emptyset)\big)\big] \\ \quad + K_\alpha(t_i)[z_1 - z_2]\big) + \sum_{i=1}^k S_\alpha(t-t_i)\Big(I_i\Big(y\big(t_i^-\big) + \emptyset\big(t_i^-\big)\Big) - [z_1 - z_2]\Big) \\ + \sum_{i=1}^k K_\alpha(t-t_i)\Big(\big(\hat{A}^\alpha\big)^{-1} \bar{I}_i\big(y(t_i^-) + \emptyset(t_i^-)\big) - \big[\varphi(0) - g(0, y+\emptyset) - h\big(0, \varphi(0) - g(0.y+\emptyset)\big)\big]\Big) \\ \qquad ,\ \text{if } t \in (t_k, t_{k+1}], k = 1,2,\dots,m \end{cases}$$

$$(\bar{\Upsilon} y)(t) = \begin{cases} 0\,, \ \text{if } t \in (-\infty, 0] \\ \int_0^t P_\alpha(t-s)\, h(s, y_s + \emptyset_s) ds + \int_0^t \int_0^s T_\alpha(t-s) f\left(c, y_c + \emptyset_c, I_c^\gamma(y+\emptyset)\right) dcds\,, \ \text{if } t \in (0, t_1] \\ \sum_{i=1}^k [S_\alpha(t-t_i) + K_\alpha(t-t_i)]\Big(\int_{t_{i-1}}^{t_i} P_\alpha(t_i - s)\, h(s, y_s + \emptyset_s) ds \\ \quad + \int_{t_{i-1}}^{t_i} \int_0^s T_\alpha(t_i - s) f\left(c, y_c + \emptyset_c, I_c^\gamma(y+\emptyset)\right) dcds\Big) + \int_{t_k}^t P_\alpha(t-s)\, h(s, y_s + \emptyset_s) ds \\ \quad + \int_{t_k}^t \int_0^s T_\alpha(t-s) f\left(c, y_c + \emptyset_c, I_c^\gamma(y+\emptyset)\right) dcds, \quad \text{if } t \in (t_k, t_{k+1}], k = 1,2,\dots,m \end{cases}$$

Now, to find the mild solution of impulsive abstract Cauchy problem (1.2), we need to prove that $\Upsilon + \bar{\Upsilon}$ has a fixed point on E_r . For any $, v \in E_r$, we have

$$\begin{aligned} \|(\Upsilon u)(t) + (\bar{\Upsilon} v)(t)\| &\le \breve{M}[\|\varphi(0)\| + \|g(0, u+\emptyset)\| + \big\|h\big(0, \varphi(0) - g(0, u+\emptyset)\big)\big\|] \\ &\quad + \breve{M}\|z_1 - z_2\| + \breve{M} t_1\big(L_1 \|v_s(s) + \emptyset_s(s)\|_{E_v} + L_2\big) \\ &\quad + \breve{M} t_1 T\left(K_1\left(\|v_c(s) + \emptyset_c(s)\|_{E_v} + \frac{t^\gamma}{\Gamma(\gamma+1)} \sup_{0<s\le t_1} \big\|\big(v(s) + \emptyset(s)\big)\big\|_{E_v}\right) + K_2\right) \\ &\le \breve{M}[\|\varphi(0)\| + \|g(0, u+\emptyset)\| + \big\|h\big(0, \varphi(0) - g(0, u+\emptyset)\big)\big\|] + \breve{M}\|z_1 - z_2\| \\ &\quad + \breve{M} t_1(L_1 r' + L_2) + \breve{M} t_1 T\left(K_1\left(r' + \frac{t^\gamma}{\Gamma(\gamma+1)} r'\right) + K_2\right) \le N \le r, \quad \text{for } t \in (0, t_1] \end{aligned}$$

