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Estimating the Reliability Function of some Stress- Strength Models for the Generalized Inverted Kumaraswamy Distribution

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Abstract

This paper discusses reliability of the stress-strength model. The reliability functions $R1$ and $R2$ were obtained for a component which has an independent strength and is exposed to two and three stresses, respectively. We used the generalized inverted Kumaraswamy distribution GIKD with unknown shape parameter as well as known shape and scale parameters. The parameters were estimated from the stress- strength models, while the reliabilities $R1$, $R2$ were estimated by three methods, namely the Maximum Likelihood, Least Square, and Regression.

A numerical simulation study a comparison between the three estimators by mean square error is performed. It is found that best estimator between the three estimators is Maximum likelihood estimators.

Keywords: Reliability Function, Generalized Inverted Kumaraswamy Distribution (GIKD), Strength-Stress model, Estimation.

تقدير دالة معولية لبعض نماذج الاجهاد والمتانة لتوزيع معكوس كيومسوامي العام

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الخلاصه

هذا البحث ناقش معولية نماذج الاجهاد والمتانة . حصلنا على دوال المعولية ر1 و ر2 لمكونة تمتلك متانة مستقلة وتتعرض الى 2 و 3 من الاجهادات على التوالي باستخدام توزيع معكوس كيومسوامي العام بمعلمة الشكل غير المعلومة ومعلمتا الشكل والقياس المعلومة . المعلمات قدرناها من نماذج الاجهاد والمتانة وقدرنا المعولية الى كل من ر1 و ر2 باستخدام ثلاث طرق هي الامكان الاعظم والمربعات الصغرى والانحدار.

وبدراسة المحاكاة العددية تم مقارنة بين المقدرات الثلاثة بواسطة متوسط مربعات الخطأ تم الاداء وقد وجدنا ان افضل مقدر بين المقدرات الثلاثة هو مقدر الامكان الاعظم.

1 .Introduction

The stress-strength model is of a special importance in reliability literature. In the statistical approach to the stress-strength model, most of the considerations depend on the assumption that the

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component strengths are independently and identically distributed and are subjected to a common stress [1].

In this paper, X and Y are independent and identical in system reliability.

Let X is a strength random variable subjected to a common stress Y, then the reliability of a system that contains one component is: $R = P(X > Y)$. The reliability of a system is the probability that the system is operating under stated
environmental conditions [2].

The stress- strength reliability describes the life of a component which has a random strength X that is subjected to a random stress Y. When the stress applied to the component exceeds the strength, the component fails instantly and will function satisfactorily till $X > Y$. Therefore, R=(Y<X) is a measure of a component reliability. According to the statistical literature, it is known as a stress- strength parameter. It has wide applications in almost all areas of knowledge, especially in engineering, such as structures, deterioration of rocket motors, static fatigue of ceramic components, aging of concrete pressure vessels, etc.(Sharma *et al.*, 2014). A component which has strength that is independently exposed to two stresses was studied by Karaday *et al.* (2011). The authors estimated the reliability for a component exposed to two and three independent stresses based on Generalized Inverted Kumaraswamy Distribution, which is a heavy-tail probability often used in reliability approaches with extended applications in economic sciences, actuarial modeling, queuing problems, biological sciences, business, and life testing [3].

2. The Mathematical Formula of Reliability

The stress-strength model defines the life of a component having X strength and is exposed to Y stress. According to this, if stress exceeds strength $(Y > X)$ it will be impossible for a component to live. The reliability of such a system with one component, which consists of stress and strength, is expressed as:

$R = P(Y < X) = \int_0^\infty F_y(x)f(x)dx$

In order to continue working with the device, its power must be greater than the pressure it is exposed to . There exist many devices in various forms where the number of the stresses and the number of the components which have the strength vary according to the kind of the device . In this research , one component which has one strength and is exposed to two stresses was used to test the performance of the device and its reliability to achieve the function. After that, one component which has one strength and exposed to three stresses was utilized to examine range reliability of the device to proceed in functioning.

In this model, strength-stress random variables of a component which has strength (X) and exposed to stresses (Yj, j= 1, 2, 3), respectively, where X and Yj are independent identically distributed GIKD random variables with common known shape and scale parameters (α, γ) and unknown shape parameters β, bj, j= 1, 2, 3.

Karadayi *et al*. (2011) showed that when a component having X strength and exposed to two independent (Y_1 and Y_2) stresses , then the stress- strength reliability is obtained as follows [4]:

$R = P[Max(Y_1, Y_2) < X] = \int_{x=0}^{\infty} P(Y_1 < X)P(Y_2 < X)f_{x(X)}\ dx$

Thus, in the case of two Stress- one Strength Component Reliability, i.e.
a component having a strength X and exposed to two independent (Y_1 , Y_2) stresses, then the stress-strength reliability is obtained as follows:

$$R_1 = P[\max\{y_1, y_2\} < x]$$

$$= \int_0^\infty \int_0^x \int_0^x f(y_1, y_2, x)dy_1 dy_2 dx$$

$$= \int_0^\infty \left[\int_0^x f(y_1)dy_1\right]\left[\int_0^x f(y_2)dy_2\right] f(x)dx$$

$$R_1 = \int_0^\infty F_{y_1}(x)F_{y_2}(x)f(x)dx \quad (1)$$

Based on the reliability formula R_1, in the case a component is exposed to two independent stresses, which were found by Karadayi *et al.* [4], we proposed a formula to find reliability R_2 in case of a component having strength and exposed to three independent stresses (Y_1, Y_2, Y_3) [5]. We obtain the stress-strength reliability as follows :