and

$\|(\Upsilon u)(t) + (\bar{\Upsilon} v)(t)\|$

$$\leq \sum_{i=1}^{k}[\|S_\alpha(t-t_i)\|+\|K_\alpha(t-t_i)\|]\left(\|S_\alpha(t_i)\|\left\|\varphi(0)-g(0,u+\emptyset)-h\big(0,\varphi(0)-g(0,u+\emptyset)\big)\right\|\right.$$
$$+\|K_\alpha(t_i)\|\|\|z_1-z_2\|\big)+\sum_{i=1}^{k}\|S_\alpha(t-t_i)\|(a_i\|u(t_i^-)+\emptyset(t_i^-)\|+b_i-\|z_1-z_2\|)$$
$$+\sum_{i=1}^{k}\|K_\alpha(t-t_i)\|\left(c_i\|u(t_i^-)+\emptyset(t_i^-)\|+d_i+\left\|\varphi(0)-g(0,u+\emptyset)-h\big(0,\varphi(0)-g(0,u+\emptyset)\big)\right\|\right)$$
$$+\sum_{i=1}^{k}[\|S_\alpha(t-t_i)\|+\|K_\alpha(t-t_i)\|]\left(\int_{t_{i-1}}^{t_i}\|P_\alpha(t_i-s)\|\,(L_1\|v_s(s)+\emptyset_s(s)\|+L_2)ds\right.$$
$$\left.+\int_{t_{i-1}}^{t_i}\int_0^s\|T_\alpha(t_i-s)\|\left(K_1\big(\|v_c+\emptyset_c\|+\left\|I_c^\gamma(v+\emptyset)\right\|\big)+K_2\right)dcds\right)$$
$$+\int_{t_k}^{t}\|P_\alpha(t-s)\|\,(L_1\|v_s(s)+\emptyset_s(s)\|+L_2)ds$$
$$+\int_{t_k}^{t}\int_0^s\|T_\alpha(t-s)\|\left(K_1\big(\|v_c+\emptyset_c\|+\left\|I_c^\gamma(v+\emptyset)\right\|\big)+K_2\right)dcds$$
$$\leq \sum_{i=1}^{k}2\breve{M}\left(\breve{M}\left\|\varphi(0)-g(0,u+\emptyset)-h\big(0,\varphi(0)-g(0,u+\emptyset)\big)\right\|+\breve{M}\|z_1-z_2\|\right)$$
$$+\sum_{i=1}^{k}\breve{M}(a_i(r'')+b_i-\|z_1-z_2\|)$$
$$+\sum_{i=1}^{k}\breve{M}\left(c_i(r'')+d_i+\left\|\varphi(0)-g(0,u+\emptyset)-h\big(0,\varphi(0)-g(0,u+\emptyset)\big)\right\|\right)$$
$$+\sum_{i=1}^{k}2\breve{M}\left((t_i-t_{i-1})\breve{M}(L_1(r')+L_2)+(t_i-t_{i-1})\breve{M}T(K_1(r'+\frac{t^\gamma}{\Gamma(\gamma+1)}(r''))+K_2\right)$$
$$+\breve{M}T(L_1(r')+L_2)+\breve{M}T^2(K_1(r'+\frac{t^\gamma}{\Gamma(\gamma+1)}(r''))+K_2)\leq N\leq r.$$

For $t\in(t_k,t_{k+1}]$, then we can get $\rho_1,\ \rho_2>0$, such that

$$\|(\Upsilon u)(t)+(\overline{\Upsilon}v)(t)\|\leq\begin{cases}\rho_1, t\in[0,t_1]\\ \rho_2, t\in[t_k,t_{k+1}], k=1,2,\dots,m\end{cases}\leq N\leq r.$$

Which means that $\Upsilon u+\overline{\Upsilon}v$ is bounded and thus verified the condition (1) in Lemma (2.2).

Now, we show that $\overline{\Upsilon}$ is compact. For this we start proving that $\{(\overline{\Upsilon}y)(t): y\in E_r\}$ is relatively compact, for $t\in[0,t_1]$, we have

$$\|(\overline{\Upsilon}y)(t)\|\leq\left\|\int_0^{t_1}P_\alpha(t-s)\,h(s,y_s+\emptyset_s)ds\right\|+\left\|\int_0^{t_1}\int_0^s T_\alpha(t-s)f\left(c,y_c+\emptyset_c,I_c^\gamma(y+\emptyset)\right)dcds\right\|$$
$$\leq\breve{M}t_1\big(L_1\big(\|\varphi\|_{E_v}+lr+\breve{M}\|[\varphi(0)-g(0,u+\emptyset)]\|\big)+L_2\big)$$
$$+\breve{M}t_1T\left(K_1\left(\|\varphi\|_{E_v}+lr+\breve{M}\|[\varphi(0)-g(0,u+\emptyset)]\|+\frac{t^\gamma}{\Gamma(\gamma+1)}\big(r+\breve{M}\|\varphi(0)-g(0,u+\emptyset)\|\big)\right)+K_2\right)$$
$$\leq\breve{M}t_1\left[(L_1(r')+L_2)+T(K_1(r'+\frac{t^\gamma}{\Gamma(\gamma+1)}(r''))+K_2)\right]$$