$R_2 = P[\max\{y_1, y_2, y_3\} < x]$

$$= \int_0^\infty \int_0^x \int_0^x \int_0^x f\,(y_1, y_2, y_3, x) dy_1 dy_2 dy_3 dx$$

$$= \int_0^\infty \left[\int_0^x f(y_1) dy_1\right]\left[\int_0^x f(y_2) dy_2\right]\left[\int_0^x f(y_3) dy_3\right] f(x) dx$$

$$R_2 = \int_0^\infty F_{y_1}(x) F_{y_2}(x) F_{y_3}(x) f(x) dx \qquad (2)$$

3. Generalized Inverted Kumaraswamy distribution ($GIKD$)

Kumaraswamy [6] obtained a distribution derived from beta distribution after fixing some parameters. The distribution is appropriated to natural phenomena whose outcomes are bounded from both sides, such as the individuals' heights, test scores, temperatures, and hydrological daily data of rain fall .

The probability density function and distribution function of Kumaraswamy distribution are given, respectively, by

$f(x;\ \alpha, \beta) = \alpha\beta\, x^{\alpha-1} \quad [1 - x^{\alpha})]^{\beta-1} \quad , \alpha, \beta > 0 \quad , \quad x \in (0,1)$

$F(x;\ \alpha, \beta) = 1 - [1 - x^{\alpha})]^{\beta}$

The inverted distribution has many applications in different fields, such as biogical sciences, life testing problems, engineering sciences, environmental studies, and econometrics.

In the last two decades, the researchers proposed many inverted distributions due to their outstanding applications; One of the distribution types investigated is the inverted Kumaraswamy distribution [7].

Abd Al-Fattah *et al*. [7] derived the inverted Kumaraswamy (IKum) distribution from Kumaraswamy (Kum) distribution using the transformation $T = x^{-1} - 1$ when $X \sim$ Kum(α ,β), where α and β are shape parameters. Then, the probability density function and distribution function of the inverted Kumaraswamy distribution are given, respectively, by:

$f(t, \alpha, \beta) = \alpha\beta(1 + t)^{-(\alpha+1)}[1 - (1 + t)^{-\alpha}]^{\beta-1}, x > 0$ And

$F(t, \alpha, \beta) = [1 - (1 + t)^{-\alpha}]^{\beta}, x \geq 0$

where $\alpha > 0\ and\ \beta > 0$ are two shape parameters.

Iqbal [6] introduced an extension of the inverted Kumaraswamy distribution, called the generalized inverted Kumaraswamy distribution, by using transformation $t = x^{\gamma}$, with probability density function and distribution function given, respectively, by:

Let X be a random variable that refers to strength, then if $X \sim GIK(\alpha, \beta, \gamma)$, then the PDF of GIK is given as follows [7, 8] :

$$f(x;\ \alpha, \beta, \gamma) = \alpha\beta\gamma\, x^{\gamma-1}\, (1 + x^{\gamma})^{-(1+\alpha)} \quad [1 - (1 + x^{\gamma})^{-\alpha}]^{\beta-1} \qquad (3)$$

The CDF of GIK is:

$F(x;\ \alpha, \beta, \gamma) = [1 - (1 + x^{\gamma})^{-\alpha}]^{\beta}$ (4)

where $\alpha, \beta, \gamma > 0\ , x > 0\ \ , \alpha, \beta$ are shape parameters.

Let y be a random variable that refers to stress, then if $Y \sim GIK(\alpha, b, \gamma)$, then the PDF of GIK is:

$$f(y;\ \alpha, b, \gamma) = \alpha b\gamma\, y^{\gamma-1}\, (1 + y^{\gamma})^{-(1+\alpha)} \quad [1 - (1 + y^{\gamma})^{-\alpha}]^{b-1} \qquad (5)$$

where $y > 0\ , \alpha, b, \gamma > 0$

and the cdf of GIK is: $F(y;\ \alpha, b, \gamma) = [1 - (1 + y^{\gamma})^{-\alpha}]^{b}$ (6)

3-1 Two Stress- one Strength Component Reliability

As mentioned earlier, Y_i is stress $\sim GIK(\alpha, b_i, \gamma)$, $i = 1,2$

, and X is strength $\sim GIK(\alpha, \beta, \gamma)$

$$F(Y_i;\ \alpha, b_i, \gamma) = [1 - (1 + {Y_i}^{\gamma})^{-\alpha}]^{b_i}, i = 1,2 \qquad (7)$$

By substituting (3)and(7) in (1), we have:

$$R_1 = \int_0^\infty F_{y_1}(x)F_{y_2}(x)f(x)dx$$

$$R_1 = \int_0^\infty [1-(1+x^\gamma)^{-\alpha}]^{b_1}\,[1-(1+x^\gamma)^{-\alpha}]^{b_2}\ \alpha\beta\gamma\, x^{\gamma-1}\,(1+x^\gamma)^{-(1+\alpha)}$$
$$[1-(1+x^\gamma)^{-\alpha}]^{\beta-1}\,dx$$

$$= \int_0^\infty \beta[1-(1+x^\gamma)^{-\alpha}]^{b_1+b_2+\beta-1}\quad \alpha(1+x^\gamma)^{-(1+\alpha)}\quad \gamma x^{\gamma-1}\ dx$$

$$R_1 = \frac{\beta}{b_1+b_2+\beta} \tag{8}$$

3-2 Three Stress- one Strength Component Reliability

Let Y_i be stress $\sim GIK(\alpha, b_i, \gamma)$, $i = 1, 2, 3$, and X be strengh $\sim GIK(\alpha, \beta, \gamma)$.