and for $t\in(t_k,t_{k+1}]$, we get

$$\|(\overline{\Upsilon}y)(t)\|=\left\|\sum_{i=1}^{k}[S_\alpha(t-t_i)+K_\alpha(t-t_i)]\left(\int_{t_{i-1}}^{t_i}P_\alpha(t_i-s)\,h(s,y_s+\emptyset_s)ds\right.\right.$$
$$\left.+\int_{t_{i-1}}^{t_i}\int_0^s T_\alpha(t_i-s)f\left(c,y_c+\emptyset_c,I_c^\gamma(y+\emptyset)\right)dcds\right)+\int_{t_k}^{t}P_\alpha(t-s)\,h(s,y_s+\emptyset_s)ds$$
$$\left.+\int_{t_k}^{t}\int_0^s T_\alpha(t-s)f\left(c,y_c+\emptyset_c,I_c^\gamma(y+\emptyset)\right)dcds\right\|$$
$$\leq\sum_{i=1}^{k}2\breve{M}\left((t_i-t_{i-1})\breve{M}\big((L_1r'+L_2)\big)+(t_i-t_{i-1})\breve{M}T\left((K_1(r'+\frac{t^\gamma}{\Gamma(\gamma+1)}(r''))+K_2)\right)\right)$$
$$+\breve{M}T\big((L_1(r')+L_2)\big)+\breve{M}T^2\left((K_1(r'+\frac{t^\gamma}{\Gamma(\gamma+1)}(r''))+K_2)\right).$$

Then the set $\{(\overline{\Upsilon}y)(t): y\in E_r\}$ is uniformly bounded. Therefore, from lemma (3.7), we have that the $\overline{\Upsilon}(E_r)$ is equicontinuous, and from lemma (3.8), we have that the $W(t)=\{(\overline{\Upsilon}y)(t): y\in E_r\}$ is relatively compact for any $t\in J\setminus\{t_1,\dots,t_m\}$. Then by the Arzela-Ascoli theorem, we get that the closure of $\{(\overline{\Upsilon}y)(t): y\in E_r\}$ is compact . Next, we shall prove that $\overline{\Upsilon}$ is continuous:

Let $\{y_n\}$ be a sequence in E_r and $y_n\to y$ for $\in E_v$. From assumptions (A_2)-(A_3), we get f and h are continuous, i.e. for all $\varepsilon>0$ there is a positive integer N such that for $n>N$

$$\left\|f\left(c,y_{c_n}+\emptyset_{c_n},I_c^\gamma(y_n+\emptyset_n)\right)-f\left(c,y_c+\emptyset_c,I_c^\gamma(y+\emptyset)\right)\right\|\leq\varepsilon \quad\text{and}\quad \left\|h\big(s,y_{s_n}+\emptyset_{s_n}\big)-h(s,y_s+\emptyset_s)\right\|\leq\varepsilon .$$

Now, for $\in(0,t_1]$, we get

$$\|(\overline{\Upsilon}y_n)(t)-(\overline{\Upsilon}y)(t)\|\leq\int_0^{t_1}\|P_\alpha(t-s)\|\left\|h\big(s,y_{s_n}+\emptyset_{s_n}\big)-h(s,y_s+\emptyset_s)\right\|ds$$

$$+\int_0^{t_1}\int_0^s\|T_\alpha(t-s)\|\left\|f\left(c,y_{c_n}+\emptyset_{c_n},I_c^\gamma(y_n+\emptyset_n)\right)-f\left(c,y_c+\emptyset_c,I_c^\gamma(y+\emptyset)\right)\right\|dcds$$

$$\leq t_1\breve{M}\left\|h\left(s,y_{s_n}+\emptyset_{s_n}\right)-h(s,y_s+\emptyset_s)\right\|$$
$$+t_1\breve{M}T\left\|f\left(c,y_{c_n}+\emptyset_{c_n},I_c^\gamma(y_n+\emptyset_n)\right)-f\left(c,y_c+\emptyset_c,I_c^\gamma(y+\emptyset)\right)\right\|$$
$$\leq t_1\breve{M}\varepsilon(1+T)$$