$$F(Y_i;\ \alpha, b_i, \gamma) = [1-(1+{Y_i}^\gamma)^{-\alpha}]^{b_i} ,\ i = 1,2,3 \tag{9}$$

By substituting (3)and(9) in (2), we obtain:

$$R_2 = \int_0^\infty F_{y_1}(x)F_{y_2}(x)F_{y_3}(x)f(x)dx$$

$$R_2 = \int_0^\infty [1-(1+x^\gamma)^{-\alpha}]^{b_1}\ [1-(1+x^\gamma)^{-\alpha}]^{b_2}\ [1-(1+x^\gamma)^{-\alpha}]^{b_3}\ \alpha\beta\gamma\, x^{\gamma-1}\ (1+x^\gamma)^{-(1+\alpha)}$$
$$[1-(1+x^\gamma)^{-\alpha}]^{\beta-1}\,dx$$

$$= \int_0^\infty \beta[1-(1+x^\gamma)^{-\alpha}]^{b_1+b_2+b_3+\beta-1}\quad \alpha(1+x^\gamma)^{-(1+\alpha)}\quad \gamma x^{\gamma-1}\ dx$$

$$R_2 = \frac{\beta}{b_1+b_2+b_3+\beta} \tag{10}$$

4- Methods of Estimating the Reliability Function

4-1 Maximum likelihood method

This is one of the important methods of estimation which aims at making the likelihood function in great end [6]. Let $(X_1, X_2, \dots, X_n)$ strength be a random sample that has a $GIK(\alpha,\beta,\gamma)$ distribution with sample size n, where β is an unknown shape parameter. Then, the likelihood function $L(x_i;\alpha,\beta,\gamma) = \prod_{i=1}^{n} f(x_i;\alpha,\beta,\gamma)$, which is the joint probability function with a general form, can be written as follows [9]:

$$L(x_i;\alpha,\beta,\gamma) = f(x_1;\alpha,\beta,\gamma)f(x_2;\alpha,\beta,\gamma)\dots\dots\dots f(x_n;\alpha,\beta,\gamma)$$

$$L(x_i,\alpha,\beta,\gamma) = \alpha^n\beta^n\gamma^n \prod_{i=1}^{n} {x_i}^{\gamma-1} \prod_{i=1}^{n}(1+{x_i}^\gamma)^{-(1+\alpha)} \prod_{i=1}^{n}[1-(1+{x_i}^\gamma)^{-\alpha}]^{\beta-1}$$

$$Lnl = nln\alpha + nln\beta + nln\gamma + (\gamma-1)\sum_{i=1}^{n} lnx_i - (1+\alpha)\sum_{i=1}^{n} ln\,(1+{x_i}^\gamma)$$
$$+ (\beta-1)\sum_{i=1}^{n} ln[1-(1+{x_i}^\gamma)^{-\alpha}] \tag{11}$$

Taking partial derivatives, respectively, from equation (11), we get:

$$\frac{dlnl}{d\alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} ln\,(1+{x_i}^\gamma) + (\beta-1)\sum_{i=1}^{n}\frac{(1+{x_i}^\gamma)^{-\alpha}\ln(1+{x_i}^\gamma)}{[1-(1+{x_i}^\gamma)^{-\alpha}]} = 0$$

$$\hat{\alpha}_{ml} = \frac{n}{\sum_{i=1}^{n} ln\,(1+{x_i}^\gamma) - (\beta-1)\sum_{i=1}^{n}\frac{(1+{x_i}^\gamma)^{-\alpha}\ln(1+{x_i}^\gamma)}{[1-(1+{x_i}^\gamma)^{-\alpha}]}}$$

$$\frac{dlnl}{d\beta}=\frac{n}{\beta}+\sum_{i=1}^{n} ln[1-(1+x_i{}^{\gamma})^{-\alpha}]=0$$

$$\hat{\beta}_{ml}=-\frac{n}{\sum_{i=1}^{n} ln[1-(1+x_i{}^{\gamma})^{-\alpha}]} \qquad (12)$$

In the same way, let Y be a stress to a random sample that has $GIK(\alpha,b,\gamma)$ distribution with a sample size m, where b is an unknown parameter. The ML estimator for b ($\hat{b}_{ml}$) can be written as:

$$\hat{b}_{ml}=-\frac{m}{\sum_{j=1}^{m} ln[1-(1+y_j{}^{\gamma})^{-\alpha}]}$$

Now, for the first and second reliability functions,
let $Y_{1_{j_1}}; j_1=1,2,\dots,m_1$, $Y_{2_{j_2}}; j_2=1,2,\dots,m_2$ and $Y_{3_{j_3}}; j_3=1,2,\dots,m_3$ are stresses on a random sample that has $GIK(\alpha,b_1,\gamma)$, $GIK(\alpha,b_2,\gamma)$, and $GIK(\alpha,b_3,\gamma)$ distributions, with sample size of $m_1, m_2,$ and m_3, respectively, where b_1, b_2 and b_3 are unknown parameters.
And by using the same manner, the ML estimators to the unknown parameters b_1, b_2 and b_3 are:

$$\hat{b}_{k\ ml}=-\frac{m_k}{\sum_{j_k=1}^{m_k} ln[1-(1+y_{kj_k}{}^{\gamma})^{-\alpha}]} \quad ,k=1,2,3\dots\dots\dots.. \qquad (13)$$

By substituting equations (12) and (13) in relations (8) and (10), respectively, we get the reliability estimators $\hat{R}_{1ml}$ and $\hat{R}_{2ml}$:

$$\hat{R}_{1ml}=\frac{\hat{\beta}_{ML}}{\hat{b}_{1ml}+\hat{b}_{2ml}+\hat{\beta}_{ml}} \qquad \dots\dots\dots..(14)$$

$$\hat{R}_{2ml}=\frac{\hat{\beta}_{ML}}{\hat{b}_{1ml}+\hat{b}_{2ml}+\hat{b}_{3ml}+\hat{\beta}_{ml}} \qquad \dots\dots\dots\dots\dots\dots(15)$$

4-2 Least square (LS) estimation method

In the least square method the minimize function , the least squares method estimators can be produce by minimizing the sum of square error between the value and its expected value [10], as follows:

$$S=\sum_{i=1}^{n}\left[F(X_{(i)})-E\big(F(X_{(i)})\big)\right]^2$$

such that $F(X_{(i)})=[(1-(1+x_i{}^{\gamma})^{-\alpha}]^{\beta}$.

By minimizing the sum of square error between the cdf of GIKD and the expected value to cdf, we obtain:

$F(x_i)=E\big(F(x_i)\big)[9]$, such that $E\big(F(x_i)\big)=p_i=\dfrac{i}{n+1}$

Which is plotting position , $i=1,2,3,\dots\dots\dots..,n$

Since CDF of the Generalized Inverted Kumaraswamy distribution does not have a linear formula given to the parameters, we get the following linear formula:

$F(x_i)=p_i$

$[(1-(1+x^{\gamma})^{-\alpha}]^{\beta}=p_i$

By taking the logarithm of both sides, we have:

$\beta Ln[(1-(1+x^{\gamma})^{-\alpha}]=Lnp_i$

$$\beta Ln[(1-(1+x^{\gamma})^{-\alpha}]-Lnp_i=o \qquad (16)$$

By taking the square and summation to equation (16), we have:

$$s=\sum_{i=1}^{n}[\beta Ln[(1-(1+x^{\gamma})^{-\alpha}]-\ Lnp_i\ ]^2=0 \qquad (17)$$

By taking partial derivatives with respect to $\alpha,\beta,$ respectively, from equation(17), we obtain:

$$\frac{ds}{d\alpha}=\sum_{i=1}^{n}2[\beta Ln[(1-(1+x^{\gamma})^{-\alpha}]-\ Lnp_i\ ]\beta\frac{(1+x^{\gamma})^{-\alpha}\ln(1+x^{\gamma}))}{1-(1+x^{\gamma})^{-\alpha}}=0$$

$$\frac{ds}{d\beta}=\sum_{i=1}^{n}2[\beta Ln[(1-(1+x^{\gamma})^{-\alpha}]-\ Lnp_i\,]\,Ln[(1-(1+x^{\gamma})^{-\alpha}]=0$$

$$=\sum_{i=1}^{n}\beta[Ln(1-(1+x^{\gamma})^{-\alpha})\,]^2-\ \sum_{i=1}^{n}Ln(p_i)\ Ln[(1-(1+x^{\gamma})^{-\alpha}]=0$$

$$\hat{\beta}_{ls}=\frac{\sum_{i=1}^{n}Ln(p_i)\,Ln[(1-(1+x_i{}^{\gamma})^{-\alpha}]}{\sum_{i=1}^{n}[Ln(1-(1+x_i{}^{\gamma})^{-\alpha})\,]^2} \tag{18}$$

Accordingly, let Y be a stress to a random sample that has $GIK(\alpha,b,\gamma)$ distribution with sample size m, where b is an unknown shape parameter. Then the LS estimator for b ($\hat{b}_{ls}$) can be written as:

$$\hat{b}_{ls}=\frac{\sum_{j=1}^{m}Ln(p_j)\,Ln[(1-(1+y_j{}^{\gamma})^{-\alpha}]}{\sum_{j=1}^{m}[Ln(1-(1+y_j{}^{\gamma})^{-\alpha})\,]^2}$$

Now, for the first and second reliability functions,
let $Y_{1_{j_1}};j_1=1,2,\dots,m_1$, $Y_{2_{j_2}};j_2=1,2,\dots,m_2$ and $Y_{3_{j_3}};j_3=1,2,\dots,m_3$ are stresses to a random sample that has $GIK(\alpha\,,b_1,\gamma)$, $GIK(\alpha\,,b_2,\gamma)$, and $GIK(\alpha\,,b_3,\gamma)$ distributions, with sample size of $m_1,m_2,$ and m_3, respectively, where $b_1,b_2,$ and b_3 are unknown parameters.
By using the same manner, the LS estimators to the unknown parameters $b_1,b_2,$ and b_3 are:

$$\hat{b}_{k\ ls}=\frac{\sum_{j_k=1}^{m_k}Ln(p_{j_k})\,Ln[(1-(1+y_{kj_k}{}^{\gamma})^{-\alpha}]}{\sum_{j_k=1}^{m_k}[Ln(1-(1+y_{kj_k}{}^{\gamma})^{-\alpha})\,]^2}\quad ,k=1,2,3\ldots.. \tag{19}$$