Moreover, for $\in(t_k,t_{k+1}]$, we have

$$\|(\overline{\Upsilon}y_n)(t)-(\overline{\Upsilon}y)(t)\|$$
$$\leq\sum_{i=1}^k\|S_\alpha(t-t_i)+K_\alpha(t-t_i)\|\left(\int_{t_{i-1}}^{t_i}\|P_\alpha(t_i-s)\|\left\|h\left(s,y_{s_n}+\emptyset_{s_n}\right)-h(s,y_s+\emptyset_s)\right\|ds\right.$$
$$+\int_{t_{i-1}}^{t_i}\int_0^s\|T_\alpha(t_i-s)\|\left\|f\left(c,y_{c_n}+\emptyset_{c_n},I_c^\gamma(y_n+\emptyset_n)\right)-f\left(c,y_c+\emptyset_c,I_c^\gamma(y+\emptyset)\right)\right\|dcds\Big)$$
$$+\int_{t_k}^t\|P_\alpha(t-s)\|\left\|h\left(s,y_{s_n}+\emptyset_{s_n}\right)-h(s,y_s+\emptyset_s)\right\|ds$$
$$+\int_{t_k}^t\int_0^s\|T_\alpha(t-s)\|\left\|f\left(c,y_{c_n}+\emptyset_{c_n},I_c^\gamma(y_n+\emptyset_n)\right)-f\left(c,y_c+\emptyset_c,I_c^\gamma(y+\emptyset)\right)\right\|dcds$$
$$\leq\sum_{i=1}^k\breve{M}\varepsilon\,(t_i-t_{i-1})\left(2\breve{M}+T\right)$$
$$+\breve{M}\varepsilon(t_k-t_{k+1})(1+T)$$

Hence , $\overline{\Upsilon}$ is continuous. By the analysis above, we can see that $\overline{\Upsilon}$ implies the condition (2) of lemma (2.2) which means that $\overline{\Upsilon}$ is completely continuous.

For ending the proof, we will show that Υ is a contraction. Let $,v\in E_r$, for $t\in(0,t_1]$, we get

$$\|(\Upsilon u)(t)-(\Upsilon v)(t)\|\leq\|S_\alpha(t)\|\left\|\left[\varphi(0)-g(0,u+\emptyset)-h\left(0,\varphi(0)-g(0,u+\emptyset)\right)\right]\right.$$
$$\left.-\left[\varphi(0)-g(0,v+\emptyset)-h\left(0,\varphi(0)-g(0,v+\emptyset)\right)\right]\right\|$$
$$\leq\|S_\alpha(t)\|\left(N_1\|u-v\|+L_1(N_1\|u-v\|)\right)$$
$$\leq\breve{M}(1+L_1)N_1\|u-v\|$$

Therefore, for $\in(t_k,t_{k+1}]$, we get

$$\|(\Upsilon u)(t)-(\Upsilon v)(t)\|\leq\sum_{i=1}^k2\breve{M}\left(\breve{M}(N_1\|u-v\|+L_1(N_1\|u-v\|))\right)$$
$$+\sum_{i=1}^k\breve{M}\,d_i\|u(t_i^-)-v(t_i^-)\|+\sum_{i=1}^k\breve{M}\left(\bar{d}_i\|u(t_i^-)-v(t_i^-)\|+N_1\|u-v\|+L_1(N_1\|u-v\|)\right)$$
$$\leq\left[\sum_{i=1}^k2\breve{M}\,N_1\left(\breve{M}(1+L_1)\right)+\sum_{i=1}^k\breve{M}\,d_i+\sum_{i=1}^k\breve{M}\left(\bar{d}_i+N_1+L_1(N_1)\right)\right]\|u-v\|$$

Since $\breve{M}(1+L_1)N_1<1$, and

$$\left[\sum_{i=1}^k2\breve{M}\,N_1\left(\breve{M}(1+L_1)\right)+\sum_{i=1}^k\breve{M}\,d_i+\sum_{i=1}^k\breve{M}\left(\bar{d}_i+N_1+L_1(N_1)\right)\right]<1\quad,\qquad\text{for}\quad k=1,2,\ldots,m.$$

Then Υ is contraction. Therefore, the three conditions of lemma (2.2) are satisfied. Therefore, for this the operator $\Upsilon+\overline{\Upsilon}$ has a fixed point in E_r . Then the impulsive abstract Cauchy problem (1.2) has a mild solution on J.

Conclusions:

We conclude that the solvability of impulsive nonlinear fractional order partial differential equations with nonlocal conditions needed to define an approach to use the fractional Laplace transforms which make important roles for computing the formula of the semigroup family operators. Hence, we thought that it is an important and basic issue for solvability. The assumptions presented in this work are needed to prove that our problem, with the specific main results, was never assumed before for solving other problems. We used this approach to simulate the theoretical approach and gain solvability results of impulsive nonlinear fractional order partial differential equations with nonlocal conditions by transforming it to an impulsive abstract Cauchy problem. We also used the nonlinearity functional analysis as a suitable analytic tool for specific spaces and domains of operators to generalize the problem of nonlinear extensible beam equations and other problems.

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