By substituting the equations (18) and (19) in relations (8) and (10), respectively, we get the following reliability estimators:
$\hat{R}_{1\,ls}$ and $\hat{R}_{2\,ls}$

$$\hat{R}_{1\,ls}=\frac{\hat{\beta}_{ls}}{\hat{b}_{1\,ls}+\hat{b}_{2\,ls}+\hat{\beta}_{ls}} \quad \ldots\ldots\ldots..(20)$$

$$\hat{R}_{2\,ls}=\frac{\hat{\beta}_{ls}}{\hat{b}_{1\,ls}+\hat{b}_{2\,ls}+\hat{b}_{3\,ls}+\hat{\beta}_{ls}}\ldots \quad \ldots\ldots\ldots\ldots..(21)$$

4-3 Regression Estimation Method

Regression is one of the important procedures that use auxiliary information to construct estimators with good efficiency [11, 12].
The standard regression equation is:

$$z_i=a+b\,u_i+ei \tag{22}$$

where z_i is a dependent variable (response variable), u_i is an independent variable (explanatory variable), and e is the error r.v. independent identically.
By taking the logarithm to equation (4), which is the cdf of GIK, we have:
$F(x;\ \alpha,\beta,\gamma)=[(1-(1+x^{\gamma})^{-\alpha}]^{\beta}$

$$LnF(x_i)=\beta Ln[(1-(1+x_i{}^{\gamma})^{-\alpha}] \tag{23}$$

By comparing (22) and (23), we get
$zi=LnF(x_i)$, such that $F(x_i)=p_i$, then $LnF(x_i)=Ln(p_i)\,,p_i=\frac{i}{n+1}\,,i=1,2,\ldots\ldots,n$ $,a=0\ \ ,b=\beta\,,ui=\ Ln[(1-(1+x_i{}^{\gamma})^{-\alpha}]$
To estimate a and b from the equation $zi\ =a+bui+ei,$ we obtain:

$$s=\sum_{i=0}^{n}\ (zi\ -a-bui)^2$$

$$\frac{ds}{da}=\sum_{i=1}^{n}\ (zi\ -\hat{a}-bui)=0$$

$$\sum_{i=1}^{n}zi-n\hat{a}-b\sum_{i=1}^{n}\ ui$$

$$\sum_{i=1}^{n} zi - b\sum_{i=1}^{n} ui = n\hat{a}$$

$$\hat{a} = \frac{\sum_{i=1}^{n} zi - b\sum_{i=1}^{n} ui}{n} \qquad (24)$$

$$\frac{ds}{db} = \sum_{i=1}^{n} (zi - a - \hat{b}ui)ui = 0$$

$$\sum_{i=1}^{n} ziui - a\sum_{i=1}^{n} ui - \hat{b}\sum_{i=1}^{n}(ui)^2 = 0$$

$$\hat{a} = \frac{\sum_{i=1}^{n} ziui - \hat{b}\sum_{i=1}^{n}(ui)^2}{\sum_{i=1}^{n} ui} \qquad (25)$$

By equalizing the two relations of (24) and (25), we get:

$$\frac{\sum_{i=1}^{n} zi - b\sum_{i=1}^{n} ui}{n} = \frac{\sum_{i=1}^{n} ziui - \hat{b}\sum_{i=1}^{n}(ui)^2}{\sum_{i=1}^{n} ui}$$

$$\hat{b} = \frac{\sum_{i=1}^{n} zi\sum_{i=1}^{n} ui - n\sum_{i=1}^{n} zi\,ui}{\left(\sum_{i=1}^{n} ui\right)^2 - n\sum_{i=1}^{n}(ui)^2}$$

$$\hat{\beta}_{reg} = \frac{\sum_{i=1}^{n} Ln(p_i)\sum_{i=1}^{n} Ln[(1-(1+{x_i}^{\gamma})^{-\alpha}] - n\sum_{i=1}^{n} Ln(p_i)\, Ln[(1-(1+{x_i}^{\gamma})^{-\alpha}]}{\left(\sum_{i=1}^{n} Ln[(1-(1+{x_i}^{\gamma})^{-\alpha}]\right)^2 - n\sum_{i=1}^{n}(Ln[(1-(1+{x_i}^{\gamma})^{-\alpha}])^2} \ldots (26)$$

such that $p_i = \frac{i}{n+1}$, $i = 1,2,\ldots\ldots,n$

In the same way, let Y be a stress to a random sample that has $GIK(\alpha, b, \gamma)$ distribution, with sample size m, where b is an unknown shape parameter. The REG estimator for b ($\hat{b}_{reg}$) can be derived as:

$$\hat{b}_{reg} = \frac{\sum_{j=1}^{m} Ln(p_j)\sum_{j=1}^{m} Ln[(1-(1+{y_j}^{\gamma})^{-\alpha}] - m\sum_{j=1}^{m} Ln(p_j)\, Ln[(1-(1+{y_j}^{\gamma})^{-\alpha}]}{\left(\sum_{j=1}^{m} Ln[(1-(1+{y_j}^{\gamma})^{-\alpha}]\right)^2 - m\sum_{j=1}^{m}\left(Ln[(1-(1+{y_j}^{\gamma})^{-\alpha}]\right)^2}$$

such that $p_j = \frac{j}{m+1}$, $j = 1,2,\ldots\ldots,m$

Now, for the first and second reliability function,

let $Y_{1_{j_1}}; j_1 = 1,2,\ldots,m_1$, $Y_{2_{j_2}}; j_2 = 1,2,\ldots,m_2$ and $Y_{3_{j_3}}; j_3 = 1,2,\ldots,m_3$ be stresses to a random sample that has $GIK(\alpha, b_1, \gamma)$, $GIK(\alpha, b_2, \gamma)$, and $GIK(\alpha, b_3, \gamma)$ distributions, with sample size of m_1, m_2 and m_3, respectively, where b_1, b_2 and b_3 are unknown parameters.

By using the same manner, the REG estimators to the unknown parameters b_1, b_2 and b_3 are:

$$\hat{b}_{k_{reg}} = \frac{\sum_{j_k=1}^{m_k} Ln(p_{j_k})\sum_{j_k=1}^{m_k} Ln[(1-(1+{y_{k_{j_k}}}^{\gamma})^{-\alpha}] - m_k\sum_{j_k=1}^{m_k} Ln(p_{j_k})\, Ln[(1-(1+{y_{k_{j_k}}}^{\gamma})^{-\alpha}]}{\left(\sum_{j_k=1}^{m_k} Ln[(1-(1+{y_{k_{j_k}}}^{\gamma})^{-\alpha}]\right)^2 - m_k\sum_{j_k=1}^{m_k}\left(Ln[(1-(1+{y_{k_{j_k}}}^{\gamma})^{-\alpha}]\right)^2} \ldots (27)$$

where $k = 1,2,3$

such that $p_{j_k} = \frac{j_k}{m_k+1}$, $j_k = 1,2,\ldots\ldots,m_k$, $k = 1,2,3$

By substitute equations (26) and (27) in relations (8) and (10), respectively, we get the reliability estimators $\hat{R}_{1\,reg}$ and $\hat{R}_{2\,reg}$, as follows:

$$\hat{R}_{1\,reg} = \frac{\hat{\beta}_{reg}}{\hat{b}_{1\,reg} + \hat{b}_{2\,reg} + \hat{\beta}_{reg}} \ldots \ldots\ldots (28)$$

$$\hat{R}_{2\,reg} = \frac{\hat{\beta}_{reg}}{\hat{b}_{1\,reg} + \hat{b}_{2\,reg} + \hat{b}_{3\,reg} + \hat{\beta}_{reg}} \ldots\ldots\ldots\ldots\ldots (29)$$

5. Simulation

A simulation study is conducted to estimate the reliability by three methods of estimation and compare the results by using Mean Square Error.

By generating random variables, assume that U is a random variable with the uniform distribution of the interval (0,1). The sample random data are generated to follow Generalized Inverted Kumaraswamy distribution using the cdf and find the inverse of the distribution function, as follows: U=F(X)→X=$F^{-1}(U)$. By using eq.(4), we have:

$$(F)^{\frac{1}{\beta}} = 1-(1+x^{\gamma})^{-\alpha}$$

$$(1+x^{\gamma})^{-\alpha} = 1-(F)^{\frac{1}{\beta}}$$

$$1+x^{\gamma} = \left[1-(F)^{\frac{1}{\beta}}\right]^{\frac{-1}{\alpha}}$$

$$x^{\gamma} = \left[1-(F)^{\frac{1}{\beta}}\right]^{\frac{-1}{\alpha}} - 1$$

$$x = \left[\left[1-(F)^{\frac{1}{\beta}}\right]^{\frac{-1}{\alpha}} - 1\right]^{\frac{1}{\gamma}}$$

by letting U = F($x;\ \alpha,\beta,\gamma$), where U is a uniform continuous random variable defined on the interval (0,1), which can be used to generate random samples following the Generalized Inverted Kumaraswamy.

The simulation process was designed in the following basic stages, which are important to find the estimation reliability of the shape parameter for the generalized Inverted Kumaraswamy distribution.

Step1: Different sizes of the sample were selected, which are proportional to the effect of sample size on the accuracy of the results obtained by using the three approaches of this paper, where the small sample size was selected to be 15, medium sample size to be 30, and large sample size to be 90.

The random samples are $x_1,x_2,\ldots,x_n$, $y_{11},y_{12},\ldots,y_{1m1}$, $y_{21},y_{22},\ldots,y_{2m2}$, $y_{31},y_{32},\ldots,y_{3m3}$, of sizes ($n,m_1,m_2,m_3$) , such that sizes ($n,m_1,m_2,m_3$) = (15,15,15,15), (30,30,30,30), (90,90,90,90) ,(15,30,30,90) ,(30,90,90,30) and (15,30,90,15).

Step 2-Real parameter values are selected for 6 experiments ($\beta, b_1, b_2, b_3, \alpha, \gamma$), as shown in the following table.

Experiment	β	B_1	B_2	B_3	α	γ
1	2	1	3	1.5	1	1
2	2	1	3	1.5	2	1
3	2	1	3	1.5	1	2
4	3	2.5	3.1	2.8	1	1
5	3	2.5	3.1	2.8	2	1
6	3	2.5	3.1	2.8	1	2

Step3: We estimate the parameters $\beta, b1, b2, b3$ by Maximum likelihood , Least Square, and Regression methods as in (12), (13), (18), (19), (26), and (27), respectively.

Step4: Estimation of reliability as in (14), (15), (20), (21), (28) and (29).

Step5: Calculation of the mean by the relation: Mean=$\frac{\sum_{i=1}^{L}\hat{R}_i}{L}$,

where$\hat{R}$ is the estimation of the parameter R.

Step6. Comparing the three different methods of estimation by using Mean Square Error : $(\hat{R}) = \frac{1}{L}\sum_{i=1}^{L}(\hat{R}_i - R)^2$.

where L represents the number of replications for any experiment, and the number of iterations is selected to be L = 500.

Simulation Results

Using the proposed estimation methods, the results presented in Table-1 were obtained.

Table 1-The ($\hat{R}1$, $\hat{R}2$ and MSE) values for experiment (1) such that R1-REAL= 0.333333 and R2-REAL= 0.266667.

Sample size	$\hat{R}1$-MLE	$\hat{R}1$-LS	$\hat{R}1$-REG	$\hat{R}2$-MLE	$\hat{R}2$-LS	$\hat{R}2$-REG
(15,15,15,15)	0.330338	0.329216	0.327791	0.248485	0.247793	0.246675
	0.005327	0.006188	0.008525	0.003756	0.004313	0.005964
(30,30,30,30)	0.336173	0.337431	0.338589	0.253824	0.254613	0.255512
	0.002637	0.003271	0.00482	0.001913	0.002285	0.003245
(90,90,90,90)	0.33162	0.331762	0.331958	0.24856	0.248793	0.249075
	0.000794	0.000969	0.001508	0.000839	0.000951	0.001296
(15,30,30,90)	0.343322	0.33203	0.330373	0.259163	0.246381	0.24448
	0.004623	0.005062	0.007206	0.003055	0.003627	0.005019
(30,90,90,30)	0.335191	0.323359	0.32034	0.251304	0.244523	0.242743
	0.001932	0.002457	0.003741	0.001641	0.002209	0.003172
(15,30,90,15)	0.335558	0.320471	0.319932	0.252089	0.243404	0.243666
	0.004167	0.00481	0.00688	0.003239	0.003956	0.005468

Table 2-The $\hat{R}1$, $\hat{R}2$, and MSE values for experiment (2) such that R1-REAL= 0.333333 and R2-REAL= 0.266667.

Sample size	$\hat{R}1$-MLE	$\hat{R}1$-LS	$\hat{R}1$-REG	$\hat{R}2$-MLE	$\hat{R}2$-LS	$\hat{R}2$-REG
(15,15,15,15)	0.329427	0.330638	0.331419	0.249051	0.25008	0.250763
	0.004962	0.005821	0.008112	0.003631	0.004099	0.005486
(30,30,30,30)	0.333157	0.334073	0.334765	0.250213	0.250757	0.251055
	0.002486	0.003006	0.004534	0.001859	0.002134	0.003046
(90,90,90,90)	0.336643	0.337039	0.337388	0.252498	0.252846	0.253174
	0.000887	0.001142	0.001792	0.000737	0.000885	0.001286
(15,30,30,90)	0.333674	0.323517	0.322215	0.251408	0.23915	0.237163
	0.004235	0.004963	0.007119	0.002975	0.003848	0.005375
(30,90,90,30)	0.337719	0.325355	0.322532	0.253005	0.24575	0.244096
	0.001859	0.00218	0.003346	0.001537	0.002015	0.002933
(15,30,90,15)	0.339758	0.321272	0.316699	0.254322	0.243191	0.240158
	0.003978	0.004383	0.006248	0.003332	0.003823	0.005092

Table 3-The $\hat{R}1$, $\hat{R}2$, and MSE values for experiment (3) such that R1-REAL= 0.333333 and R2-REAL= 0.266667.

Sample size	$\hat{R}1$-MLE	$\hat{R}1$-LS	$\hat{R}1$-REG	$\hat{R}2$-MLE	$\hat{R}2$-LS	$\hat{R}2$-REG
(15,15,15,15)	0.33636	0.336631	0.337509	0.253909	0.254239	0.25506
	0.005321	0.00601	0.008337	0.003735	0.004253	0.005854
(30,30,30,30)	0.330924	0.330235	0.329882	0.248916	0.248508	0.24839
	0.002626	0.003169	0.004702	0.001974	0.002327	0.003349
(90,90,90,90)	0.333168	0.332473	0.331905	0.249342	0.248884	0.248445
	0.000804	0.001061	0.001752	0.00082	0.000993	0.001437
(15,30,30,90)	0.33796	0.327048	0.325247	0.256261	0.24393	0.242101
	0.004488	0.005232	0.007558	0.003156	0.003928	0.005497
(30,90,90,30)	0.335162	0.322867	0.319262	0.251106	0.243501	0.240984
	0.001972	0.002503	0.003768	0.001699	0.002277	0.003208
(15,30,90,15)	0.340291	0.324282	0.322065	0.255103	0.24653	0.245794
	0.003873	0.004439	0.006427	0.002987	0.003627	0.005095

Table 4-The $\hat{R}1$, $\hat{R}2$, and MSE values for experiment (4) such that R1-REAL= 0.348837 and R2-REAL= 0.263158.

Sample size	$\hat{R}1$-MLE	$\hat{R}1$-LS	$\hat{R}1$-REG	$\hat{R}2$-MLE	$\hat{R}2$-LS	$\hat{R}2$-REG
(15,15,15,15)	0.331933	0.332946	0.333852	0.25	0.250677	0.251507
	0.004769	0.005511	0.007767	0.003178	0.003647	0.005091
(30,30,30,30)	0.333142	0.333509	0.334124	0.250322	0.250146	0.250183
	0.002637	0.003127	0.004567	0.001736	0.002048	0.002991
(90,90,90,90)	0.333289	0.333539	0.334002	0.250808	0.250881	0.251094
	0.001066	0.001238	0.001783	0.000681	0.000828	0.001217
(15,30,30,90)	0.342338	0.333349	0.333268	0.259142	0.247921	0.247033
	0.004598	0.005352	0.007287	0.003116	0.003511	0.00472
(30,90,90,30)	0.33679	0.324215	0.320841	0.252401	0.244964	0.24286
	0.001998	0.002868	0.004291	0.001472	0.001974	0.002965
(15,30,90,15)	0.33964	0.325266	0.325288	0.255196	0.24748	0.248509
	0.003976	0.004922	0.006825	0.002976	0.003487	0.004904

Table 5-The $\hat{R}1$, $\hat{R}2$, and MSE values for experiment (5) such that R1-REAL= 0.348837 and R2-REAL= 0.263158.

Sample size	$\hat{R}1$-MLE	$\hat{R}1$-LS	$\hat{R}1$-REG	$\hat{R}2$-MLE	$\hat{R}2$-LS	$\hat{R}2$-REG
(15,15,15,15)	0.337145	0.33595	0.335088	0.253195	0.252231	0.251771
	0.005779	0.006916	0.009814	0.003636	0.00445	0.006412
(30,30,30,30)	0.336671	0.33734	0.338021	0.25186	0.252007	0.252289
	0.002365	0.003053	0.004672	0.001585	0.001996	0.003019
(90,90,90,90)	0.333206	0.331821	0.33039	0.249552	0.248522	0.247479
	0.001039	0.001355	0.002035	0.000684	0.000862	0.001269
(15,30,30,90)	0.332862	0.322564	0.321574	0.251062	0.239547	0.238445
	0.003844	0.004819	0.006928	0.002472	0.003195	0.004558
(30,90,90,30)	0.336693	0.322485	0.317342	0.251806	0.243311	0.240144
	0.002076	0.002894	0.004281	0.001612	0.002074	0.003023
(15,30,90,15)	0.345634	0.329734	0.328269	0.260482	0.251525	0.251299
	0.003938	0.004786	0.006901	0.002944	0.003448	0.005086

Table 6-The $\hat{R}1$, $\hat{R}2$, and MSE values for experiment (6). R1-REAL= 0.348837 and R2-REAL= 0.263158.

Sample size	$\hat{R}1$-MLE	$\hat{R}1$-LS	$\hat{R}1$-REG	$\hat{R}2$-MLE	$\hat{R}2$-LS	$\hat{R}2$-REG
(15,15,15,15)	0.332434	0.330994	0.330222	0.24869	0.247226	0.246396
	0.005851	0.006842	0.009465	0.004027	0.004829	0.00678
(30,30,30,30)	0.332943	0.331943	0.331262	0.250358	0.249622	0.249054
	0.002713	0.00319	0.004651	0.001761	0.002135	0.00312
(90,90,90,90)	0.332436	0.331834	0.331685	0.249536	0.249291	0.24928
	0.001158	0.001499	0.002249	0.000749	0.000938	0.001387
(15,30,30,90)	0.338511	0.329981	0.330658	0.255991	0.245575	0.245639
	0.004387	0.004968	0.006772	0.002953	0.003333	0.004507
(30,90,90,30)	0.340829	0.328831	0.326334	0.255488	0.248272	0.246734
	0.001998	0.002716	0.004009	0.001455	0.001882	0.002802
(15,30,90,15)	0.339539	0.323895	0.32198	0.254347	0.245838	0.245167
	0.004171	0.005392	0.00757	0.003066	0.003765	0.00522

Conclusions

The reliability, $R1 = P[\max\{Y_1, Y_2\} < X]$, of a component having X Generalized Inverted Kumaraswamy strength and exposed to (Y1 and Y2) Generalized Inverted Kumaraswamy stresses with an unknown shape parameter (β) and common known shape (α) and scale (γ) parameters was obtained. Then, the reliability, $R2= P[\max\{Y_1, Y_2, Y_3\} < X]$, of a component having X Generalized Inverted Kumaraswamy strength and was derived and exposed to Y1, Y_2 and Y3 Generalized Inverted Kumaraswamy stresses.

MSE criteria were used to make a comparison between three different estimators for each reliability function. All tables in the simulation were noticed and studied . $\hat{R}1$ and $\hat{R}2$ were found in all cases where MSE was the least in the MLE method. Therefore, based on the estimation of the best method, MLE had the first order, followed by LS in the second order and REG in the third order.

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